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LINEAR CONNECTIONS ON ALMOST COMMUTATIVE ALGEBRAS

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Abstract. In this paper we study linear connection on bimodules over almost commutative algebras using the framework of noncommutative geometry. We also present its curvature and its torsion. As an example there are presented linear connections on N-dimmensional quantum hyperplane over two kind of bimodules.

1. Introduction

There are some ways to introduce the notion of linear connection in noncommutative geometry. In [**8**] there is introduced the notion of linear connection in noncommutative geometry on an algebra A, in [**5**] there are presented linear connections on central bimodules, in [**6**] there are studied the $E-$connections over a graded Lie-Cartan pair (L, A) and in [**2**] there are linear connection on colour bimodules over a colour algebra.

In this paper we use the framework of noncommutative geometry to present linear connections on bimodules over an almost commutative algebra. These connections are a natural generalizations of $E-$connections over a graded Lie-Cartan pair (L, A) from [**6**] in the case $L = \rho$-Der$A$.

The paper is organized as follows: In the second paragraph we review the basic notions concerning almost commutative algebras, the derivations on almost commutative algebras (see [**1**]). In the third paragraph we introduce linear connection on a bimodule over an almost commutative algebra and we define its curvature. Then we study some properties of the curvature and thus we deduce the Bianchi identity. In the particular case when the bimodule M is ρ-Der$A$ we obtain linear connections on an almost commutative algebra and in this situation we present its torsion. Some examples of almost commutative algebras and connections on these algebras are presented.

In the last section we exemplified shortly the N-dimensional quantum hyperplane (as in [**1**]) and we show our construction of linear connections on N-dimensional quantum hyperplane over two kind of bimodules: ρ-Der(S_N^q) and $\Omega^1(S_N^q)$.

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2. Almost commutative algebra

In this section we present shortly a class of noncommutative algebras which are almost commutative algebras, for more details see [**1**].

Let G be an abelian group, additively written, and let A be a G-graded algebra. This means that as a vector space A has a G-grading $A = \oplus_{a\in G} A_a$, and that one has moreover $A_a A_b \subset A_{a+b}$ $(a, b \in G)$. The G-degree of a (nonzero) homogeneous element f of A is denoted as $|f|$. Futhermore let $\rho : G \times G \to k$ be a map which satisfies

$$\rho(a,b) = \rho(b,a)^{-1}, \qquad a, b \in G, \tag{1}$$

$$\rho(a+b,c) = \rho(a,c)\rho(b,c), \qquad a, b, c \in G. \tag{2}$$

This implies $\rho(a,b) \neq 0$, $\rho(0,b) = 1$, and $\rho(c,c) = \pm 1$, for all $a, b, c \in G$, $c \neq 0$. We define for homogeneous elements f and g in A an expression, which is *ρ-commutator* of f and g as:

$$[f,g]_\rho = fg - \rho(|f|\,|g|)gf. \tag{3}$$

This expression as it stands make sense only for homogeneous element f and g, but can be extended linearly to general elements. The ρ-bracket has the following properties:

$$[A_a, A_b] \subset A_{a+b}, \qquad a, b \in G, \tag{4}$$

$$[f,g]_\rho = -\rho(|f|\,|g|)[g,f]_\rho, \tag{5}$$

$$\begin{aligned} \rho(|f|,|h|)^{-1}[f,[g,h]_\rho]_\rho \quad &+ \quad \rho(|g|,|f|)^{-1}[g,[h,f]_\rho]_\rho + \\ &+ \quad \rho(|h|,|g|)^{-1}[h,[f,g]_\rho]_\rho = 0, \qquad f, g, h \in A. \end{aligned} \tag{6}$$

Eq. (5) may be called ρ-antisymmetric and Eq. (6) the *ρ-Jacobi identity.*

A *ρ-derivation* X of A, of degree $|X|$, is a bilinear map $X : A \to A$, of G-degree $|X|$, such that one has for all homogeneous elements f and g in A

$$X(fg) = (Xf)g + \rho(|X|,|f|)f(Xg). \tag{7}$$

It is known that the ρ-commutator of two derivations is again a ρ-derivation and the linear space of all ρ-derivations is a ρ-Lie algebra, denoted by ρ-Der$A$.

