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"MORE OR LESS" FIRST-RETURN RECOVERABLE FUNCTIONS

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Abstract. It is known that a real-valued function defined on the unit interval is first-return recoverable if and only if itbelongs to Baire class one. Further, it is known that if first-return recoverability is replaced by stronger notions, such as universal or consistent first-return recoverability, then familiar subclasses of the Baire one functions are obtained. Likewise, if first-return recoverability is weakened to first-return recoverability except on a set of measure zero [first category], then one obtains precisely the class of Lebesgue measurable functions [functions having the Baire property]. Here we examine the situation where even smaller exceptional sets (countable or scattered) are excluded, and then explore possibility of combining these various methods for strengthening and weakening recoverability.

1. Introduction

It has been over a decade since the class of real-valued functions on the unit interval $I \equiv [0,1]$ which are first-return recoverable was shown to be identical to the class of Baire one functions [**5**]. In subsequent years both strengthenings and weakenings of the notion of first-return recoverability have been used to characterize some standard classes of functions in real analysis. For example, the universally first-return recoverable functions are the quasicontinuous Baire one functions [**4**] and the almost everywhere first-return recoverable functions are the Lebesgue measurable functions [**6**]. In this paper we wish to pursue a systematic investigation of classifications of functions which arise when strengthening and weakening the notion of first-return recovery in various natural ways. Before beginning, we need to recall the terminology and notation associated with first-return recoverability.

Underlying most of our subsequent definitions is the notion of what we call a *trajectory.* A *trajectory* is any sequence $\overline{x} = \{x_n\}$ of distinct points in I, whose range is dense in I. Any countable dense set $D \subset I$ is called a *support set* and, of course, any enumeration of D becomes a trajectory. For a given trajectory

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$\overline{x} = \{x_n\}$ and a finite union H of intervals, we let $r(\overline{x}, H)$ denote the first x_n that belongs to H.

For $x \in [0,1]$ and $\rho > 0$ we let $B_\rho(x) = \{y \in [0,1] : |y-x| < \rho\}$. As is standard, we denote the restriction of a function $f : I \to \mathbb{R}$ to a set $E \subseteq I$ by $f|_E$.

Definition 1.1. Let $x \in I$ and let $\overline{x} = \{x_n\}$ be a fixed trajectory. The *first return route to* x, $R(\overline{x}, x) = \{w_{\overline{x},k}(x)\}_{k=1}^{\infty}$ (or $\{w_k(x)\}_{k=1}^{\infty}$ when the trajectory is understood), is defined recursively via

$$w_1(x) = x_1,$$

$$w_{k+1}(x) = \begin{cases} r\left(\overline{x}, B_{|x-w_k(x)|}(x)\right) & \text{if } x \neq w_k(x) \\ x & \text{if } x = w_k(x). \end{cases}$$

We say that f is *first return recoverable with respect to* $\overline{x}$ at x provided that

$$\lim_{k\to\infty} f(w_k(x)) = f(x),$$

and if this happens for each $x \in I$, we say that f is *first return recoverable with respect to* $\overline{x}$. Finally, we say that f is *first-return recoverable* if it is first-return recoverable with respect to some trajectory.

2. Functions which are recoverable except on small sets

Here we shall consider functions $f : I \to \mathbb{R}$ which are recoverable except at points in a set which is small in one sense or another.

Definition 2.1. Let $f : I \to \mathbb{R}$. We say that f is

1. *almost recoverable* ($f \in \mathcal{AR}$) if there exists a trajectory $\overline{x}$which recovers f at each point of $I \setminus S$, where S is of measure zero.
2. *typically recoverable* ($f \in \mathcal{TR}$) if there exists a trajectory $\overline{x}$ which recovers f at each point of $I \setminus S$, where S is of first category.
3. *nearly recoverable* ($f \in \mathcal{NR}$) if there exists a trajectory $\overline{x}$ which recovers f at each point of $I \setminus S$, where S is countable.
4. *very nearly recoverable* ($f \in \mathcal{SR}$) if there exists a trajectory $\overline{x}$ which recovers f at each point of $I \setminus S$, where S is scattered. (Recall that a set $S \subset \mathbb{R}$ is *scattered* if it contains no nonempty dense-in-itself subset, or equivalently, if S is a countable $\mathcal{G}_\delta$.)

In [**6**] it was shown that $f \in \mathcal{AR}$ if and only if f is measurable, and that $f \in \mathcal{TR}$ if and only if f has the Baire property. Our first immediate goal for this section is to classify the smaller class $\mathcal{NR}$. We shall utilize a few simple lemmas.

Lemma 2.1. *If $E = \{e_n\}$ is scattered, and $\{y_n\} \subset \mathbb{R}$ is an arbitrary countable set, then*

$$h(x) \equiv \begin{cases} 0 & \text{if } x \notin E \\ y_n & \text{if } x = e_n \end{cases}$$

is a Baire 1 function.

Proof. We actually show that h is in the first Borel class which in our case, is equivalent. Let U be open. Then $h^{-1}(U) = E_U \cup Z_U$ where $E_U \subset E$ and Z_U is either $\emptyset$ or $[0,1]\backslash E$ depending on whether U contains 0 or not. In either case, $E_U \in \mathcal{F}_\sigma$. □

The next lemma is an immediate consequence.

Lemma 2.2. *Suppose f is a Baire 1 function, $\{y_n\}$ is countable and $E = \{e_n\}$ is scattered. Then*

$$g(x) \equiv \begin{cases} f(x) & \text{if } x \notin E \\ y_n & \text{if } x = e_n \end{cases}$$

is a Baire 1 function.

Proof. Since E is scattered, it follows from Lemma 2.1 that $h \in B_1$ where

$$h(x) \equiv \begin{cases} 0 & \text{if } x \notin E \\ y_n - f(e_n) & \text{if } x = e_n \end{cases}.$$

But, $g(x) = f(x) + h(x)$ and as the class of Baire 1 functions is closed under addition, the result follows. □

With this lemma we are able to establish the following result, which may be of independent interest.

Lemma 2.3. *If $f : [0,1] \to \mathbb{R}$ belongs to honorary Baire class two, then there exists a Baire class one function g^* such that the set $E \equiv \{x : f(x) \neq g^*(x)\}$ is countable and such that the graph of g^* restricted to the complement of E is dense in the graph of g^*.*

Proof. Let f belong to honorary Baire class two. Then there is a Baire 1 function g for which $\{x : f(x) \neq g(x)\} \equiv E$ is countable. Let A denote $[0,1]\backslash E$.

If $\epsilon > 0$, we say that a point x is *ϵ-isolated* from a set S if the distance between x and S is at least ϵ. Using the notation $C(f|_A, x)$ to denote the cluster set of $f|_A$ at x, we let

$$E_1 = \{x \in E : g(x) \in C(f|_A, x)\}$$
$$E_2 = \{x \in E : g(x) \notin C(f|_A, x)\} \equiv \{x \in E : g(x) \notin C(g|_A, x)\}.$$

For each natural number n we set

$$E_{2,n} \equiv \{x \in E_2 : g(x) \text{ is } \frac{1}{n} - \text{isolated from } C(g|_A, x)\},$$

and note that $E_2 = \bigcup_{n=1}^{\infty} E_{2,n}$. We claim that each $E_{2,n}$ is scattered. To see this, fix an n and suppose that $E_{2,n}$ contains a dense-in-itself subset D. Then $\overline{D}$ is perfect and since g belongs to Baire class one, there is a point $s \in \overline{D}$ at which the function $g|_{\overline{D}}$ is continuous at s. Choose $0 < \epsilon < 1/4n$ and $\delta > 0$ so that if $x \in \overline{D}$ and $|x - s| < \delta$, then $|g(x) - g(s)| < \epsilon$. Next, choose $x^* \in D$ such that $|x^* - s| < \delta$.

