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UNIVALENT HARMONIC MAPPINGS CONVEX IN ONE DIRECTION

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Abstract. In this work some distortion theorems and relations between the coefficients of normalized univalent harmonic mappings from the unit disc onto domains on the direction of imaginary axis are obtained.

1. Introduction

J. Clunie and T. Sheil-Small studied the class S_H of all harmonic, complex-valued, sense-preserving, univalent mappings defined on the unit disc U, which are normalized by $f(0) = f_z(0) - 1 = 0$. Such functions f can be written in the form $f = h + \bar{g}$ where $h(z) = z + a_2 z^2 + \dots$ and $g(z) = b_1 z + b_2 z^2 + \dots$ are analytic in U and $|g'(z)| < |h'(z)|$ for z in U. It follows that $|b_1| < 1$ and hence $f - \overline{b_1 f}$ also belongs to S_H. Thus we often restrict ourselves to the subclass S_H^0 of S_H consisting of those functions in S_H with $f_{\bar{z}}(0) = 0$. It is proven that S_H^0 is a compact and normal family and many other fundamental properties of S_H^0 and some of its subclasses are obtained [**2**].

But the general coefficient problems for the functions in the classes S_H and S_H^0 are not yet solved. For this reason many mathematicians have tried to solve coefficient problems in the subclasses of S_H [**1**], [**2**], [**3**], [**5**].

This paper is concerned with the subclass $K_H^0(\theta)$ of S_H^0 with the images $f(U)$ convex in the direction of θ, $(0 \le \theta < \pi)$. In this subclass we shall obtain distortion theorems and coefficient estimates.

Lemma 1.1. [**5**, Theorem 5.7] *First we give two important results that will be used during our work,* [**4**], [**5**]*. A function $f = h + \bar{g}$ in S_H maps U onto a convex domain if and only if the analytic function $h - e^{2i\theta} g$ is univalent and maps U onto a domain convex in direction for all θ, $0 \le \theta < \pi$.*

Lemma 1.2. [**4**, Theorem 1] *Let $\varphi(z)$ be a non-constant function regular in U. The function $\varphi(z)$ maps univalently onto a domain convex in direction of imaginary axis if and only if there are numbers μ and ν, $0 \le \mu < 2\pi$ and $0 \le \nu \le \pi$,*

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such that

$$\mathrm{Re}\{-ie^{i\mu}(1-2\,z\,e^{-i\mu}\cos\nu+z^2e^{-2i\mu})\varphi'(z)\}\geq 0,\quad z\in U. \tag{1}$$

2. Univalent Harmonic Mappings Convex in the Direction of the Imaginary Axis

Instead of studying a class of functions in the direction of any θ, $0\leq\theta<\pi$, it is enough to study the class of harmonic univalent functions convex in the direction of the imaginary axis. That is because, if the harmonic univalent function $f=h+\bar{g}$ is convex in the direction of some θ, there is a real α so that $F(z)=e^{i\alpha}f(e^{-i\alpha}z)$ is convex in the direction of the imaginary axis.

Let $K_H(i)$ and $K_H^0(i)$ denote the subclasses of S_H and S_H^0, respectively, which are convex on the direction of the imaginary axis.

Remark 2.1. *A harmonic function $f=h+\bar{g}$ maps U univalently onto a domain convex in the direction of the imaginary axis if and only if the analytic function $h+g$ is univalent and maps U onto a domain convex in the direction of the imaginary axis.*

We obtain the following result from Lemma 1 and Remark 1:

Remark 2.2. *A harmonic function $f=h+\bar{g}$ in $K_H(i)$ if and only if there numbers μ, $(0\leq\mu<2\pi)$ and ν, $(0\leq\nu\leq\pi)$, such that*

$$\mathrm{Re}\{-i\,e^{i\mu}(1-2\,z\,e^{-i\mu}\cos\nu+z^2e^{-2i\mu})[h'(z)+g'(z)]\}\geq 0,\quad z\in U. \tag{2}$$

