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ON THE PERRON PROBLEM FOR THE EXPONENTIAL DICHOTOMY OF C_0-SEMIGROUPS

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Abstract. In the present paper we give a sufficient condition for the exponential dichotomy of a C_0-semigroup in terms of "Perron-type" theorems in the case when we don't have the requirement of invertibility on the kernel of the dichotomic projection.

1. Introduction and Preliminaries

Over the past ten years the asymptotic theory of one parameter semigroups of operators has witnessed an explosive development. A number of long-standing open problems have recently been solved and the theory seems to have obtained a certain degree of maturity. There are various conditions characterizing exponentially stable or dichotomic semigroups on Banach or Hilbert spaces.

The concept of exponential dichotomy of linear differential equations was introduced by O. Perron [**9**], which is concerned with the problem of conditional stability of a system $x' = A(t)x + f(t,x)$ in a finite-dimensional space. After seminal researches of O. Perron, relevant results concerning the extension of Perron's problem in the more general framework of infinite-dimensional Banach spaces were obtained by M. G. Krein [**2**], J. L. Daleckij [**2**], J. L. Massera[**4**] and J. J. Schäffer [**4**], and recently by van Neerven [**7**], van Minh [**5, 6**], F. Räbiger [**6**], R. Schnaulbelt [**6**] and Vu Quoc Phong [**11**].

The first aim of this paper is to propose a new and direct way to deal with the connections between some "Perron-type" conditions and the exponential dichotomy of a C_0-semigroup, more easier to verify from our point of view.

Let X be a Banach space and $B(X)$ the Banach algebra of all bounded linear operators acting on X. The norm on X and on $B(X)$ will be denoted by $\|\cdot\|$.

We recall that a family $\mathbf{T} = \{T(t)\}_{t\geq 0}$ of bounded linear operators from X into itself is a C_0-semigroup on X, if

(s_1) $T(0) = I$ (where I is the identity operator on X);
(s_2) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$;
(s_3) $\lim\limits_{t\to 0_+} T(t)x = x$, for all $x \in X$

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It is well-known that every C_0-semigroup is exponentially bounded i.e.

$$\|T(t)\| \leq Me^{\omega t}, \text{ for all } t \geq 0$$

for some $M, \omega > 0$. See for instance [**7, 8**].

Therefore it makes sense to define

$$\omega(\mathbf{T}) = \inf\{\alpha \in \mathbb{R} : \exists\, \beta \geq 1 \text{ such that } \|T(t)\| \leq \beta e^{\alpha t}, \text{ for all } t \geq 0\}.$$

For the spectral radius of the operator $T(t)$ we have the formula (see [**7**])

$$r(T(t)) = e^{t\omega(\mathbf{T})}.$$

We denote by X_1 the space of all $x \in X$ with the property that $T(\cdot)x$ is bounded. In what follows X_1 will be assumed complemented (i.e. X_1 is closed and there exists X_2 a closed subspace such that $X = X_1 \oplus X_2$).Also we denote by P a projection along X_2 (that is $P \in B(X), P^2 = P$ and $Ker(P) = X_2$).

It is easy to see that X_1 is $T(t)$-invariant for all $t \geq 0$ (that is equivalent to $PT(t)P = T(t)P$ for each $t \geq 0$) and so the application $T_1 : \mathbb{R}_+ \to B(X_1)$, $T_1(t) = T(t)_{|X_1}$ is also a C_0-semigroup, acting on X_1.

Let us denote by $C(\mathbb{R}_+, X)$ the space of all bounded continuous functions from $\mathbb{R}_+$ to X, which is a Banach space endowed with the norm

$$|||f||| = \sup_{t\geq 0} \|f(t)\|.$$

Definition 1.1. *The C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is exponentially dichotomic if there exist the constants $N_1, N_2, \nu_1, \nu_2 > 0$ such that*

(d$_1$) $\|T(t)x\| \leq N_1 e^{-\nu_1 t}\|x\|$, *for all $t \geq o$ and all $x \in X_1$;*
(d$_2$) $\|T(t)x\| \geq N_2 e^{\nu_2 t}\|x\|$, *for all $t \geq 0$ and all $x \in X_2$.*

Remark 1.1. The condition d_1) is equivalent with $\omega(T_1) < 0$.

We note that in domain's literature almost all authors (see for instance [**1, 10**]) defined the concept of exponential dichotomy in the case of C_0-semigroups in the following way: A strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ is said to be exponentially dichotomic if there exists a projection operator P on X (so-called *dichotomic projection*) such that the following statements hold:

(i) $PT(t) = T(t)P$ for all $t \geq 0$
(ii) There are positive constants M, ν such that $\|T(t)x\| \leq Me^{-\nu t}\|x\|$ for all $x \in P(X)$ and $t \geq 0$
(iii) The restriction $T(t)|_{Ker(P)}$ is an invertible operator (so extends to a C_0-group) and $\|T^{-1}(t)x\| \leq Me^{-\nu t}\|x\|$ for all $x \in Ker(P)$ and $t \geq 0$.