Since $\overline{D}$ is perfect and E is countable, there exists a sequence $\{x_k\}$ in $\overline{D} \cap A$ such that $x_k \to x^*$. Since each $x_k \in A$ and since the distance from $g(x^*)$ to $C(g|_A, x^*)$ is at least $1/n$, there exists a natural number K such that for all $k > K$,

$|g(x_k) - g(x^*)| > 1/2n$ and $|x_k - x^*| < \delta - |x^* - s|$. For such a $k > K$ we have $x_k \in \overline{D}$, $|x_k - s| < \delta$, and

$$|g(x_k) - g(s)| \geq |g(x_k) - g(x^*)| - |g(x^*) - g(s)| > \frac{1}{2n} - \epsilon > \epsilon,$$

and this contradiction completes the proof of our claim that $E_{2,n}$ is scattered.

Next, define $H_0 = E_1$ and for $n \in \mathbb{N}$, $H_n = E_{2,n+1} \backslash E_{2,n}$. Now let $h_0(x) = g(x)$ and

$$h_{n+1}(x) = \begin{cases} h_n(x) & \text{if } x \notin H_n \\ y^*(x) & \text{if } x \in H_n \end{cases}$$

where $y^*(x)$ is any point of $C(g|_A, x)$ with $\frac{1}{n+1} < |y^*(x) - g(x)| \leq \frac{1}{n}$ in the case that $n \geq 1$ and any point of $C(g|_A, x)$ whatsoever if $n = 0$. It follows from Lemma 2.2 that h_n belongs to Baire class one for each $n \in \mathbb{N}$. Moreover, it is easy to see that $\{h_n\}$ is uniformly Cauchy and hence converges to a Baire 1 function g^*. Finally, $g^*(x) = g(x)$ whenever $x \in A$ and for $x \in E$, $g^*(x) \in C(g|_A, x) = C(g^*|_A, x)$ and as such, the graph of $f|_A = g^*|_A$ is dense in the graph of the Baire 1 function, g^*. □

Theorem 2.1. *A function $f : I \to \mathbb{R}$ belongs to $\mathcal{NR}$ if and only if f belongs to honorary Baire class two.*

Proof. Let $f : I \to \mathbb{R}$ belong to honorary Baire class two and let g^* be the Baire one function from Lemma 2.3. Let A be the co-countable set on which f and g^* agree. Since $g^*|_A$ is dense in the graph of g^*, we may find a support set $D \subset A$ for which $g^*|_D$ is dense in g^*. Then Theorem 1 in [**4**] assures that there is an ordering $\overline{x}$ of D that recovers g^* everywhere. Since f and g^* agree on A and, in particular, agree on D, $\overline{x}$ recovers f at each point of A. Thus, f is recoverable nearly everywhere.

Conversely, suppose that $f \in \mathcal{NR}$. Let $\overline{x}$ be a trajectory which recovers f at each point of a co-countable set A. Note that without loss of generality we may assume each $\overline{x}(n)$ belongs to A. F. Hausdorff [**9**] has shown that a function belongs to honorary Baire class two if and only if the inverse image of each open set differs from an $\mathcal{F}_\sigma$ set by a countable set. We shall show that f has this property.

First, viewing A as a metric space, we have that the function $f|_A : A \to R$ is recoverable everywhere on A. In [**3**] it was shown that if a function from a metric space to a separable metric space is recoverable everywhere, then the function is of Borel class one. (See the comment following the proof of Theorem 1 in [**3**].)

Now, let U be an open set in $\mathbb{R}$. Since $f|_A$ is Borel class one, there is an $\mathcal{F}_\sigma$ subset F of $[0,1]$ such that $(f|_A)^{-1}(U) = F \cap A$. Then

$$\begin{aligned} f^{-1}(U) &= (f|_A)^{-1}(U) \cup \left(f^{-1}(U) \cap A^c\right) \\ &= (F \cap A) \cup \left(f^{-1}(U) \cap A^c\right) \\ &= \left[(F \cap A) \cup (F \cap A^c) \cup \left(f^{-1}(U) \cap F^c \cap A^c\right)\right] \setminus \left[A^c \setminus f^{-1}(U)\right] \\ &= F \cup \left(f^{-1}(U) \cap F^c \cap A^c\right) \setminus \left(A^c \setminus f^{-1}(U)\right). \end{aligned}$$

Since both $f^{-1}(U) \cap F^c \cap A^c$ and $A^c \setminus f^{-1}(U)$ are countable sets, we have that $f^{-1}(U)$ differs from an $\mathcal{F}_\sigma$ set by a countable set. $\square$

If we further restrict the exceptional set for recoverability to not only being countable, but having countable closure, then we are back to precisely the class of Baire one functions, as was observed in [**2**]. More generally, we have

Theorem 2.2. *Let $f : I \to \mathbb{R}$. The following are equivalent:*

1. *f belongs to Baire class one.*
2. *f is recoverable*
3. *f is recoverable except on a scattered set.*

Proof. Since 1. and 2. were shown to be equivalent in [**5**], and since 2. $\Rightarrow$ 3., only 3. $\Rightarrow$ 2. requires proof here. To this end, suppose D is a support set, $E \subset I \setminus D$ is scattered, and $\overline{x} = \{x_n\}$ is an ordering of D which recovers f except on E. We shall produce an ordering $\overline{y}$ of $D \cup E$ which recovers f on I. More specifically, we shall define $\overline{y}$ in such a way that for each $x \in I \setminus E$, the first return route to x based on the trajectory $\overline{y}$, $R(\overline{y}, x)$, and the first return route to x based on the trajectory $\overline{x}$, $R(\overline{x}, x)$ have a common tail sequence. Indeed, we shall arrange things so that for each $x \in I \setminus E$, $R(\overline{y}, x)$ contains only finitely many points of E.

Enumerate E as $\{e_k\}$. We shall define the modified trajectory $\overline{y}$ inserting each e_k between two terms in $\overline{x}$. Since E is scattered, it is a countable $\mathcal{G}_\delta$ and we may write $E = \cap_{i=1}^{\infty} G_i$, where each G_i is open and $G_1 \supset G_2 \supset \dots$. Let n_1 be sufficiently large that if (a, b) is the component of G_1 containing e_1, then there exist $k_1, k_2 < n_1$ such that $a < x_{k_1} < e_1 < x_{k_2} < b$. Then choose n_2 larger than n_1 such that if (a, b) is the component of G_2 containing e_2, then there exist $k_1, k_2 < n_2$ such that $a < x_{k_1} < e_2 < x_{k_2} < b$. Continue this process and order $D \cup E$ as $\overline{y} = \{x_1, x_2, \dots, x_{n_1}, e_1, x_{n_1+1}, \dots, x_{n_2}, e_2, x_{n_2+1}, \dots\}$. Now, if $\{e_{k_j}\} \subseteq R(\overline{y}, x)$, then $x \in \cap_{j=1}^{\infty} G_{i_j} = E$. Thus, if $x \in I \setminus E$, then $R(\overline{y}, x)$ can contain only finitely many points of E and, thus, from some point on $R(\overline{y}, x)$ and $R(\overline{x}, x)$ agree. Thus, $\overline{y}$ recovers f on I. $\square$

3. Functions which are universally recoverable except on small sets

In the previous section we explored the possibility of weakening the condition of recoverablility. One way to strengthen the condition is as follows.

Definition 3.1. Let $f : I \to \mathbb{R}$. The function is called *universally recoverable* ($f \in \mathcal{UR}$) if for every support set D there is an ordering $\overline{x}$ of D, such that f is first-return recoverable with respect to $\overline{x}$.

In [**4**] it was shown that $f \in \mathcal{UR}$ if and only if f is a quasicontinuous function in Baire class one. Let us recall the definition of quasicontinuity:

Definition 3.2. A function $f : I \to \mathbb{R}$ is *quasicontinuous at x* if every neighborhood of $(x, f(x))$ contains a point of the graph of $f|_{C(f)}$, where $C(f)$ denotes the set of points of continuity of f. We let $Q(f)$ denote the set of points of quasicontinuity of f and $NQ(f) = [0,1] \setminus Q(f)$. If $Q(f) = I$, we say that f is a quasicontinuous function.

