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TRIANGULAR MAPS WITH THE CHAIN RECURRENT POINTS PERIODIC

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Abstract. Forti and Paganoni [Grazer Math. Ber. **339** (1999), 125–140] found a triangular map $F(x, y) = (f(x), g_x(y))$ from $I \times I$ into itself for which closed set of periodic points is a proper subset of the set of chain recurrent points. We asked whether there is a characterization of triangular maps for which every chain recurrent point is periodic. We answer this question in positive by showing that, for a triangular map with closed set of periodic points and any positive real ε, every ε-chain from a chain recurrent point to itself may be represented as a finite union of ε-chains whose all points either are periodic or form a nontrivial ε-chain of some one-dimensional map g_x.

1. Introduction and the theorem

Denote by I the closed interval $[0, 1] \subseteq \mathbb{R}$ with the induced topology, by I^2 the Cartesian product $I \times I \subseteq \mathbb{R}^2$, by X an arbitrary compact metric space. If $A, B \subseteq X$ then $\overline{A}$, $\mathrm{int}(A)$ and $\mathrm{dist}(A, B)$ is the closure, the interior of A and the distance of A and B, respectively.

The set of continuous mappings of a compact metric space X into itself is denoted by $C(X, X)$. For $\varphi \in C(X, X)$ and $x \in X$, define inductively the n**th iteration** of φ by $\varphi^0(x) = x$, and $\varphi^n(x) = \varphi(\varphi^{n-1}(x))$; n is a member of positive integer set $\mathbb{N}$. The **orbit** $Orb(A)$ of a set $A \subseteq X$ is the set of its images under φ. The **trajectory** of x is the sequence $\{\varphi^n(x)\}_{n=0}^{\infty}$, and the set $\omega_\varphi(x)$ of limit points of the trajectory of x is the ω**-limit set** of x. Let $\omega(\varphi) = \bigcup_{x \in I} \omega_\varphi(x)$. The α-limit point of x is the limit point of some sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_0 = x$ and $\varphi(x_n) = x_{n-1}$. Let $\alpha_\varphi(x)$ be the set of all α-limit points of x and let $\mathrm{Fix}(\varphi) = \{x \in I \mid \varphi(x) = x\}$ be the set of fixed points of φ. A point x is a **periodic** point of φ, if $\varphi^p(x) = x$ for some $p \in \mathbb{N}$. The orbit of a periodic point x is a **cycle** and its cardinality is its **period**. Denote by $P(\varphi)$ the set of periodic points of φ. A point x is **nonwandering** if for every open neighborhood U of x there is an $n \in \mathbb{N}$ such that $\varphi^n(U) \cap U \neq \emptyset$. Let $\Omega(\varphi)$ denote the set of nonwandering points of φ.

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Let $\varepsilon > 0$ be given and let x, y be points of X. An **ε-chain** from x to y is a finite sequence $\{x_0, x_1, \ldots, x_n\}$ of points of X with $x = x_0$, $y = x_n$ and $d_X(\varphi(x_{k-1}), x_k) < \varepsilon$ for $k = 1, 2, \ldots, n$. An ε-chain E is **nontrivial** if at least one point of E is nonperiodic. A point x is **chain recurrent** if and only if, for every $\varepsilon > 0$, there is an ε-chain from x to itself ($x \to_\varepsilon x$). Let $CR(\varphi)$ denote the set of all chain recurrent points of φ.

The relations between the above-mentioned sets are given by the next proposition which can be found, e.g., in [**1**].

Theorem 1.1. *If X is a compact metric space and $\varphi \in C(X, X)$ then*

$$\mathrm{Fix}(\varphi) \subseteq P(\varphi) \subseteq \omega(\varphi) \subseteq \Omega(\varphi) \subseteq CR(\varphi). \tag{1}$$

The mentioned sets are equal if $f \in C(I, I)$ satisfies some assumptions; a long list of such assumptions can be found, e.g. in [**4**]. The next proposition which can be found, e.g. in [**3**], displays a few of them.