Thus, in this spirit, the concept of exponential dichotomy is in fact exponential stability, first for the restriction $T_1(t)$ and second for $T_2^{-1}(t)$ where $T_2(t) = T(t)|_{Ker(P)}$.

We remark that if P is a dichotomic projection then $P(X) = X_1$ and the conditions of Definition 1.1 are satisfied. The converse statement seems to be an open question until now. However in the spirit of the Definition 1.1 our main result do not refer on the additional requirement of invertibility on X_2, so is more easier to verify.

Definition 1.2. *The C_0 semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ satisfy the Perron condition if for all $f \in C(I\!R_+, X)$ exists $x \in X$ such that $u(\cdot; x, f) \in C(I\!R_+, X)$ where*

$$u(t; x, f) = T(t)x + \int_0^t T(t-s)f(s)ds.$$

Proposition 1.1. *If the C_0 semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ satisfy the Perron condition then for every $f \in C(I\!R_+, X)$ exists a unique $x_2 \in X_2$ such that $u(\cdot; x_2, f) \in C(I\!R_+, X)$.*

Proof. Consider $f \in C(I\!R_+, X)$, x given by Definition 1.2 $x_1 \in X_1$, $x_2 \in X_2$ with $x = x_1 + x_2$. Then

$$u(\cdot, x_2, f) = u(\cdot, x, f) - T(\cdot)x_1 \in C(I\!R_+, X)$$

and so we have the existence part.

If we assume that $y_2, z_2 \in X_2$ and $u(\cdot; y_2, f), u(\cdot, z_0, f) \in C(I\!R, X)$ it is clear that

$$T(\cdot)(y_2 - z_2) = u(\cdot; y_2, f) - u(\cdot; z_2, f) \in C(I\!R_+, X)$$

which implies that $y_2 - z_2 \in X_1 \cap X_2$ and hence $y_2 = z_2$. □

The unique element $x_2 \in X_2$ for $f \in C(I\!R_+, X)$ will be denoted in what follows by x_f.

Proposition 1.2. *If the C_0-semigroup $\mathbf{S} = \{S(t)\}_{t\geq 0}$ has the property $\sup\limits_{t\geq 0} t\|S(t)\| < \infty$ then $\omega(\mathbf{S}) < 0$.*

Proof. It is obvious that from the hypothesis we have that there exists $a > 0$ with $\|S(a)\| < 1$.

It follows that

$$e^{a\omega(\mathbf{S})} = r(S(a)) \leq \|S(a)\| < 1$$

and so we obtain $\omega(\mathbf{S}) < 0$. □

2. The main result

Proposition 2.1. *If the C_0 semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ satisfy the Perron condition then there exists $K > 0$ such that*

$$|||u(\cdot; x_f, f)||| \leq K|||f|||, \text{ for all } f \in C(I\!R_+, X).$$

Proof. Define $U : C(I\!R_+, X) \to C(I\!R_+, X)$, $Uf = u(\cdot; x_f, f)$. We note that U is a linear operator. In order to prove that in addition U is also bounded, consider (f_n) a sequence of elements belonging to $C(I\!R_+, X)$ and $f, g \in C(I\!R_+, X)$ such that

$$f_n \xrightarrow{|||\cdot|||} f \quad \text{and} \quad Uf_n \xrightarrow{|||\cdot|||} g.$$

Since $x_{f_n} = (Uf_n)(0)$, for all $n \in N$ it follows that

$$x_{f_n} \to g(0) \quad \text{and so} \quad g(0) \in X_2.$$

Using the fact that

$$\begin{aligned}\left\|\int_0^t T(t-s)f_n(s)ds - \int_0^t T(t-s)f(s)ds\right\| &\leq \int_0^t \|T(t-s)(f_n(s)-f(s))\|ds \\ &\leq Me^{\omega t}|||f_n - f|||,\end{aligned}$$

for all $t \geq 0$ and all $n \in \mathbb{N}$ we obtain that

$$u(\cdot; g(0), f) = g \in C(\mathbb{R}_+, X)$$

which implies that $x_f = g(0)$ and hence $Uf = g$.

It is now clear that

$$|||u(\cdot; x_f, f)||| = |||Uf||| \leq \|U\||||f|||, \text{ for all } f \in C(\mathbb{R}_+, X)$$

Now we can state the main result of this paper. □

Theorem 2.1. *If the C_0 semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ satisfy the Perron condition then $\mathbf{T}$ is exponentially dichotomic.*

Proof. For $\delta > 0$, $x \in X_2 \setminus \{0\}$ we define $\chi : \mathbb{R}_+ \to \mathbb{R}_+$

$$\chi(t) = \begin{cases} 1, & t \in [0, \delta] \\ 1 + \delta - t, & t \in (\delta, \delta + 1) \\ 0, & t \geq \delta + 1 \end{cases}$$

and $f : \mathbb{R}_+ \to X$, $f(t) = -\dfrac{\chi(t)}{\|T(t)x\|}T(t)x$. Then $f \in C(\mathbb{R}_+, X)$, $|||f||| \leq 1$ and

$$\begin{aligned}\int_0^t T(t-s)f(s)ds &= -\int_0^t \frac{\chi(s)}{\|T(s)x\|}dsT(t)x = \\ &= -\int_0^\infty \frac{\chi(s)}{\|T(s)x\|}dsT(t)x + \int_t^\infty \frac{\chi(s)}{\|T(s)x\|}dsT(t)x,\end{aligned}$$

for all $t \geq 0$.