In this section we investigate the various classes resulting from mixing the notion of universality with those of the previous section. Here are the definitions we utilize:

Definition 3.3. Let $f : I \to \mathbb{R}$. We say that f is

1. *almost universally recoverable* ($f \in \mathcal{AUR}$) if there is a measure zero set S such that every support set D has an ordering which recovers f at each point of $I \setminus S$.
2. *universally almost recoverable* ($f \in \mathcal{UAR}$) if for each support set D, there is an ordering $\overline{x}$ of D and a measure zero set $S(\overline{x})$ such that $\overline{x}$ recovers f at each point of $I \setminus S(\overline{x})$.
3. *typically universally recoverable* ($f \in \mathcal{TUR}$) if there is a first category set S such that every support set D has an ordering which recovers f at each point of $I \setminus S$.
4. *universally typically recoverable* ($f \in \mathcal{UTR}$) if for each support set D, there is an ordering $\overline{x}$ of D and a first category set $S(\overline{x})$ such that $\overline{x}$ recovers f at each point of $I \setminus S(\overline{x})$.
5. *nearly universally recoverable* ($f \in \mathcal{NUR}$) if there is a countable set S such that every support set D has an ordering which recovers f at each point of $I \setminus S$.
6. *universally nearly recoverable* ($f \in \mathcal{UNR}$) if for each support set D, there is an ordering $\overline{x}$ of D and a countable set $S(\overline{x})$ such that $\overline{x}$ recovers f at each point of $I \setminus S(\overline{x})$.
7. *very nearly universally recoverable* ($f \in \mathcal{SUR}$) if there is a scattered set S such that every support set D has an ordering which recovers f at each point of $I \setminus S$.
8. *universally very nearly recoverable* ($f \in \mathcal{USR}$) if for each support set D, there is an ordering $\overline{x}$ of D and a scattered set $S(\overline{x})$ such that $\overline{x}$ recovers f at each point of $I \setminus S(\overline{x})$.

We proceed to classify each of these eight function classes. Along the way we shall show that there are really only four distinct classes since some of the adverbs "commute." For example, we shall see that a function is almost universally recoverable if and only if it is universally almost recoverable. We begin with a few straightforward lemmas.

Lemma 3.1. *Suppose f belongs to either $\mathcal{UAR}$ or $\mathcal{UTR}$ and $r < s$. Then there does not exist an interval J in which both $E_1 = f^{-1}((-\infty, r])$ and $E_2 = f^{-1}([s, +\infty))$ are dense in J.*

Proof. We will prove this lemma for $f \in \mathcal{UAR}$, the proof for $f \in \mathcal{UTR}$ being similar. Suppose such an interval J exists and choose support sets D_1 and D_2 such that $D_i \subset E_i \cap J$ for $i = 1, 2$. Since $f \in \mathcal{UAR}$, there are enumerations $\overline{x}_1$

of D_1 and $\overline{x}_2$ of D_2 such that for almost every $x \in [0,1]$, $\{f(w_{\overline{x}_1,k}(x))\} \to f(x)$ and for almost every $x \in [0,1]$, $\{f(w_{\overline{x}_2,k}(x))\} \to f(x)$. It follows that for almost every $x \in J$, $f(x) \le r$ and also for almost every $x \in J$, $f(x) \ge s$. Since $r < s$, this contradiction completes the proof. □

Lemma 3.2. *Suppose f belongs to either $\mathcal{UAR}$ or $\mathcal{UTR}$ and $r < s$. Then $\{x : \lim_{t\to x} f(t)$ exists$\}$ is residual.*

Proof. Set $A = \{x : \lim_{t\to x} f(t)$ does not exist$\}$, and define

$$A_{rs} = \{x \in A : \liminf_{t\to x} f(t) \le r < s \le \limsup f_{t\to x} f(t)\}.$$

Then $A = \bigcup_{r<s\in\mathbb{Q}} A_{rs}$ and Lemma 3.1 assures that for fixed $r < s$, the set A_{rs} is nowhere dense. Thus, A is of first category, establishing the lemma. □

Lemma 3.3. *If f belongs to either $\mathcal{UTR}$ or $\mathcal{UAR}$, then $C(f)$ is residual.*

Proof. Define $B = ([0,1] \setminus C(f)) \setminus A$, where A is the set defined in the proof of Lemma 3.2. In other words, B is the set of points where f has a removable discontinuity. For $r < s$ define

$$B_{rs} = \{x \in B : f(x) \le r < s \le \lim_{t\to x} f(t)\}, \text{ and}$$

$$B'_{rs} = \{x \in B : \lim_{t\to x} f(t) \le r < s \le f(x)\}.$$

It follows that $B = \bigcup_{r<s\in\mathbb{Q}} (B_{rs} \cup B'_{rs})$ and from Lemma 3.1 that for fixed $r < s$, both B_{rs} and B'_{rs} are nowhere dense. Hence B is of first category. This observation and Lemma 3.2 complete the proof. □

Theorem 3.1. *Let $f : I \to \mathbb{R}$. The following are equivalent.*

1. *$f \in \mathcal{AUR}$.*
2. *$f \in \mathcal{UAR}$.*
3. *f is measurable and $NQ(f)$ has measure zero.*

Proof. Clearly, 1. ⇒ 2. Next, suppose that $f \in \mathcal{UAR}$. Then $f \in \mathcal{AR}$ and is, therefore, measurable according to Theorem 2.2. in [**6**]. Furthermore, Lemma 3.3 assures that $C(f)$ is residual. Suppose that $NQ(f)$ has positive outer measure. Let D be a support set lying entirely in $C(f)$. No ordering of D will recover f at any point of $NQ(f)$, contradicting $f \in \mathcal{UAR}$, verifying that 2. ⇒ 3.

Next, assume f is measurable and $NQ(f)$ has measure zero. Let $\mathcal{L}(f)$ denote the set of points of f. (Recall that z_0 is a Lebesgue point of f if $\lim_{h\to 0} \frac{1}{h} \int_{z_0}^{z_0+h} |f - f(z_0)| = 0$.) As f is measurable, $I \setminus \mathcal{L}(f)$ is of measure zero and, hence, there is a $\mathcal{G}_\delta$ set T of measure zero such that $I \setminus \mathcal{L}(f) \subset T$. As T is $\mathcal{G}_\delta$, there are open sets $G_1 \supseteq G_2 \supseteq \ldots$, such that $\lambda(G_n) < \frac{1}{2^n}$ for each $n \in \mathbb{N}$ and $T = \cap_{n=1}^{\infty} G_n$. Throughout the remainder of this proof we shall adopt the following notation. If J is any interval in I, then $\overset{\star}{J}$ denotes the interval of length $|J|/2$ which is centered in J. Furthermore, we let $A(J) = \frac{1}{|J|} \int_J f$.

Let D be any support set. We shall find an ordering $\{x_n\}$ of D which recovers f at each point of $\mathcal{L}(f) \setminus T$. Without loss of generality assume that both 0 and 1 belong to D and enumerate D as $\{d_1 = 0, d_2 = 1, d_3, d_4, \dots\}$, where the ordering of d_n for $n \geq 3$ is arbitrary but fixed for the remainder of the proof. We shall reorder $\{d_n\}$ as $\{x_n\}$ inductively in steps. At the conclusion of the n^{th} step, we will have selected points $x_1, x_2, \dots, x_{q(n)}$ from D, where $q(n) > n$. Furthermore, d_n will be one of the selected points; i.e., $d_n \in X_n \equiv \{x_1, x_2, \dots, x_{q(n)}\}$.

Step 1. Let $x_1 = 0$, $x_2 = 1$, $q(1) = 2$, and $X_1 = \{x_1, x_2\}$.