For the functions

$$h(z)=z+\sum_{n=2}^{\infty}a_nz^n\quad\text{and}\quad g(z)=\sum_{n=2}^{\infty}b_nz^n$$

analytic in U, let $f=h+\overline{g}$ in $K_H^0(i)$. If we take

$$q(z)=-i\,e^{i\mu}(1-2\,z\,e^{-i\mu}\cos\nu+z^2e^{-2i\mu})[h'(z)+g'(z)] \tag{3}$$

and

$$p(z)=\frac{q(z)+i\cos\mu}{\sin\mu}; \tag{4}$$

then $\mathrm{Re}\,p(z)>0$ and $p(0)=1$. Therefore the function $p(z)$ belongs the class P of the analytic functions with positive real part. Furthermore, since $\sin\mu\geq 0$ for $\mu\in[0,\pi]$, $\mathrm{Re}\,q(z)\geq 0$. From (3) and (4)

$$\phi'(z)=h'(z)+g'(z)=\frac{\cos\mu+i\sin\nu\,p(z)}{1-2\,z\,e^{-i\mu}\cos\nu+z^2e^{-2i\mu}} \tag{5}$$

can be obtained.

Theorem 2.1. *A harmonic function f in $K_H^0(i)$ if and only if there is analytic function $p_1 \in P$ and two constant $\mu, \nu \in [0, \pi]$ such that*

$$f(z) = \operatorname{Re}\phi(z) + i \operatorname{Im} \int_0^z \phi'(\varsigma)\, p_1(\varsigma)\, d\varsigma. \tag{6}$$

Proof. Let $f = h + \bar{g}$ is in $K_H^0(i)$ then we can write

$$f(z) = \operatorname{Re}(h+g) + i \operatorname{Im}(h-g) \tag{7}$$

and

$$h' - g' = (h' + g')\frac{h' - g'}{h' + g'} = \phi' \frac{h' - g'}{h' + g'}. \tag{8}$$

We set $w = -g'/h'$ then the function w is analytic in U, $w(0) = 0$ and $|w(z)| < 1$. If we take

$$p_1(z) = \frac{h'(z) - g'(z)}{h'(z) + g'(z)} = \frac{1 + w(z)}{1 - w(z)}$$

then p_1 is analytic in U and $p_1(0) = 1$, $\operatorname{Re} p_1 > 0$ and $p_1 \in P$. If we consider (5), (7) and (8) altogether then we obtain (6). □

Theorem 2.2. *If $f = h + \bar{g}$ in $K_H^0(i)$ and*

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \textit{and} \quad g(z) = \sum_{n=2}^{\infty} b_n z^n\,, \quad z \in U,$$

then

$$|a_n| \le \frac{(n+1)(2n+1)}{6}, \quad |b_n| \le \frac{(n-1)(2n-1)}{6} \tag{9}$$

and

$$||a_n| - |b_n|| \le n.$$

Equality occurs for the harmonic Koebe function $k_0 = h + \bar{g}$, where

$$h(z) = \frac{6z - 3z^2 + z^3}{6(1-z)^3} \quad \textit{and} \quad g(z) = \frac{72z^2 + z^3}{6(1-z)^3}.$$

Proof. From $h' + g' = \phi'$ and $h' - g' = \phi'\, p_1$ we get

$$\begin{aligned} h'(z) &= e^{-i\mu}[\cos\mu + i\, p(z)\sin\mu]\frac{1}{1 - 2\,z\,e^{-i\mu}\cos\nu + z^2 e^{-2i\mu}}\frac{1 + p_1(z)}{2} \\ &\ll \frac{1+z}{1-z}\frac{1}{(1-z)^2}\frac{1}{1-z}. \end{aligned}$$

Here $\ll$ means that the moduli of the function on the left are bounded by the corresponding coefficients of the function on the right. Thus,

$$h'(z) \ll \sum_{n=0}^{\infty} \frac{(n+1)(n+2)(2n+3)}{6} z^n$$

i.e.

$$|na_n| \le \frac{n(n+1)(2n+1)}{6} \qquad \text{and} \qquad |a_n| \le \frac{(n+1)(2n+1)}{6}.$$