A G-graded algebra A with a given cocycle ρ will be called *ρ-commutative* if $fg = \rho(|f|,|g|)gf$ for all homogeneous elements f and g in A.

One verifies immediately that for such an A, ρ-Der$A$ is not only a ρ-Lie algebra but also a left A-module with the action of A on ρ-Der$A$ defined by

$$(fX)g = f(Xg) \qquad f, g \in A, X \in \rho\text{-Der}A. \tag{8}$$

Let M be a G-graded left module over a ρ-commutative algebra A, with the usual properties, in particular $|f\psi| = |f| + |\psi|$ for $f \in A$, $\psi \in M$. Then M is also a right A-module with the right action on M defined by

$$\psi f = \rho(|\psi|,|f|)f\psi. \tag{9}$$

In fact M is a bimodule over A, i.e.

$$f(\psi g) = (f\psi)g \qquad f, g \in A, X \in M. \tag{10}$$

Because there is no essential difference between left and right modules over a ρ-commutative algebra and because of the properties of ρ we have the possibility of doing multilinear algebra in terms of A modules, starting from ρ-Der$A$, and that is all that the general pseudogeometric formalism that we will develop.

In this context it can be introduce p-forms: $\Omega^0(A) := A$, and $\Omega^p(A)$ for $p = 1, 2, \ldots$, as the G-graded vector space of p-linear maps $\alpha_p : \times(\rho\text{-Der}A) \to A$, p-linear in sense of A-modules

$$\alpha_p(fX_1, \ldots, X_p) = f\alpha_p(X_1, \ldots, X_p), \tag{11}$$

$$\alpha_p(X_1, \ldots, X_j f, X_{j+1}, \ldots, X_p) = \alpha_p(X_1, \ldots, X_j, fX_{j+1}, \ldots X_p) \tag{12}$$

and ρ-alternating

$$\begin{aligned} &\alpha_p(X_1, \ldots, X_j, X_{j+1}, \ldots, X_p) = \\ &\qquad = -\rho(|X_j|, |X_{j+1}|)\alpha_p(X_1, \ldots, X_{j+1}, X_j, \ldots, X_p) \end{aligned} \tag{13}$$

for $j = 1, \ldots, p-1$; $X_k \in \rho$-Der$A$, $k = 1, \ldots, p$; $f \in A$ and Xf is the right A-action on ρ-Der$A$ defined by (9).

$\Omega^p(A)$ is in natural way a G-graded right A-module with

$$|\alpha_p| = |\alpha_p(X_1, \ldots, X_p)| - (|X_1| + \ldots + |X_p|) \tag{14}$$

and with the right action of A defined as

$$(\alpha_p f)(X_1, \ldots, X_p) = \alpha_p(X_1, \ldots, X_p)f. \tag{15}$$

The direct sum $\Omega(A) = \oplus_{p=0}^{\infty}\Omega^p(A)$ is again a G-graded A-module.

One defines exterior differentiation as a linear map $d : \Omega(A) \to \Omega(A)$, carrying $\Omega^p(A)$ into $\Omega^{p+1}(A)$, as

$$d\alpha_0(X) = X(f),$$

and for $p = 1, 2, \ldots$,

$$\begin{aligned} d\alpha_p(X_1, \ldots, X_{p+1}) := &\sum_{j=1}^{p+1}(-1)^{j-1}\rho(\sum_{i=1}^{j-1}|X_i|, |X_j|)X_j\alpha_p(X_1, \ldots, \widehat{X}_j, \ldots, X_{p+1}) + \\ &+ \sum_{1\le j<k\le p+1}(-1)^{j+k}\rho(\sum_{i=1}^{j-1}|X_i|, |X_j|)\rho(\sum_{i=1}^{j-1}|X_i|, |X_k|)\times \\ &\times \rho(\sum_{i=j+1}^{k-1}|X_i|, |X_k|)\alpha_p([X_j, X_k], \ldots, X_1, \ldots, \widehat{X}_j, \ldots, \widehat{X}_k, \ldots, X_{p+1}). \end{aligned} \tag{16}$$

One can show that d has degree 0, and that $d^2 = 0$.