Inductive Step. Let $n \in \mathbb{N}$ and assume X_n has been chosen. We let $X_n^* = X_n \cup \{d_{n+1}\}$. Set $\rho_n = \min\{|x - y| : x, y \in X_n^*, x \neq y\}$, and select δ_n so small that

- $\delta_n < \rho_n/2$,
- If $x \in X_n^* \cap \mathcal{L}(f)$, then whenever $x - \delta_n < a < x < b < x + \delta_n$, $|A([a,b]) - f(x)| < \frac{1}{n+1}$, and
- If $x \in X_n^* \setminus \mathcal{L}(f)$, then $(x - \delta_n, x + \delta_n) \subseteq G_{n+1}$.

Fix $x < y$ where x and y are consecutive points of X_n^* in the usual ordering of $[0,1]$. We shall hierarchically identify finitely many points of $(D \setminus X_n^*) \cap (x, y)$. (Upon doing this for each consecutive pair $x < y$ in X_n^*, we shall append these points in a specific order to X_n^* to form X_{n+1}.)

Let $J = [x, y]$ and set

$$V(J) = \inf\{|f(t) - A(J)| : t \in D \cap \overset{\star}{J}\}.$$

Let $p \equiv p(x,y) \in D \cap \overset{\star}{J}$ be such that $|f(p) - A(J)| < V(J) + \frac{1}{n}$. Now, set $J_0 = [x, p]$, and $J_1 = [p, y]$.

Inductively, suppose $k \in \mathbb{N}$ and intervals $J_{\overline{0}(k)} = [x, p_{\overline{0}(k-1)}]$ and $J_{\overline{1}(k)} = [p_{\overline{1}(k-1)}, y]$ have been defined, where $\overline{0}(j) = \underbrace{00\dots0}_{j}$ and $\overline{1}(j) = \underbrace{11\dots1}_{j}$. Let

$$V(J_{\overline{0}(k)}) = \inf\{|f(t) - A(J_{\overline{0}(k)})| : t \in D \cap \overset{\star}{J}_{\overline{0}(k)}\}$$

and

$$V(J_{\overline{1}(k)}) = \inf\{|f(t) - A(J_{\overline{1}(k)})| : t \in D \cap \overset{\star}{J}_{\overline{1}(k)}\}.$$

Then choose $p_{\overline{0}(k)} \in D \cap \overset{\star}{J}_{\overline{0}(k)}$ and $p_{\overline{1}(k)} \in D \cap \overset{\star}{J}_{\overline{1}(k)}$ such that

$$|f(p_{\overline{0}(k)}) - A(J_{\overline{0}(k)})| < V(J_{\overline{0}(k)}) + \frac{1}{n}$$

and

$$|f(p_{\overline{1}(k)}) - A(J_{\overline{1}(k)})| < V(J_{\overline{1}(k)}) + \frac{1}{n}.$$

It is easy to see that as $k \to \infty$, both $p_{\overline{0}(k)} \to x$ and $p_{\overline{1}(k)} \to y$. Hence, there exist k_0 and k_1 such that both

$$k \geq k_0 \Rightarrow 0 < p_{\overline{0}(k)} - x < \delta_n,$$

and

$$k \geq k_1 \Rightarrow 0 < y - p_{\overline{1}(k)} < \delta_n.$$

Let $P(x,y) = \{p(x,y)\} \cup \{p_{\overline{0}(k)} : 1 \leq k \leq k_0\} \cup \{p_{\overline{1}(k)} : 1 \leq k \leq k_1\}$ and set $P = \cup P(x,y)$ where the union is taken over all pairs $x < y$ where x and y are consecutive elements of X_n^* in the usual ordering of $[0,1]$. Now, let $X_{n+1} = X_n^* \cup P$ and order the elements of X_{n+1} according to the following scheme:

1. The initial portion $X_n = \{x_0, x_1, \ldots, x_{q(n)}\}$ retains its original order.
2. First, points of the form $p(x,y)$ are appended according to the usual order of $[0,1]$ with the leftmost such point being denoted $x_{q(n)+1}$.
3. Then, all points of the form $p_{\overline{0}(k)}$ and $p_{\overline{1}(k)}$ are appended next, ordered lexicographically: first according to "k", then according to the usual ordering of $[0,1]$.
4. Finally we append d_{n+1} if $d_{n+1} \notin X_n$.

We let $q(n+1)$ be the subscript of the final point appended in the scheme, completing the inductive step.

We now proceed to show that this ordering $\{x_n\}$ of D recovers f at each $x \in S \equiv \mathcal{L}(f) \setminus T$. Suppose not; i.e., suppose there is an $y_o \in S$ for which $\{f(r_n)\}$ fails to converge to $f(y_o)$, where $\{r_n\}$ is the first-return route to y_o based on the ordering $\{x_n\}$. Then there exists an $\epsilon_o > 0$ and an increasing sequence $\{n_i\}$ such that $|f(r_{n_i}) - f(y_o)| \geq \epsilon_o$ for each i.

Since $y_o \in \mathcal{L}(f)$, there exists a $\eta_1 > 0$ such that if H is any interval containing y_o with $|H| < \eta_1$, then $|f(y_o) - A(H)| < \epsilon_o/10$. Furthermore, since y_o must be a point of approximate continuity of f, there exists a $0 < \eta_2 < \eta_1$ such that if $\delta \leq \eta_2$, then there is a set $E(\delta) \subseteq (y_o - \delta, y_o + \delta)$ with $\lambda(E(\delta)) > 9\delta/5$ such that $|f(x) - f(y_o)| < \epsilon_o/10$ for all $x \in E(\delta)$. Furthermore, each r_{n_i} was appended to the trajectory $\{x_n\}$ at some Stage N_{n_i}. There is a K such that for all $i > K$, $N_{n_i - 1} > \frac{5}{\epsilon_o}$ and thus, we may assume this true for all i. For a fixed i we have that r_{n_i} was appended to the trajectory $\{x_n\}$ at Stage N_{n_i} for one of two reasons: Either $r_{n_i} = d_{N_{n_i}}$ or it is a member of the collection P of Stage N_{n_i}.

Suppose that the former case applies. Then we must have $y_o \in (d_{N_{n_i}} - \delta_{N_{n_i}-1}, d_{N_{n_i}} + \delta_{N_{n_i}-1})$. If $d_{N_{n_i}} \notin \mathcal{L}(f)$, then $y_o \in G_{N_{n_i}}$. But if this happens for infinitely many $i's$, we would have $y_o \in T$, a contradiction. Thus, we may assume that if $r_{n_i} = d_{N_{n_i}}$, then $d_{N_{n_i}} \in \mathcal{L}(f)$. Let H be any closed subinterval of $(y_o - \eta_2, y_o + \eta_2) \cap (d_{N_{n_i}} - \delta_{N_{n_i}-1}, d_{N_{n_i}} + \delta_{N_{n_i}-1})$, containing both y_o and $d_{N_{n_i}} = r_{n_i}$. Then $|f(y_o) - A(H)| < \epsilon_o/10$ and $|f(r_{n_i}) - A(H)| < 1/N_{n_i} < \epsilon_o/5$, contradicting the assumption that $|f(r_{n_i}) - f(y_o)| \geq \epsilon_o$.