Theorem 1.2. [**3**] *$f \in C(I, I)$. Then the following conditions are equivalent:*
(i) *$P(f) = \overline{P(f)}$.*
(ii) *$P(f) = \Omega(f)$.*
(iii) *$P(f) = CR(f)$.*

Our paper is devoted to **triangular maps of the square**, i.e., continuous maps $F : I^2 \to I^2$ of the form $F(x, y) = \big(f(x), g_x(y)\big)$; the function f is called the **base** function of F. Denote by $T(I^2, I^2)$ the set of triangular maps $I^2 \to I^2$ and by d the max metric on I^2. In 1990 L. S. Efremova proved the next result (in a more strong form).

Theorem 1.3. [**2**] *Let $F \in T(I^2, I^2)$ and $P(F)$ be closed. Then $P(F) = \Omega(F)$.*

Let (i)–(iii) be the properties displayed in Proposition 1.2 considered for a triangular map F. Since the set $\Omega(F)$ is closed, the Proposition 1.3 and (1) give (i)$\Leftrightarrow$(ii). Moreover, Forti and Paganoni in [**3**] found a triangular map F with closed set of the periodic points such that $CR(F) \setminus \omega(F) \neq \emptyset$. By (1), $\omega(F) \subseteq \Omega(F) \subseteq CR(F)$, so there is a problem to characterize triangular maps F with closed set $P(F)$ such that $P(F) = CR(F)$. The wanted characterization follows from the next theorem which is the main result of our paper.

Definition 1.4. Let $z_1 = (x, y_1), z_2 = (x, y_2)$ be periodic points of F. The point z_1 is **accessible** from z_2 ($z_2 \to^a z_1$) if there is an ε-chain from z_2 to z_1 by the map F restricted to $I_{Orb(x)} = \bigcup_{y \in Orb(x)} \{y\} \times I$ for any $\varepsilon > 0$. The point z_1 is **nontrivially accessible** from z_2 ($z_2 \to^n z_1$) if $z_2 \to^a z_1$ and for any sufficiently small $\varepsilon > 0$ any ε-chain from z_2 to z_1 by the map F restricted to $I_{Orb(x)}$ is nontrivial.

Let K_1, K_2 be subsets of $P(F)$. Then K_1 is **accessible** from K_2 ($K_2 \to^a K_1$) if $z_2 \to^a z_1$ for some $z_1 \in K_1$ and $z_2 \in K_2$.

Definition 1.5. Points $z_1, z_2 \in P(F)$ form a **t-pair** if **(i)** $z_2 \to^n z_1$ and, for any $\delta > 0$, **(ii)** there exists a finite number of connected components K_i,

$i = 1, 2, \ldots, m$, of $P(F)$ such that $z_1 \in K_1, z_2 \in K_m$ and, for all $i = 1, 2, \ldots, m-1$, either $K_i \to^a K_{i+1}$ or $\operatorname{dist}(K_i, K_{i+1}) < \delta$.

Theorem 1.6. Main theorem *Let $F \in T(I^2, I^2)$ and $P(F)$ be closed. Then $CR(F) \setminus P(F) \neq \emptyset$ if and only if there exists a t-pair.*

2. Known facts

For a map $F \in T(I^2, I^2)$, let $A(F)$ Denote any of the sets $\operatorname{Fix}(F), P(F), \omega(F)$, $\Omega(F)$ and $CR(F)$ and let π be the canonical projection $(x, y) \mapsto x$.

Theorem 2.1. [**4**] *Let $F \in T(I^2, I^2)$, let f be the base function of F. Then $\pi\big(A(F)\big) = A(f)$.*

We already know that $CR(\varphi)$ is closed for any $\varphi \in C(X, X)$. Of course, $CR(\varphi)$ enjoy a number of other general properties used later.