In additional to d there are other linear operators in $\Omega(A)$. One has, for $X \in \rho$-Der$A$, a contraction i_X defined as

$$i_X\alpha_p(X_1, \ldots, X_{p-1}) := \rho(\sum_{i=1}^{p-1}|X_i|, |X|)\alpha_p(X, X_1, \ldots, X_{p-1}) \tag{17}$$

and *Lie derivative* L_X given by

$$\begin{aligned} L_X\alpha_p(X_1,\ldots,X_p) \quad := \quad & \rho(\sum_{i=1}^{p}|X_i|,|X|)X(\alpha_p(X,X_1,\ldots,X_{p-1})) - \\ & -\sum_{i=1}^{p}\rho(\sum_{i=1}^{p}|X_i|,|X|)\alpha_p(X_1,\ldots,[X,X_i],\ldots,X_p), \end{aligned} \tag{18}$$

with of course $i_X\alpha_0 = 0$, $\alpha_0 \in \Omega^0(A)$. Note that $|i_X| = |L_X| = |X|$.

3. Linear connections on bimodules over almost commutative algebras

In this paragraph we present our theory on the connections on bimodules over an almost commutative algebra A. We present the curvature of these connections and we show that the curvature is a ρ-symmetric operator from ρ-Der$A \times \rho$-Der$A$ into End(M). In the particular case when the bimodule M is ρ-Der$A$ we present the torsion of a connection. We give some examples of almost commutative algebras and connections on these algebras.

In this paragraph A=$\sum_{\alpha\in G} A$ is almost commutative algebra. M is a G-graded left A-bimodule.

Definition 1. A **connection** on M is a linear map of ρ-Der$A$ into the linear endomorphisms of M

$$\nabla : \rho\text{-Der}A \to \mathrm{End}(M),$$

such that one has:

$$\nabla_X : M_p \to M_{p+|X|}, \tag{19}$$

$$\nabla_{aX}(m) = a\nabla_X(m), \tag{20}$$

and

$$\nabla_X(am) = \rho(|X|,|m|)X(a)m + a\nabla_X(m), \tag{21}$$

for all $p \in G$, $a \in A$, and homogeneous elements $X \in \rho$-Der$A$ and $m \in M$.

Let ∇ be a connection as above. Its *curvature* R is the map

$$R : (\rho\text{-Der}A) \times (\rho\text{-Der}A) \to \mathrm{End}(M)$$

$$(X,Y) \longmapsto R_{X,Y}$$

defined by:

$$R_{X,Y}(m) = [\nabla_X,\nabla_Y](m) - \nabla_{[X,Y]}(m) \tag{22}$$

for any $X, Y \in \rho$-Der$A$, and $m \in M$, where the brackets are:

$$[\nabla_X,\nabla_Y] = \nabla_X\nabla_Y - \rho(|X|,|Y|)\nabla_Y\nabla_X$$

and

$$[X,Y] = X\circ Y - \rho(|X|,|Y|)Y\circ X.$$

Theorem 1. *The curvature of any connection ∇ has the following properties:*
1. A-linearity:

$$R_{aX,Y(m)} = aR_{X,Y}, \tag{23}$$

2. $R_{X,Y}$ is $A-linear$:

$$R_{X,Y}(am) = aR_{X,Y}(m) \tag{24}$$

3. R is a ρ-symmetric map:

$$R_{X,Y} = -\rho(|X|, |Y|)R_{Y,X} \tag{25}$$

for any $a \in A$, $m \in M$, $X, Y \in \rho$-Der$A$.