Thus, it must be the case that for each i, r_{n_i} is a member of the collection P of Stage N_{n_i}. Thus, r_{n_i} be a point of the form $p(x,y)$ or $p_{\overline{0}(k)}$ or $p_{\overline{1}(k)}$. Let J denote the interval of Stage N_{n_i} which determined r_{n_i}. Then we know J contains y_o, $r_{n_i} \in \overset{\star}{J} \subset (y_o - \eta_2, y_o + \eta_2)$, and $|f(r_{n_i}) - A(J)| < V(J) + \frac{1}{N_{n_i}-1} < V(J) + \epsilon_o/5$. Now, since $NQ(f)$ is of measure zero, there is a point $x^* \in Q(f) \cap \overset{\star}{J} \cap E(\eta_2)$. Hence, there is a point $x^{**} \in C(f) \cap \overset{\star}{J}$ such that $|f(x^{**}) - f(y_o)| < \epsilon_o/10$. Thus, there is an interval $H \subset \overset{\star}{J}$ such that $|f(x) - f(y_o)| < \epsilon/10$ for all $x \in H$. Since D is dense, there is a $d \in H \cap D$, and hence $|f(d) - f(y_o)| < \epsilon/10$. This implies that

$V(J) \leq |f(d) - A(J)| \leq |f(d) - f(y_o)| + |f(y_o) - A(J)| < \epsilon_o/10 + \epsilon_o/10 = \epsilon_o/5$. Consequently, $|f(r_{n_i}) - f(y_o)| \leq |f(r_{n_i}) - A(J)| + |A(J) - f(y_o)| < \epsilon_o/5 + \epsilon_o/5 + \epsilon_o/10 < \epsilon_o$, contradicting $|f(r_{n_i}) - f(y_o)| \geq \epsilon_o$, and completing the proof. □

Theorem 3.2. *Let $f : I \to \mathbb{R}$. The following are equivalent.*

1. *There is a residual set T such that every ordering of every support set recovers f at each point of T.*
2. *$f \in \mathcal{TUR}$.*
3. *$f \in \mathcal{UTR}$.*
4. *$C(f)$ is residual.*

Proof. It is obvious that 1. ⇒ 2. ⇒ 3., and Lemma 3.3 assures that 3. ⇒ 4. Finally, assume that $C(f)$ is residual. Letting $T = C(f)$, it is clear that every ordering of every support set recovers f at each point of T. □

Theorem 3.3. *Let $f : I \to \mathbb{R}$. The following are equivalent.*

1. *$f \in \mathcal{NUR}$ [$\mathcal{SUR}$].*
2. *$f \in \mathcal{UNR}$ [$\mathcal{USR}$].*
3. *f belongs to Baire class one and $NQ(f)$ is countable [scattered].*

Proof. Clearly, 1. ⇒ 2. Next to see that 2. ⇒ 3., suppose that $f \in \mathcal{UNR}$ [$\mathcal{USR}$]. Then $f \in \mathcal{UTR}$ and Lemma 3.3 assures that $C(f)$ is dense. Let $D \subset C(f)$ be a support set and let $\overline{x}_D$ be an ordering of D which recovers f at each point of $I \setminus S(\overline{x}_D)$, where $S(\overline{x}_D)$ is countable [scattered]. Since $\overline{x}_D$ cannot recover $f(x)$ at any $x \in NQ(f)$, we have $NQ(f) \subseteq S(\overline{x}_D)$ and, hence, $NQ(f)$ is countable [scattered].

We shall show that f belongs to Baire class one by observing that it is recoverable. (Since $\mathcal{USR} \subset \mathcal{UNR}$ we need only provide the proof for when $f \in \mathcal{UNR}$.) We first recall that J. Borsík [**1**] has shown that $Q(f) \setminus BQ(f)$ is countable, where $BQ(f)$ denotes the set of points at which f is bilaterally quasicontinuous. Let D be a support set containing the countable set $NQ(f) \cup ([Q(f) \setminus BQ(f)]$ and let $\overline{x}$ be an ordering of D which recovers f except at points in a countable set $S(\overline{x}) = \{s_1, s_2, \dots\}$. By systematically inserting each s_n between two terms in the trajectory $\overline{x} = \{x_j\}$ we shall produce a trajectory $\overline{y}$ which is an enumeration of the support set $D \cup S(\overline{x})$ and which recovers f everywhere. We proceed inductively:
Step 1. Let $\epsilon_1 = 1$. Since $s_1 \in BQ(f)$, there exists an $x_{j(1,l)}$ and an $x_{j(1,r)}$ such that

$$0 < s_1 - x_{j(1,l)} < \epsilon_1 \text{ and } \left|f(s_1) - f\left(x_{j(1,l)}\right)\right| < \epsilon_1,$$

and

$$0 < x_{j(1,r)} - s_1 < \epsilon_1 \text{ and } \left|f(s_1) - f\left(x_{j(1,r)}\right)\right| < \epsilon_1,$$

where $j(1,l)$ and $j(1,r)$ are the minimal subscripts yielding these results. Let $j(1)$ denote the larger of $j(1,l)$ and $j(1,r)$, set $k_1 = j(1) + 1$ and define the initial segment of the new trajectory $\{y_k\}$ by

$$y_k = \begin{cases} x_k & \text{if } k \leq j(1) \\ s_1 & \text{if } k = j(1) + 1 = k_1 \end{cases}.$$

In other words, we have inserted s_1 into the sequence $\{x_j\}$ between $x_{j(1)}$ and $x_{j(1)+1}$.

Step n. Assume that $\{y_k\}_{k=1}^{k_{n-1}}$ has been specified. Let

$$\epsilon_n = \min\left(\left\{\frac{1}{n}\right\} \cup \{|y_k - y_j| : k \neq j,\, k, j \leq k_{n-1}\}\right).$$

Since $s_n \in BQ(f)$, there exists an $x_{j(n,l)}$ and an $x_{j(n,r)}$ such that

$$0 < s_n - x_{j(n,l)} < \epsilon_n \text{ and } \left|f(s_n) - f\left(x_{j(n,l)}\right)\right| < \epsilon_n, \tag{1}$$

and

$$0 < x_{j(n,r)} - s_n < \epsilon_n \text{ and } \left|f(s_n) - f\left(x_{j(n,r)}\right)\right| < \epsilon_n, \tag{2}$$

where $j(n,l)$ and $j(n,r)$ are the minimal subscripts yielding these results. Let $j(n)$ denote the larger of $j(n,l)$ and $j(n,r)$. Set $k_n = j(n) + n$ and define y_k for $k_{n-1} < k \leq k_n$ by

$$y_k = \begin{cases} x_{k-n+1} & \text{if } k_{n-1} < k < k_n \\ s_n & \text{if } k = k_n \end{cases}.$$

In other words, we have inserted s_n into the sequence $\{x_j\}$ between $x_{j(n)}$ and $x_{j(n)+1}$.

In this manner we have completed our definition of the expanded trajectory $\overline{y} = \{y_k\}$, which is an enumeration of $D \cup S(\overline{x})$. Letting $U(\overline{y}, y_k)$ denote the interval of influence of y_k based on the trajectory $\overline{y}$, and $U(\overline{x}, x_j)$ denote the interval of influence of x_j based on the trajectory $\overline{x}$, we have that for each n, $s_n = y_{k_n}$, and $U(\overline{y}, s_n) \subset U(\overline{x}, x_{j(n,l)}) \cup U(\overline{x}, x_{j(n,r)})$. This observation, together with inequalities (1) and (2) shows that $\overline{y}$ recovers f everywhere.

Finally, to see that 3. $\Rightarrow$ 1., suppose that f belongs to Baire class one and $NQ(f)$ is countable [scattered]. Let D be any support set. Then $f|_D$ is dense in $f|Q(f)$. Let $S = NQ(f) \setminus D$ and define a function $g : I \to \mathbb{R}$ by

$$g(x) \equiv \begin{cases} f(x) & \text{if } x \in I \setminus S \\ \sup\left(C(f|_{C(f)}, x)\right) & \text{if } x \in S \end{cases},$$

where $C(f|_{C(f)}, x))$ denotes the cluster set at x of the restriction of f to its set of continuity points. Then g belongs to Baire class one and $g|D$ is dense in g. Thus, Theorem 1 in [**4**] guarantees the existence of an ordering $\overline{x}$ of D which recovers g everywhere. Then, since $f|_D = g|_D$ and $g(x) = f(x)$ for all $x \in I \setminus S$, we have that $\overline{x}$ recovers f at each point of $I \setminus S$. In particular, $\overline{x}$ recovers f at each point of $I \setminus NQ(f)$, showing that $f \in \mathcal{NUR}$ [$\mathcal{SUR}$]. $\square$

4. Functions which are consistently recoverable except on small sets

Besides universality, a second natural means of strengthening the notion of recoverability is to require the existence of a support set every ordering of which recovers the function. More specifically we have the following.