Similarly,

$$g'(z) = \phi'(z)\frac{1+p_1(z)}{2} \ll \frac{1+z}{1-z}\frac{1}{(1-z)^2}\frac{-z}{1-z} = \sum_{n=0}^{\infty} -\frac{n(n+1)(2n+1)}{6}\, z^n$$

i.e.

$$|nb_n| \le \frac{(n-1)\,n\,(2n-1)}{6} \quad \text{and} \quad |b_n| \le \frac{(n-1)(2n-1)}{6}.$$

From (9), we get

$$||a_n| - |b_n|| \le |a_n + b_n| \le n.$$

□

Theorem 2.3. *If $f = h + \bar{g}$ in $K_H^0(i)$, then for $|z| = r < 1$, and $b = |\cos\nu|$, $0 \le \nu \le \pi$,*

(10)

$$\frac{1-r}{(1+r)^2(1+2br+r^2)} \le |h'(z)| \le \begin{cases} \dfrac{1+r}{(1-r)^2(1-2br+r^2)} & ; r < \dfrac{1-\sin\nu}{b} \\ \dfrac{1}{(1-r)^3\sin\nu} & ; \dfrac{1-\sin\nu}{b} \le r < 1 \end{cases}$$

and

(11)

$$\frac{r(1-r)}{(1+r)^2(1+2br+r^2)} \le |g'(z)| \le \begin{cases} \dfrac{r(1+r)}{(1-r)^2(1-2br+r^2)} & ; r < \dfrac{1-\sin\nu}{b} \\ \dfrac{1}{(1-r)^3\sin\nu} & ; \dfrac{1-\sin\nu}{b} \le r < 1 \end{cases}$$

Both inequalities are sharp.

Proof. Since f is sense-preserving, the Jacobian of f $J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 > 0$ or $|g'(z)| < |h'(z)|$, $z \in U$. If we define $a(z) = g'(z)/h'(z)$, $a(z)$ satisfies the conditions of Schwarz Lemma. Then by (5)

$$zh'(z)[1 + a(z)] = [\cos\mu + ip(z)\sin\mu]k_\nu(z) \tag{12}$$

where

$$k_\nu(z) = \frac{z}{1 - 2z\cos\nu + z^2}. \tag{13}$$

Since $p \in P$, by [**4**, Lemma 2]

$$\frac{1-r}{1+r} \le |\cos\mu + i\,p(z)\sin\mu| \le \frac{1+r}{1-r}$$

for $|z| = r < 1$, and equality occurs for $\mu = \pi/2$ and for the function $p(z) = (1+z)/(1-z)$. Furthermore by [**4**, Lemma 2]

$$\frac{r}{1+2br+r^2} \le |k_\nu'(z)| \le \begin{cases} \dfrac{r}{1-2br+r^2} & ; r < \dfrac{1-\sin\nu}{b} \\ \dfrac{1}{(1-r^2)\sin\nu} & ; \dfrac{1-\sin\nu}{b} \le r < 1 \end{cases} \tag{14}$$

(12), (13) and (14) together gives (10). The inequality $|g'(z)| \le |z||h'(z)|$ together with (10) gives (11). □

Theorem 1, for $\nu = 0$ or $\nu = \pi$ the top one of the inequalities (10) and (11), and for $\nu = \pi/2$, the bottom one is valid for every r, $0 \le r < 1$.

The following is a result of Theorem 1:

Remark 2.3. *If $f = h + \bar{g}$ in $K_H^0(i)$ and $\nu = 0, \pi$, then for $|z| = r < 1$*

$$\frac{1-r}{(1+r)^3} \le |h'(z)| \le \frac{1+r}{(1-r)^4}$$

and

$$\frac{r(1-r)}{(1+r)^3} \le |g'(z)| \le \frac{r(1+r)}{(1-r)^4}.$$

If $\nu = \pi/2$, then

$$\frac{1-r}{(1+r)(1+r^2)} \le |h'(z)| \le \frac{1}{(1-r)^3}$$

and

$$\frac{r(1-r)}{(1+r)(1+r^2)} \le |g'(z)| \le \frac{1}{(1-r)^3}.$$

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