Proof. The relation (23) is obvious from the definition of the connection.
Let us prove the relation (24)

$$\begin{aligned}
R_{X,Y}(am) &= [\nabla_X, \nabla_Y](am) - \nabla_{[X,Y]}(am) = \\
&= \nabla_X \nabla_Y(am) - \rho(|X|, |Y|)\nabla_Y \nabla_X(am)\text{-}\nabla_{[X,Y]}(am) = \\
&= \nabla_X(\rho(|Y|, |m|)Y(a)m + a\nabla_Y(m)) - \\
&\qquad - \rho(|X|, |Y|)\nabla_Y(\rho(|X|, |m|)X(a)m + a\nabla_X(m)) - \\
&\qquad - \rho(|X| + |Y|, |m|)[X, Y](a)m - a\nabla_{[X,Y]}(m) = \\
&= \rho(|X|, |m|)X(\rho(|Y|, |m|)Y(a))m + \rho(|Y|, |m|)Y(a)\nabla_X(m) + \\
&\qquad + \rho(|X|, |Y| + |m|)X(a)\nabla_Y(m) + a\nabla_X\nabla_Y(m) - \\
&\qquad - \rho(|X|, |Y|)\rho(|Y|, |m|)Y(\rho(|X|, |m|)X(a))m - \\
&\qquad - \rho(|X|, |Y|)\rho(|X|, |m|)X(a)\nabla_Y(m) - \\
&\qquad - \rho(|X|, |Y|)\rho(|Y|, |X| + |m|)Y(a)\nabla_X(m) - \\
&\qquad - \rho(|X|, |Y|)a\nabla_Y\nabla_X(m) - \\
&\qquad - \rho(|X| + |Y|, |m|)[X, Y](a)m - a\nabla_{[X,Y]}(m) = \\
&\qquad = aR_{X,Y}(m).
\end{aligned}$$

In the last equality we used the followings relations:

$$X(\rho(|Y|, |m|)a) = \rho(|Y|, |m|)X(a)$$

and

$$\rho(|X|, |m|)X(\rho(|Y|, |m|)Y(a)) - \rho(|X|, |Y|)\rho(|Y|, |m|)Y(\rho(|X|, |m|)X(a)) = \\
= \rho(|X| + |Y|, |m|)[X, Y](a). \qquad \square$$

From previous theorem and from (6) we obtain the following Bianchi identity of linear connection over an almost commutative algebra A.

Theorem 2. *The curvature R of the connection ∇ satisfies the following Bianchi identity:*

$$\rho(|Z|,|X|)[\nabla_X, R_{Y,Z}] + \rho(|X|,|Y|)[\nabla_Y, R_{Z,X}] + \rho(|Y|,|Z|)[\nabla_Z, R_{X,Y}] = \rho(|Z|,|X|)R_{[X,Y],Z} + \rho(|X|,|Y|)R_{[Y,Z],X} + \rho(|Y|,|Z|)R_{[Z,X],Y}.$$

Next we present the case when the bimodule M is ρ-Der$A$.

By definition a *linear connection* on an almost commutative algebra A is a linear connection over the bimodule ρ-Der$A$.

If ∇ is a linear connection on an almost commutative algebra A we define its *torsion* as follows:

$$T : \rho\text{-Der}A \times \rho\text{-Der}A \to \rho\text{-Der}A,$$

$$T_\nabla(X,Y) = \nabla_X Y - \rho(|X|,|Y|)\nabla_Y X - [X,Y] \tag{26}$$

for any homogeneous $X, Y \in \rho$-Der$A$.

Remark 1. From the properties of the ρ-bracket and from the definition of the torsion T of an almost commutative algebra A, we have that $T \in \Omega^2(A)$.

Example 1. In the case when the function ρ is trivial ($G = \mathbf{Z}$, $\rho(\alpha,\beta) = 1$ for any $\alpha, \beta \in G$) the algebra A is commutative and we obtain the classical notion of connection on a smooth vector bundle E of finite rank over a smooth finite-dimensional paracompact manifold V by taking the algebra $A = C^\infty(V)$ of smooth functions on V and the module $\Gamma(E)$ of smooth sections of E.

Example 2. We consider the case when $A = \sum_{\alpha\in\mathbf{Z}} A_\alpha$ is a $\mathbf{Z}$-graded algebra and the function ρ is

$$\rho : \mathbf{Z} \times \mathbf{Z} \to k,\ \rho(\alpha,\beta) = (-1)^{\alpha\beta},$$

for any $\alpha, \beta \in \mathbf{Z}$.