Definition 4.1. Let $f : I \to \mathbb{R}$. Let D be a support set. We shall say that D *consistently recovers* f provided that f is first-return recoverable with respect to every ordering of D. A function is said to be *consistently recoverable* ($f \in \mathcal{CR}$) if there exists a support set D which consistently recovers f.

In [**8**] it was shown that $f \in \mathcal{CR}$ if and only if f has only countably many discontinuities.

Paralleling Definition 3.3, we make the following definitions.

Definition 4.2. Let $f : I \to \mathbb{R}$. We say that f is

1. *almost consistently recoverable* ($f \in \mathcal{ACR}$) if there is a measure zero set S and a support set D every ordering of which recovers f at each point of $I \setminus S$.
2. *consistently almost recoverable* ($f \in \mathcal{CAR}$) if there is a support set D, such that every ordering $\overline{x}$ of D recovers f at each point of $I \setminus S(\overline{x})$, where $S(\overline{x})$ is of measure zero.
3. *typically consistently recoverable* ($f \in \mathcal{TCR}$) if there is a first category set S and a support set D every ordering of which recovers f at each point of $I \setminus S$.
4. *consistently typically recoverable* ($f \in \mathcal{CTR}$) if there is a support set D, such that every ordering $\overline{x}$ of D recovers f at each point of $I \setminus S(\overline{x})$, where $S(\overline{x})$ is of first category.
5. *nearly consistently recoverable* ($f \in \mathcal{NCR}$) if there is a countable set S and a support set D every ordering of which recovers f at each point of $I \setminus S$.
6. *consistently nearly recoverable* ($f \in \mathcal{CNR}$) if there is a support set D, such that every ordering $\overline{x}$ of D recovers f at each point of $I \setminus S(\overline{x})$, where $S(\overline{x})$ is countable.
7. *very nearly consistently recoverable* ($f \in \mathcal{SCR}$) if there is a scattered set S and a support set D every ordering of which recovers f at each point of $I \setminus S$.
8. *consistently very nearly recoverable* ($f \in \mathcal{CSR}$) if for each support set D, such that every ordering $\overline{x}$ of D recovers f at each point of $I \setminus S(\overline{x})$, where $S(\overline{x})$ is scattered.

We proceed to investigate these eight classes of functions.

Theorem 4.1. *Let $f : I \to \mathbb{R}$. The following are equivalent.*

1. $f \in \mathcal{ACR}$.
2. $f \in \mathcal{CAR}$.
3. *f is almost everywhere equal to a function g which is continuous almost everywhere.*

Proof. Clearly, 1. $\Rightarrow$ 2. To see that 2. $\Rightarrow$ 3., let $f \in \mathcal{CAR}$. Since $f \in \mathcal{AR}$, f is measurable according to Theorem 2.2 in [**6**]. Let D be a support set such that every ordering of D recovers f almost everywhere. Let $F = \arctan(f)$ and note that every ordering of D recovers the bounded measurable function F almost everywhere. Thus, in the terminology of [**6**], Theorem 2.1 of that paper asserts that every ordering of D yields the Lebesgue integral of F. Then Theorem 1 in [**7**] assures the existence of a Riemann integrable function G, which equals F almost everywhere. Letting $g = \tan(G)$, we have that $f = g$ almost everywhere and g is continuous almost everywhere, completing the proof that 2. $\Rightarrow$ 3.

Finally, we show that 3. $\Rightarrow$ 1. To this end, assume that $f = g$ almost everywhere and that g is continuous at each point of a full measure set S. Let D be a support set lying entirely in the set $S^* \equiv S \cap \{x : f(x) = g(x)\}$. Then, clearly every ordering of D recovers g at each point of S and, consequently, recovers f at each point of the full measure set S^*. $\square$

Theorem 4.2. *Let $f : I \to \mathbb{R}$. The following are equivalent.*

1. *f has the Baire property.*
2. *$f \in \mathcal{TR}$.*
3. *$f \in \mathcal{CTR}$.*
4. *$f \in \mathcal{TCR}$.*

Proof. That 4. $\Rightarrow$ 3. $\Rightarrow$ 2. follows directly from the definitions. The equivalence of 1. and 2. was shown in Theorem 2.3 of [**6**]. It remains to show that 1. $\Rightarrow$ 4. To this end, let f have the Baire property. Then there is a residual set S such that $f|_S$ is continuous. Let D be any support set lying entirely in S. Clearly every ordering of D recovers f at each point of S. Hence $f \in \mathcal{TCR}$. $\square$

Before establishing the next theorem, we need the following definition.

Definition 4.3. If $f : [0,1] \to \mathbb{R}$, then the strong oscillation of f at a point $x \in [0,1]$ is

$$\begin{aligned}\text{s-osc}(f,x) \equiv \sup\{r \geq 0 : \ &\text{there are sets } S_1(x) \text{ and } S_2(x) \text{ such that for every } \epsilon > 0 \\ &\text{both } S_1 \cap B_\epsilon(x) \text{ and } S_2 \cap B_\epsilon(x) \text{ are uncountable and} \\ &|f(x_1) - f(x_2)| \geq r, \text{ whenever } x_1 \in S_1 \text{ and } x_2 \in S_2\}.\end{aligned}$$

Theorem 4.3. *Let $f : I \to \mathbb{R}$. The following are equivalent.*

1. *$f \in \mathcal{NCR}$.*
2. *$f \in \mathcal{CNR}$.*
3. *There is a co-countable set $T \subseteq I$ such that $f|_T$ is continuous.*

Proof. That 1. $\Rightarrow$ 2. is immediate. Next, to see that 3. $\Rightarrow$ 1., suppose that S is co-countable in I and that $f|_S$ is continuous. Let D be a support set lying in S such that D contains every isolated point of S. Then every ordering of D recovers f at each point of S, so that 3. $\Rightarrow$ 1.

The bulk of the work to be done here is is showing the remaining implication that 2. $\Rightarrow$ 3. We first establish a couple of claims:

Claim 1. *Let $f : I \to \mathbb{R}$ and suppose $f \in \mathcal{CNR}$. Then $E_n \equiv \{x : \text{s-osc}(f,x) \geq \frac{1}{n}\}$ is countable.*

Proof. Suppose that for some $n \in \mathbb{N}$, E_n is uncountable. Let D be a support set, every ordering $\overline{x}$ of which recovers f on $I \setminus S(\overline{x})$, where $S(\overline{x})$ is countable. We shall determine an enumeration $\overline{x}$ of D with the property that $\overline{x}$ fails to recover f at uncountably many points, yielding a contradiction.

Let S_1 and S_2 be as in the definition of strong oscillation, i.e., for each $x \in E_n$ and for each $\epsilon > 0$ both $S_1 \cap B_\epsilon(x)$ and $S_2 \cap B_\epsilon(x)$ are uncountable and $|f(x_1) - f(x_2)| \geq 1/n$, whenever $x_1 \in S_1$ and $x_2 \in S_2$. Next, let Σ denote the collection of all finite sequences of 0's and 1's, and Σ^* denote the collection of all infinite sequences of 0's and 1's. For each $\sigma \in \Sigma$, we let $|\sigma|$ denote the length of σ and let $\Sigma_k = \{\sigma \in \Sigma : |\sigma| = k\}$. For each $\sigma \in \Sigma^*$ and each natural number k, we let $\sigma|k \in \Sigma_k$ denote the sequence consisting of the initial k terms in σ. We shall proceed by induction on k to define an enumeration $\overline{x}$ of D and to identify closed intervals H_σ for each $\sigma \in \Sigma_k$. The enumeration $\overline{x}$ and these intervals will be chosen such that the following conditions are satisfied:

- If $\sigma \neq \tau$ both belong to Σ_k, then $H_\sigma \cap H_\tau = \emptyset$.
- For each $\sigma \in \Sigma$, $H_{\sigma 0} \cup H_{\sigma 1} \subset H_\sigma$.
- If $\sigma \in \Sigma_k$, then $0 < |H_\sigma| < 1/2^k$.
- If $\sigma \in \Sigma_k$ and $x \in H_\sigma$, then there are two numbers u_x and w_x in $R(\overline{x}, x) \cap B_{\frac{1}{2^k}}(x)$ such that $|f(u_x) - f(w_x)| \geq 1/n$.