Theorem 2.2. [**1**] *Let $\varphi \in C(X, X)$. Then, for any $n \in \mathbb{N}$, $CR(\varphi) = CR(\varphi^n)$ and $P(\varphi) = P(\varphi^n)$.*

Theorem 2.3. [**1**] *Let $\varphi \in C(X, X)$. Then $CR(\varphi) = CR(\varphi|_{CR(\varphi)})$, i.e. every chain recurrent point remains chain recurrent for the restriction of φ to $CR(\varphi)$.*

Let $\varphi \in C(X, X)$. A nonempty closed set A contained in X is **Lyapunov stable** if, for each open set U containing A, there exists an open set V containing A such that $\varphi^n(V) \in U$ for all $n \in \mathbb{N}$. A nonempty closed set A is an **attractor** if there exists an open set B containing A such that $\omega_\varphi(x) \subseteq A$ for every $x \in B$. If a nonempty closed set A is both Lyapunov stable and an attractor, we say that A is **asymptotically stable**.

Theorem 2.4. [**1**] *Let $\varphi \in C(X, X)$. If $A \subseteq X$ is an asymptotically stable set, then there exists an open set W containing A such that $\varphi(\overline{W}) \subseteq W$ Moreover, for any open upset U of A we can choose W so that $\overline{W} \subseteq U$.*

Denote by $Q(x, \varphi)$ the intersection of all asymptotically stable sets containing $\omega_\varphi(x)$.

Theorem 2.5. [**1**] *Let $\varphi \in C(X, X)$. If $y \in Q(x, \varphi)$, then $Q(y, \varphi) \subseteq Q(x, \varphi)$.*

Theorem 2.6. [**1**] *Let $\varphi \in C(X, X)$. If $y \in Q(x, \varphi)$, then there is an ε-chain from x to y by φ for any $\varepsilon > 0$.*

3. Lemmas and proofs

Let $\varphi \in C(X, X)$ and $K, L \subseteq X$. Denote by $S_\varepsilon(K, L)$ the closure of $\{z \in X \mid z$ is a member of an ε-chain from a point of K to a point of L by $\varphi\}$. If $K \to_\varepsilon L$ for any $\varepsilon > 0$, the set $S_\varphi(K, L) = \bigcap_{\varepsilon > 0} S_\varepsilon(K, L)$ is clearly nonempty and closed. If $z_0 \in CR(\varphi)$, the set $S_\varphi(z_0, z_0)$ is the **independent set** of z_0.

Lemma 3.1. *Let $\varphi \in C(X, X)$ and $z_0 \in CR(\varphi)$. Then the independent set $S := S_\varphi(z_0, z_0)$ is strongly invariant, i.e. $\varphi(S) = S$.*

Proof. This easily follows from the continuity of φ and the definition of S. $\square$

Lemma 3.2. *Let $\varphi \in C(X,X)$ and $z_0 \in CR(\varphi)$. Then any point of the independent set $S := S_\varphi(z_0,z_0)$ is chain recurrent for the restriction of φ to S, i.e. $S = CR(\varphi|_S)$.*

Proof. It is easy to see, $CR(\varphi|_S) \subseteq S$. We prove the converse inclusion.

Clearly $S \subseteq CR(\varphi)$. Let z be a chain recurrent point lying in S. At first we show that for any open neighborhood U of S and any $\varepsilon > 0$ there is an ε-chain from z to z in U. Assume, contrary to what we wish to show, that there exists an open neighborhood U of S and $\varepsilon > 0$ such that every ε-chain from z to itself contains a point in the complement of U. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers tending to 0. Then, for each $n \in \mathbb{N}$, there is a point z_n of an ε_n-chain lying outside U. Denote by $\tilde{z}$ an accumulation point of $\{z_n\}_{n\in\mathbb{N}}$. Evidently, $\tilde{z} \notin U$. But $z \rightarrow_\varepsilon \tilde{z}$ and $\tilde{z} \rightarrow_\varepsilon z$ for every $\varepsilon > 0$ and, by the definition of S and $z \in S$, this implies $\tilde{z} \in S$, which is a contradiction.