Then A is a **super-commutative algebra** and the connection on A has the following form:

1. $\nabla_{aX_\lambda}(m) = a\nabla_{X_\lambda}(m)$,
2. $\nabla_{X_\lambda}(am_\alpha) = (-1)^{\lambda\alpha} X_\lambda(a)m_\alpha + a\nabla_{X_\lambda}(m_\alpha)$,

for any $m \in M$, $X_\lambda \in \rho$-Der$A$, $a \in A$, and $\alpha, \lambda \in \mathbf{Z}$.

Then the bracket is:

$$[X_\alpha, X_\beta] = X_\alpha X_\beta - (-1)^{\alpha\beta} X_\beta X_\alpha$$

and the curvature of the connection ∇ has the form:

$$R(X_\alpha, X_\beta) = \nabla_{X_\alpha}\nabla_{X_\beta} - (-1)^{\alpha\beta}\nabla_{X_\beta}\nabla_{X_\alpha} + \nabla_{[X_\alpha, X_\beta]}$$

The Bianchi identity is:

$$(-1)^{\alpha\lambda}[\nabla_{X_\alpha}, R_{X_\beta, X_\lambda}] + (-1)^{\alpha\beta}[\nabla_{X_\beta}, R_{X_\lambda, X_\alpha}] + (-1)^{\lambda\beta}[\nabla_{X_\lambda}, R_{X_\alpha, X_\beta}] =$$

$$(-1)^{\alpha\lambda} R_{[X_\alpha, X_\beta], X_\lambda} + (-1)^{\beta\alpha} R_{[X_\beta, X_\lambda], X_\alpha} + (-1)^{\lambda\beta} R_{[X_\lambda, X_\alpha], X_\beta}.$$

The torsion T of ∇ is given by:

$$T_\nabla(X_\alpha, X_\beta) = \nabla_{X_\alpha} X_\beta - (-1)^{\alpha\beta} \nabla_{X_\beta} X_\alpha - [X_\alpha, X_\beta].$$

Example 3. In the case when the group G is $\mathbf{Z}_2$, $A = A^0 \oplus A^1$ and the map $\rho : \mathbf{Z}_2 \times \mathbf{Z}_2 \to k$ is given by:

$$\rho(a, b) = (-1)^{ab}$$

then A is $\mathbf{Z}_2$-*graded commutative algebra* (see [**6**]).
ρ-Der$A$ is a $\mathbf{Z}_2$-*Lie-superalgebra* with the usual bracket:

$$[X_a, Y_b] = X_a Y_b - (-1)^{ab} Y_b X_a.$$

It follows that (ρ-DerA, A) is a graded Lie-Cartan pair and if E is an A-bimodule, then the notion of E-*connection* from [**6**] is a particular case of our connection over the bimodule E. In this case the *curvature* of a E-*connection* is curvature of connections from formula (22).

In the particular case when the bimodule E is ρ-Der$A$ then the torsion of ρ-Der$A$-connection is the torsion from formula (26).

4. Linear connections on quantum hyperplane

In this paragraph we study linear connections on N-dimensional quantum hyperplane using the notion of linear connection introduced in section 3.

First we present the definition of N-dimensional quantum hyperplane and the principal aspects of ρ-derivations and one-forms on the quantum hyperplane.

We follow the notations from [**1**] for the basic notions concerning N-dimensional quantum hyperplane.

4.1. N-dimensional quantum hyperplane

The N-dimensional quantum hyperplane is characterized by the algebra S_N^q generated by the unit element and N linearly independent elements $x_1, \ldots, x_N$ satisfying the relations:

$$x_i x_j = q x_j x_i, \qquad i < j$$

for some fixed $q \in k$, $q \neq 0$.

S_N^q is a $\mathbf{Z}^N-$graded algebra

$$S_N^q = \bigoplus_{n_1,\ldots,n_N}^{\infty} (S_N^q)_{n_1 \ldots n_N},$$

with $(S_N^q)_{n_1 \ldots n_N}$ the one-dimensional subspace spanned by products $x^{n_1} \ldots x^{n_N}$. The $\mathbf{Z}^N$-degree of these elements is denoted by

$$|x^{n_1} \cdots x^{n_N}| = n = (n_1, \ldots, n_N).$$

Define a function $\rho : \mathbf{Z}^N \times \mathbf{Z}^N \to k$ as

$$\rho(n, n') = q^{\sum_{j,k=1}^N n_j n'_k \alpha_{jk}}, \tag{27}$$

with $\alpha_{jk} = 1$ for $j < k$, 0 for $j = k$ and -1 for $j > k$. The N-dimensional quantum hyperplane S_N^q is an almost commutative algebra with the function ρ from (27).