For each $\sigma \in \Sigma^*$ we let $x(\sigma)$ be the unique point in $\cap_{k=1}^{\infty} H_{\sigma|k}$. There are uncountably many such points and clearly $\overline{x}$ cannot recover f at any such point because of the last property listed above.

Before describing the inductive process, we first let $\{d_1, d_2, \ldots\}$ be an arbitrary but fixed enumeration of D and let

$$E^* = E_n \setminus (D \cup \{x \in E_n : x \text{ is not a condensation point of } E_n\}).$$

Then E^* is uncountable.

Stage 0: Let $\epsilon_1 = 1$ and let $c_\emptyset \in E^*$. Then there are two elements $x_1, x_2 \in D$ such that $|f(x_1) - f(x_2)| \geq \frac{1}{n}$. This is because the sets $S_1(c_\emptyset)$ and $S_2(c_\emptyset)$ are uncountable and D must recover f at all but countably many points. We suppose that $|c_\emptyset - x_1| > |c_\emptyset - x_2|$ and let $H_\emptyset = \overline{B_{|c_\emptyset - x_2|}(c_\emptyset)}$. We set $n_0 = 2$ and specify the initial segment of n_0 terms in $\overline{x}$ as $\{x_1, x_2\}$.

Stage k: Assume that Stage $k-1$ has been completed. If d_k has not been selected as one of the $\{x_1, x_2, \ldots, x_{n_{k-1}}\}$, append it now as $x_{n_{k-1}+1}$ and let m_k denote the number of terms selected of $\overline{x}$ to this point. (Thus, m_k is either n_{k-1} or $n_{k-1}+1$.) For each $\sigma \in \Sigma_{k-1}$ let J_σ be a subinterval of $H_\sigma \setminus \{d_k\}$ containing c_σ with $0 < |J_\sigma| < |H_\sigma|/2$. In J_σ select two distinct points $c_{\sigma 0}, c_{\sigma 1} \in J_\sigma \cap E^*$. Next, inside J_σ choose disjoint open intervals $I_{\sigma 0}$ centered on $c_{\sigma 0}$ and $I_{\sigma 1}$ centered on $c_{\sigma 1}$. Find two points $a_{\sigma 0}$ and $b_{\sigma 0}$ in $[D \setminus \{x_1, x_2, \ldots, x_{m_k}\}] \cap I_{\sigma 0}$ with $|a_{\sigma 0} - c_{\sigma 0}| < |b_{\sigma 0} - c_{\sigma 0}|$ and $|f(a_{\sigma 0}) - f(b_{\sigma 0})| \geq 1/n$. Let $H_{\sigma 0} = \overline{B_{|c_{\sigma 0} - a_{\sigma 0}|}(c_{\sigma 0})}$. Likewise, find two points $a_{\sigma 1}$ and $b_{\sigma 1}$ in $[D \setminus \{x_1, x_2, \ldots, x_{m_k}\}] \cap I_{\sigma 1}$ with $|a_{\sigma 1} - c_{\sigma 1}| < |b_{\sigma 1} - c_{\sigma 1}|$ and $|f(a_{\sigma 1}) - f(b_{\sigma 1})| \geq 1/n$. Then let $H_{\sigma 1} = \overline{B_{|c_{\sigma 1} - a_{\sigma 1}|}(c_{\sigma 1})}$. We do this for each $\sigma \in \Sigma_{k-1}$ and have thus selected points a_σ and b_σ for each $\sigma \in \Sigma_k$. We append these 2^{k+1} points to $\{x_1, x_2, \ldots x_{m_k}\}$ taking care to insert each b point before its corresponding a point. We let n_k denote the number of elements of D selected up to this point. This completes Stage k.

The resulting ordering $\overline{x}$ and the intervals H_σ have the four properties noted above, completing the proof of Claim 1. $\square$

With this claim we have that the set $E \equiv \cup_{n=1}^{\infty} E_n$ is countable. Next we wish to establish:

Claim 2. *If $x_o \notin E_n$, then there is an $\delta(x_o) > 0$ and an interval J with $|J| = \frac{1}{n}$ such that $f(x) \in J$ for all but countably many $x \in B_{\delta(x_o)}(x_o)$.*

Proof. Since $x_o \notin E_n$, there is an $\delta = \delta_{x_o} > 0$ so that whenever $S_1, S_2 \subset B_\delta(x_o)$ with $|f(x_1) - f(x_2)| \geq \frac{1}{n}$ for every $x_1 \in S_1$ and $x_2 \in S_2$, then either S_1 or S_2 is countable. It is easy to see that there is a y_o such that $f^{-1}((-\infty, y_o)) \cap B_\delta(x_o)$ is uncountable and thus it follows that the set $f^{-1}\left((y_o + \frac{1}{n}, +\infty)\right) \cap B_\delta(x_o)$ is countable. Let

$$y^* \equiv \inf\{y : f^{-1}((-\infty, y)) \cap B_\delta(x_o) \text{ is uncountable}\}.$$

Then $y^* > -\infty$ and for every $\epsilon > 0$, $f^{-1}((-\infty, y^* - \epsilon)) \cap B_\delta(x_o)$ is countable and so $A_1 = f^{-1}((-\infty, y^*)) \cap B_\delta(x_o)$ is countable. Since $A_2 = f^{-1}\left((y^* + \frac{1}{n}, +\infty)\right) \cap B_\delta(x_o))$ is countable, it follows that for every $x \in B_\delta(x_o) \backslash (A_1 \cup A_2)$, $f(x) \in [y^*, y^* + \frac{1}{n}]$ which verifies Claim 2. $\square$

For each $x_o \notin E_n$ let $I(x_o, n)$ denote the interval $B_{\delta(x_o)}(x_o)$, where $\delta(x_o)$ is from Claim 2 and let $A(x_o, n)$ denote the countable set $A_1 \cup A_2$ described in the proof of Claim 2. Then if n is fixed, $E_n^c \subset \cup_{x \in E_n^c} I(x_o, n)$ and by Lindeloff's Theorem, there is a countable set $\{I(x_{i,n}, n)\}$ such that $E_n^c \subset \cup_{i=1}^{\infty} I(x_{i,n}, n)$. Set

$$A = E \cup \left(\cup_{n=1}^{\infty} \cup_{i=1}^{\infty} A(x_{i,n}, n)\right).$$

We are now in a position to complete the proof that 2. $\Rightarrow$ 3. To this end, assume $f \in \mathcal{CNR}$ and let A denote the countable set defined above and let $T = I \setminus A$. We show that $f|_T$ is continuous. Let $x_o \in T$ and $\epsilon > 0$ be given. Choose n so that $\frac{1}{n} < \epsilon$. Since $x_o \in T$, $x_o \notin E_n$ and so there is an $i \in \mathbb{N}$ such that $x_o \in I(x_{i,n}, n)$. Choose $\delta > 0$ sufficiently small that $(x_o - \delta, x_o + \delta) \subset I(x_{i,n}, n)$. Then if $|x - x_o| < \delta$ and $x \in A^c$, $x \in I(x_{i,n}, n) \backslash A(x_{i,n}, n)$ and by Claim 2 there is an interval J depending only on the indices i and n such that $|J| = \frac{1}{n}$ and $f(x) \in J$. But $x_o \in I(x_{i,n}, n) \backslash A(x_{i,n}, n)$ as well as x so that $f(x_o) \in J$. Hence, $|f(x) - f(x_o)| \leq \frac{1}{n} < \epsilon$. This completes the proof of the Theorem 4.3. $\square$

At this point the reader should fully expect that we will conclude this paper by showing that $\mathcal{SCR} = \mathcal{CSR}$ and providing a characterization of this subclass of the Baire 1 functions. Unfortunately, the best we can state is that

$$A \subset \mathcal{SCR} \subseteq \mathcal{CSR} \subset B, \tag{3}$$

where A is the set of all functions f having the property that there is a co-scattered set $T \subseteq I$ such that $f|_T$ is continuous and B is the set of Baire 1 functions f having the property that there is a co-countable set $T \subseteq I$ such that $f|_T$ is continuous. The first two inclusions in (3) are immediate and the third follows from Theorem 2.2 and Theorem 4.3. Furthermore, notice that Dirichlet's familiar example of a function which is continuous at precisely the irrationals shows that the first inclusion is proper. To see that the third inclusion is proper, we offer the following

Example 4.1. *There is a Baire one function f and a co-countable set T such that $f|_T$ is continuous, yet $f \notin \mathcal{CSR}$.*

Proof. Let $K_1 \equiv K \setminus \{0,1\}$, where K denotes the usual middle-thirds Cantor set. Enumerate the component intervals of $(0,1) \setminus K_1$ as $\{(a_{1,j}, b_{1,j})\}_{j=1}^{\infty}$ and let $c_{1,j}$ denote the midpoint of $(a_{1,j}, b_{1,j})$. Set $C_1 = \{c_{1,j} : j \in \mathbb{N}\}$.