It remains to show that for any $\varepsilon > 0$ there is an ε-chain from z to itself within S. By the continuity of φ, we can choose $\delta > 0$, $0 < \delta < \frac{\varepsilon}{3}$, so that $d_X(x,y) < \delta$ implies $d_X(\varphi(x),\varphi(y)) < \frac{\varepsilon}{3}$. Let U be an open δ-neighborhood of S and let $\{z_0, z_1, \dots, z_k\}$ be an $\frac{\varepsilon}{3}$-chain from z to z lying in U. There exist points $s_0, s_1, \dots, s_k$ in S with $s_0 = s_k = z$ and $d_X(z_j, s_j) < \delta$ for $1 \le j \le k$. Then for each $j = 1, 2, \dots, k$

$$d_X(\varphi(s_{j-1}), s_j) < d_X(\varphi(s_{j-1}), \varphi(z_{j-1})) + d_X(\varphi(z_{j-1}), z_j) + d_X(z_j, s_j) < \varepsilon,$$

which already proves $S = CR(\varphi|_S)$. $\square$

Lemma 3.3. *Let $F \in T(I^2, I^2)$, $P(F)$ be closed and $z = (x,y) \in CR(F) \setminus P(F)$. If x is a fixed point of the base map f, then the independent set $S_f(x,x)$ is a nondegenerate closed interval of $P(f)$.*

Proof. Let the assumptions be fulfilled. Clearly,

$$\pi(S_F(z,z)) \subseteq S_f(x,x). \tag{2}$$

Denote $S := S_F(z,z)$ and $D := S_f(x,x)$. Since any independent set is closed, strongly invariant and the fixed point x belongs to D, D is closed interval.

We show that D is nondegenerate. Assume, on the contrary, $D = \{x\}$. By (2), $S \subseteq \{x\} \times I$ and hence, with respect to Lemma 3.2, $y \in CR(g_x)$ since $z \in S$. On the other hand, $z \notin P(F)$ implies $y \notin P(g_x)$, i.e. $y \in CR(g_x) \setminus P(g_x)$. So, by Proposition 1.2, $P(g_x)$ is not closed, which contradicts $P(F) = \overline{P(F)}$.

Finally, we show $D \subseteq P(f)$. By Lemma 3.2, any point of D is chain recurrent. Recall that $P(F)$ is closed and the projection of any closed set is closed. Hence, by Propositions 1.2 and 2.1, $D \subseteq P(f)$. $\square$

Lemma 3.4. *Let $F \in T(I^2, I^2)$, $P(F)$ be closed and f be the base map of F. Let $z = (x,y) \notin P(F)$ be such that $x \in \mathrm{Fix}(f)$. If any $z_1 \in \omega_F(z)$, $z_2 \in \alpha_F(z)$ lie in the same connected component of $P(F)$, then z_1, z_2 form a t-pair. Otherwise $z_2 \rightarrow^n z_1$.*

Proof. Let the assumptions be fulfilled. Clearly, $z_1 = (x, y_1), z_2 = (x, y_2)$ are periodic points of F. We show that $y_1 \neq y_2$. Really, by the choice of z_1, z_2, for any $\varepsilon > 0$, $y \to_\varepsilon y_1$ and $y_2 \to_\varepsilon y$ by g_x. Hence $y_1 = y_2$ implies $y \in CR(g_x)$ and, accordingly, $CR(g_x) \setminus P(g_x) \neq \emptyset$. So, by Proposition 1.2, $P(g_x)$ is not closed, a contradiction.

It is easy to see that $z_2 \to^a z_1$. Hence z_1, z_2 form a t-pair if they lie in the same connected component of $P(F)$. In the opposite case, $z_2 \to^n z_1$. □

Proof of the main theorem. Let F be a triangular map and f its base map. The existence of a t-pair clearly implies the existence of nonperiodic chain recurrent point. Let us prove the converse implication.