We are in a special case where we have coordinate vector fields, the ρ-derivations $\partial/\partial x_i$, $j = 1,...N$, of $\mathbf{Z}^N$-degree $|\partial/\partial x_i|$, with $|\partial/\partial x_i| = -|x_i|$ and defined by $\partial/\partial x_i(x_j) = \delta_{ij}$. One has

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} = q\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}, \qquad j < k \tag{28}$$

and

$$\frac{\partial}{\partial x_j}(x_1^{n_1}\dots x_N^{n_N}) = n_j q^{(n_1+\dots+n_j)}(x_1^{n_1}\dots x_j^{n_{j-1}}\dots x_N^{n_N}), \tag{29}$$

which follows from the Leibniz relation

$$\frac{\partial}{\partial x_j}(fg) = \left(\frac{\partial}{\partial x_j}f\right)g + \rho\left(\left|\frac{\partial}{\partial x_j}\right|, |f|\right)f\left(\frac{\partial}{\partial x_j}g\right).$$

The ρ-derivations ρ-DerS_N^q is a free S_N^q-module of rank N with $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}$ as the basis. An arbitrary ρ-derivation X can be written as

$$X = \sum_{i=1}^{N} X_i \frac{\partial}{\partial x_i}, \tag{30}$$

with $X_i = X(x_i) \in S_N^q$, for $i = 1, \dots, N$.

$\Omega^1(S_N^q)$ is the S_N^q-module of one-forms and is also free of rank N. The coordinate of one-forms $dx_1, \dots, dx_N$, defined by

$$dx_i(X) = X(x_i) \tag{31}$$

or

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}, \tag{32}$$

form a basis in $\Omega^1(S_N^q)$, dual to the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}$ in $\rho-$DerS_N^q. Note that $|dx_i| = |x_i|$. For any $f \in S_N^q$ there is the following relation:

$$df = \sum_{i=1}^{N}(dx_i)\frac{\partial}{\partial x_i}f. \tag{33}$$

An arbitrary one-form can be written as

$$\alpha_1 = \sum_{i=1}^{N}(dx_i)A_i, \tag{34}$$

with

$$A_i = \alpha_1\left(\frac{\partial}{\partial x_i}\right) \in S_N^q. \tag{35}$$

4.2. Linear connections on N-dimensional quantum hyperplane

Remark that any linear connection along the field $X=\sum_{i=1}^{N}X_i\frac{\partial}{\partial x_i}$ is given by

$$\bigtriangledown_X=\sum_{i=1}^{N}X_i\bigtriangledown_{\frac{\partial}{\partial x_i}}$$

so any linear connections is well defined if is given along the field $\frac{\partial}{\partial x_i}$ for $i=1,\dots,N$. We use the notation

$$\bigtriangledown_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}=\Gamma^k_{i,j}\frac{\partial}{\partial x_k},\qquad \text{for } i,j=1,\dots,N, \tag{36}$$

where the coefficients $\Gamma^k_{i,j}\in S^q_N,\ \ i,j,k=1,\dots,N$ are denoted by *connection coefficients*. It follows that

$$\bigtriangledown_{\frac{\partial}{\partial x_i}}\left(x_j\frac{\partial}{\partial x_k}\right)=-q^{\alpha_{ij}}\delta_{ij}\frac{\partial}{\partial x_k}+x_j\Gamma^l_{i,j}\frac{\partial}{\partial x_l}=-q\frac{\partial}{\partial x_k}+x_j\Gamma^l_{i,j}\frac{\partial}{\partial x_l},$$

for $i,j,k=1,...,N$. In general

$$\bigtriangledown_{\frac{\partial}{\partial x_i}}\left(x_1^{n_1}\dots x_N^{n_N}\frac{\partial}{\partial x_k}\right)=-n_iq^{(n_1+\dots+n_i)}q^{\alpha_{ik}}(x_1^{n_1}...x_i^{n_{i-1}}\dots x_N^{n_N})\frac{\partial}{\partial x_k}+$$
$$+x_1^{n_1}\dots x_N^{n_N}\Gamma^l_{ik}\frac{\partial}{\partial x_l}$$