Place a copy of K_1 in each interval $(a_{1,j}, c_{1,j})$ and $(c_{1,j}, b_{1,j})$ by mapping $(0,1)$ affinely onto each of these intervals. Let K_2 be the union of all these copies of K_1 and enumerate the open components of $(0,1) \setminus K_2$ as $\{(a_{2,j}, b_{2,j})\}_{j=1}^{\infty}$. For each j let $c_{2,j}$ denote the midpoint of $(a_{2,j}, b_{2,j})$ and set $C_2 = \{c_{2,j} : j \in \mathbb{N}\}$.

Next, place a copy of K_1 in each interval $(a_{2,j}, c_{2,j})$ and $(c_{2,j}, b_{2,j})$ by mapping $(0,1)$ affinely onto each of these intervals. Let K_3 be the union of all these copies of K_1 and enumerate the open components of $(0,1) \setminus K_3$ as $\{(a_{3,j}, b_{3,j})\}_{j=1}^{\infty}$. For each j let $c_{3,j}$ denote the midpoint of $(a_{3,j}, b_{3,j})$ and set $C_3 = \{c_{3,j} : j \in \mathbb{N}\}$, and continue this process inductively. Finally, let $C = \cup_{n=1}^{\infty} C_n$ and note that

1. $C_n \cap C_m = \emptyset$ for $n \neq m$;
2. for each n, C_n is an isolated, and therefore scattered, set;
3. for each n, C_n is dense in K_n;
4. C is dense in I and consequently is not scattered.

Let $T = I \setminus C$ and define $f : I \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in C_n \\ 0 & \text{if } x \in T \end{cases}.$$

Since each C_n is scattered and $f|_T \equiv 0$, we clearly have that f is Baire one and $f|_T$ is continuous. To argue by contradiction, suppose $f \in \mathcal{CSR}$. Let D be a support set which consistently very nearly recovers f. For each n, let $S_n = C_n \cap D$ and $W_n = C_n \setminus D$.

As a first case, suppose that for each n we have W_n dense in K_n. Then $\cup_{n=1}^{\infty} W_n$ is dense in I and is consequently not scattered, but clearly no ordering of D can recover $f(x)$ at any $x \in \cup_{n=1}^{\infty} W_n$, yielding a contradiction.

Thus, there must exist an n for which W_n is not dense in K_n. Fix such an n for the remainder of this proof and let P be a nonempty portion of K_n for which $K_n \cap W_n = \emptyset$. Since C_n is dense in K_n, we must have that S_n is dense in P. Let $E = \{e_1, e_2, \ldots, e_j, \ldots\}$ be a non-scattered denumerable subset of $P \setminus D$. We shall define an ordering $\overline{x}$ of D such that $\overline{x}$ fails to recover $f(x)$ for each $x \in E$. To this end, let $\overline{d} = \{d_k\}_{k=1}^{\infty}$ be an arbitrary but fixed enumeration of D. Let $\overline{s}$ denote the infinite subsequence of $\overline{d}$ lying in S_n. Note that even though $\overline{s}$ is not a true trajectory, the symbol $r(\overline{s}, B_\epsilon(x))$ is well-defined for each $x \in E$ and each $\epsilon > 0$. We shall construct $\overline{x}$ in inductively in stages.

Stage 1: Let $x_1 = d_1$, $\epsilon_1 = \min\{1/2, |e_1 - x_1|/2\}$, $x_2 = r(\overline{s}, B_{\epsilon_1}(e_1))$, and $i_1 = 2$.

Inductive Stage: Let $m \in \mathbb{N}$ and assume that the segment $\{x_1, x_2, \ldots x_{i_m}\}$ of $\overline{x}$ has been determined. If $d_{m+1} \notin \{x_1, x_2, \ldots x_{i_m}\}$, append it as x_{i_m+1} and let

$i_m^* = i_m + 1$. Otherwise, just let $i_m^* = i_m$. Let ϵ_{m+1} denote 1/2 times the minimum of the set of numbers $\{|e_i - e_j| : 1 \le i \ne j \le m+1\} \cup \{|e_j - x_i| : 1 \le j \le m+1, 1 \le i \le i_m^*\}$. Then append the $m+1$ numbers $r(\overline{s}, B_{\epsilon_{m+1}}(e_j))$, $j = 1, 2, \dots m+1$, in any order to $\{x_1, x_2, \dots x_{i_m^*}\}$, beginning with $x_{i_m^*+1}$. Set $i_{m+1} = i_m^* + m + 1$. Thus we have defined the initial segment $\{x_1, x_2, \dots x_{i_{m+1}}\}$ of $\overline{x}$ and this completes the inductive stage.

In this manner we have defined an ordering $\overline{x}$ of D with the property that for each $x \in E$ the first return route to x contains a subsequence from S_n. Thus $\overline{x}$ does not recover $f(x)$ at each such x, completing the proof. □

Hence, the question of the equality of $\mathcal{SCR}$ and $\mathcal{CSR}$ remains open as does the problem of characterizing this class (or these classes).

5. Open questions

In addition to the open problems mentioned at the end of the previous section, several others naturally emerge. For example, in sections 2 through 4 we chose four specific types of small sets: the measure zero sets, the first category sets, countable sets, and scattered sets. These seemed the be the natural exceptional sets to initially consider and we were not disappointed with the richness of the results obtained. However, there are numerous other candidates for classes of small sets: σ-porous sets, sets with small dimension in one sense or another, sets with countable closures, etc. Perhaps an investigation of recovery or universal recovery or consistent recovery except on some of these types of small sets will yield equally interesting and diverse results.

References

1. Borsík J., *On the points of bilateral quasicontinuity of functions*, Real Anal. Exch. **19** (1993-94), 529–536.
2. Ćwiek I., Pawlak R. and Świątek B., *On some subclasses of Baire 1 functions*, Real Anal. Exch. **27** (2001/2002), 415–422.
3. Darji U. B. and Evans M. J., *Recovering Baire 1 functions*, Mathematika **42** (1995), 43–48.
4. Darji U. B., Evans M. J. and Humke P. D., *First return approachability*, J. Math. Anal. and Appl. **199** (1996), 545–557.
5. Darji U. B., Evans M. J., and O'Malley R. J., *A first return characterization of Baire 1 functions*, Real Anal. Exch. **19** (1993-94), 510–515.
6. Evans M. J. and Humke P. D., *Almost everywhere first-return recovery* (submitted for publication)
7. ______, *Consistent first-return Riemann sums for Lebesgue integrals*, Acta Math. Hungar. (accepted for publication)
8. Evans M. J., Humke P. D. and O'Malley R. J., *Consistent recovery and polygonal approximation of functions*, Real Anal. Exch. **28** (2002-03), 641-648.
9. Hausdorff, *Über halfstetige Funktionen and deren Verallgemeimerung*, Math. Z. **5** (1919), 292–309. F.

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