Let $z_0 = (x_0, y_0)$ be a nonperiodic chain recurrent point, $S := S_F(z_0, z_0)$ and $D := S_f(x_0, x_0)$. We show that there is a t-pair in S. Clearly, any t-pair of some iteration of F is t-pair of F. Moreover, by Proposition 2.2, $z_0 \in CR(F^n) \setminus P(F^n)$ for any $n \in \mathbb{N}$. So we may consider some convenient iteration of F instead of F. Thus we may assume $x_0 \in \mathrm{Fix}(f)$, since, by Propositions 1.2 and 2.1, $x_0 \in P(f)$. Obviously, $S \subseteq D \times I$, where, by Lemma 3.3, D is a closed nondegenerate interval of $P(f)$. But any interval of $P(f)$ containing a fixed point has only periodic points of period ≤ 2. So we find more convenient iteration of F and may assume $\pi(S) \subseteq \mathrm{Fix}(f)$. Moreover, by Lemma 3.2, $z_0 \in CR(F|_S) \setminus P(F|_S)$.

In the rest of the proof put $F := F|_S$ and $P := P(F) \cap S$. Since the complement of P is open, there is a $\delta > 0$ such that an open δ-neighborhood of z_0 contains no periodic point. Since S is strongly invariant, P is nonempty and hence we can define a relation $\sim := \sim(\delta)$ on the set of all connected components of P in the following way: $K \sim L$ if and only if $K \to_\delta L$ by F restricted to P. It is easy to see that $\sim$ is an equivalence relation on P. Members of the decomposition $P|_\sim$ are **δ-components**. The distance $d(v, w) \geq \delta$ if $v \in K, w \in L$ are points of distinct δ-components K, L and hence, by the compactness of S, there is only a finite number of δ-components H_i, $i = 1, 2, \ldots k$.

Lemma 3.5. *Let H be a δ-component and $u, w \in H$. Then there are connected components $L_1, L_2, \ldots, L_r$ of H such that $u \in L_1$, $w \in L_r$ and $\mathrm{dist}(L_i, L_{i+1}) < \delta$ for every $i = 1, 2, \ldots, r-1$.*

Proof. Obvious. □

By Lemma 3.1, $\omega_F(z_0) \subseteq P$ and $\alpha_F(z_0) \cap P \neq \emptyset$ since $z_0 \in S$. Choose $z_1 \in \omega_F(z_0)$. We show that there is a $z_2 := z_2(\delta) \in \alpha_F(z_0) \cap P$ such that z_1 and z_2 satisfy the condition (ii) of Definition 1.5 ($z_2 \to^t_\delta z_1$). With respect to Lemma 3.4, we may assume $z_2 \to^n z_1$. By Lemma 3.5, it is sufficient to find δ-components $K_1, K_2, \ldots, K_m$ such that $z_1 \in K_1$, $z_2 \in K_m$ and

$$K_i \to^a K_{i+1} \text{ for } i = 1, 2, \ldots, m-1. \tag{3}$$

Without loss of generality, assume $z_1 \in H_1$. If $\alpha_F(z_0) \cap H_1 \neq \emptyset$, by Lemmas 3.4 and 3.5, $z_2 \to^t_\delta z_1$ for every z_2 from this intersection. So let $\alpha_F(z_0) \cap (P \setminus H_1) \neq \emptyset$ and $K_1 := H_1$. Clearly $K \to^a L$ implies $\pi(K) \cap \pi(L) \neq \emptyset$. Hence we may restrict our attention to the set $S_1 := S \cap (\pi(K_1) \times I)$ to show that there is a δ-component H such that $K_1 \to^a H$.