The curvature R of the linear connection $\bigtriangledown$ is given by the *curvature coefficients*: $R^l_{i,j,k}\in S^q_N$ defined by:

$$R_{\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k}=R^l_{i,j,k}\frac{\partial}{\partial x_l}\qquad \text{for } i,j,k=1,\dots,N.$$

From the properties of curvature we obtain that $R^l_{i,j,k}=q^{\alpha_{ij}}R^l_{j,i,k}$, for any $i,j,k,l\in 1,\dots,N$. The relation between curvature coefficient and connection coefficients is given by

$$\begin{aligned} R^l_{i,j,k} &= -\rho(|x_i|,\left|\Gamma^p_{j,k}\right|)\frac{\partial\Gamma^p_{j,k}}{\partial x_i}+\Gamma^p_{j,k}\Gamma^l_{i,p}- \\ &\quad -q^{\alpha_{i,j}}(-\rho(|x_j|,\left|\Gamma^p_{i,k}\right|)\frac{\partial\Gamma^p_{j,k}}{\partial x_j}+\Gamma^p_{i,k}\Gamma^l_{j,p})-\Gamma^l_{i,j,k}+q^{\alpha_{i,j}}\Gamma^l_{j,i,k}, \end{aligned} \tag{37}$$

where $\Gamma^l_{i,j,k}=\bigtriangledown_{\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k}$ and it satisfies $\Gamma^l_{i,j,k}=q^{\alpha_{i,j}}\Gamma^l_{j,i,k}$, for any $i,j,k,l=1,\dots,N$.

The torsion of any connection is given by the *torsion coefficients* $T^k_{i,j}$ defined as follows:

$$T_{\bigtriangledown}(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})=T^k_{i,j}\frac{\partial}{\partial x_k} \tag{38}$$

Evidently, the relations between torsion coefficients and connection coefficients are:

$$T^k_{i,j}\frac{\partial}{\partial x_k} = (\Gamma^k_{i,j} - q^{\alpha_{i,j}}\Gamma^k_{j,i})\frac{\partial}{\partial x_k} - \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]$$

4.3. Linear connections on quantum hyperplane over the bimodule $\Omega^1(S^q_N)$

We follow ideas from the previous section we present linear connections over bimodule $\Omega^1(S^q_N)$ over the quantum hyperplane. Without any confusion we use the notation

$$\nabla_{\frac{\partial}{\partial x_i}} dx_j = \Gamma^k_{i,j} dx_k, \qquad \text{for } i,j = 1,\ldots,N, \tag{39}$$

where the coefficients $\Gamma^k_{i,j} \in S^q_N$, $i,j,k = 1,\ldots,N$ are again denoted *connection coefficients* over bimodule $\Omega^1(S^q_N)$. We obtain that

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x_i}}(x_1^{n_1}\ldots x_N^{n_N} dx_k) &= -n_i q^{(n_1+\ldots+n_i)} q^{\alpha_{ik}}(x_1^{n_1}\ldots x_i^{n_{i-1}}\ldots x_N^{n_N})dx_k + \\ &\quad + x_1^{n_1}\ldots x_N^{n_N}\Gamma^l_{ik}dx_l\end{aligned}$$

The curvature R of the linear connection ∇ is given by the *curvature coefficients*: $R^l_{i,j,k} \in S^q_N$ defined by:

$$R_{\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}} dx_k = R^l_{i,j,k} dx_l \qquad \text{for } i,j,k = 1,\ldots,N.$$

We have that $R^l_{i,j,k} = q^{\alpha_{ij}} R^l_{j,i,k}$, for any $i,j,k,l \in 1,\ldots,N$ and the relation between curvature coefficients and connection coefficients is given by same relation like (37).

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