Lemma 3.6. *$K_1 \to^a H$ for some $H \in P|_\sim$.*

Proof. Assume, on the contrary, that

$$K_1 \not\to^a H \text{ for each } H \in P|_\sim \setminus K_1. \tag{4}$$

Fix $x \in \pi(S_1) \subseteq \mathrm{Fix}(f)$. By Proposition 2.6, $Q(y, g_x) \cap (P \setminus K_1) = \emptyset$ for any $(x, y) \in K_1(x) := K_1 \cap I_x$. Hence $Q(x) \cap P = K_1$, where $Q(x)$ denotes $\bigcup_{(x,y)\in K_1(x)} Q(y, g_x)$. It is well known fact, that $Q(\cdot, \cdot)$ is an asymptotically stable set. Generally, only a finite union of asymptotically stable sets must be asymptotically stable, see [**1**]. Consequently, it suffices to clarify that $Q(x)$ is "generated" by a finite number of points of $K_1(x)$ to show that $Q(x)$ is asymptotically stable.

Since $P(F)$ is closed, K_1 is closed and hence, any sequence $\{y_i\} \subseteq K_1$ has an limit point $y \in K_1(x)$. Because $Q(y, g_x)$ is asymptotically stable, there exists $N \in \mathbb{N}$ such that $y_n \in Q(y, g_x)$ for every $n \geq N$. Moreover, if J is subinterval of $K_1(x)$, $J \subseteq Q(y, g_x)$ for any $(x, y) \in J$. By these two facts, Proposition 2.5 and the compactness of $K_1(x)$, $Q(x)$ is asymptotically stable.

Accordingly, by (4), $Q(x)$ is asymptotically stable for any $x \in \pi(S_1)$. Since we have restricted F to S_1, there is, by Proposition 2.4, an open set W containing K_1 such that

$$F(\overline{W}) \subseteq W. \tag{5}$$

Clearly, again by Proposition 2.4, we may assume $z_0 \notin W$. But we have shown that $\omega_F(z_0) \subseteq K_1$. Hence, for a sufficiently small $\varepsilon > 0$, some point of any ε-chain from z_0 to itself lies in W. On the other hand, since W is a subset of any open neighborhood of K_1, by (5), there is no ε-chain from H_1 to z_0 for some sufficiently small $\varepsilon > 0$, a contradiction. □

Denote $H'_1 := K_1$. By Lemma 3.6, a set $H'_2 = \{H \in P|_\sim \setminus H'_1 \,|\, K_1 \to^a H\}$ is nonempty. If $\alpha_F(z_0) \cap H'_2 \neq \emptyset$, by Lemmas 3.4 and 3.5, there is a $z_2 := z_2(\delta) \in \alpha_F(z_0) \cap H'_2$ such that $z_2 \to^t_\delta z_1$. If $\alpha_F(z_0) \cap H'_2$ is empty, we analogously define the set $H'_3 = \{H \in P|_\sim \setminus (H'_1 \cup H'_2) \,|\, \exists L \in H'_2 : L \to^a H\}$ and similarly show that $H'_3 \neq \emptyset$. Accordingly, since the cardinality of $P|_\sim$ is finite, there are nonempty sets $H'_1, H'_2, \ldots, H'_m$, $m \leq k$, of δ-components of P such that $\alpha_F(z_0) \cap K \neq \emptyset$ for some δ-component $K \in H'_m$ and, for any $j \in \{1, 2, \ldots, m-1\}$, any member of H'_{j+1} is accessible from some member of H'_j.

This already proves that for any $\delta > 0$ there is a $z_2 := z_2(\delta) \in \alpha_F(z_0) \cap S$ such that $z_2 \to^t_\delta z_1$. It is easy to see that $z_2(\delta) \to^t_\tau z_1$ for any $\tau > \delta$. Accordingly, for any decreasing sequence of positive reals $\{\delta_i\}$ tending to 0 the limit point $\tilde{z}_2$ of the sequence $\{z_2(\delta_i)\}$ and z_1 is a t-pair. □

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