From the argument that χ has compact support we have that the function $t \mapsto \displaystyle\int_t^\infty \frac{\chi(s)}{\|T(s)x\|}dsT(t)x : \mathbb{R}_+ \to X$ has compact support too, and hence

$$u\left(\cdot; \int_0^\infty \frac{\chi(s)ds}{\|T(s)\|}x, f\right) \in C(\mathbb{R}_+, X)$$

which implies that $x_f = \displaystyle\int_0^\infty \frac{\chi(s)ds}{\|T(s)x\|}x$ and

$$u(t, x_f, f) = \int_t^\infty \frac{\chi(s)}{\|T(s)x\|}dsT(t)x, \text{ for all } t \geq 0.$$

By Proposition 2.1 it follows that

$$\int_t^\infty \frac{\chi(s)}{\|T(s)x\|}ds\|T(t)x\| \leq K, \text{ for all } t \geq 0.$$

Using the definition of χ we can state that $\displaystyle\int_t^\delta \frac{ds}{\|T(s)x\|} \leq \frac{K}{\|T(t)x\|}$, for all $\delta > 0$, $t \geq 0$, with $t \leq \delta$,$x \in X_2 \setminus \{0\}$.

Making δ to tend toward ∞ we obtain that

$$\int_t^\infty \frac{ds}{\|T(s)x\|} \leq \frac{K}{\|T(t)x\|}, \text{ for all } t \geq 0, \text{ and all } x \in X_2 \setminus \{0\}.$$

If, for $x \in X_2 \setminus \{0\}$ we denote by $\varphi_x : I\!R_+ \to I\!R_+$, the function defined by $\varphi_x(t) = \displaystyle\int_t^\infty \frac{ds}{\|T(s)x\|}$, it is easy to see that φ_x is a differentiable function and

$$\varphi_x(t) \leq -K\varphi_x'(t), \text{ for all } t \geq 0 \text{ and all } x \in X_2 \setminus \{0\}$$

It results that

$$\int_t^{t+1} \frac{ds}{\|T(s)x\|} e^{\frac{t}{K}} \leq e^{\frac{t}{K}} \varphi_x(t) \leq \varphi_x(0) \leq \frac{K}{\|x\|},$$

for all $t \geq 0$ and all $x \in X_2 \setminus \{0\}$.

If we combine this with the fact that

$$\|T(s)x\| \leq \|T(s-t)\| \|T(t)x\| \leq Me^\omega \|T(t)x\|, \text{ for all } t \geq 0,$$

$s \in [t, t+1]$, $x \in X$.

We can conclude that

$$\|T(t)x\| \geq \frac{1}{Me^\omega K} e^{\frac{t}{K}} \|x\|, \text{ for all } t \geq 0 \text{ and all } x \in X_2$$

and hence the condition d_2) holds for $N_2 = \dfrac{1}{Me^\omega K}$ and $\nu_1 = \dfrac{1}{K}$.

Put $g : I\!R_+ \to X$, $g(t) = T(t)x$, for $x \in X_1$.

By the definition of X_1 it follows that $g \in C(I\!R_+, X)$ and

$$u(t; x_g, g) = T(t)x_g + tT(t)x, \text{ for all } t \geq 0.$$

If we assume that $x_g \neq 0$ then

$$K|||g||| \geq \|u(t; x_g, g)\| \geq \|T(t)x_g\| - t\|T(t)x\| \geq N_2 e^{\nu_2 t} - t|||g|||$$

for all $t \geq 0$, which is a contradiction. It follows that $x_g = 0$ and so

$$tT(t)x = u(t; x_g, f), \text{ for all } t \geq 0.$$

Now it is obvious that

$$\sup_{t \geq 0} t\|T_1(t)x\| < \infty, \text{ for all } x \in X_1$$

and hence $\sup\limits_{t \geq 0} t\|T_1(t)\| < \infty$. By Proposition 1.2 and Remark 1.1. we have that the condition d_1) holds, and with this the proof is complete. □

References

1. Chicone C. and Latushkin Y., *Evolution semigroups in dynamical systems and differential equations*, Mathematical Surveys and Monographs, vol. **70**, Providence, RO: American Mathematical Society, 1999.
2. Daleckij J. L. and Krein M. G., *Stability of solutions of Differential Equations in Banach spaces*, Trans. Am. Math. Society (1974).
3. Datko R., *Uniform asymptotic stability of evolutionary processes in Banach spaces*, SIAM J. Math. Anal. **3** (1972), 428–445.
4. Massera J. L. and Schäffer J. J., *Linear differential equations and function spaces*, Academic Press, New York, 1966.
5. van Minh N., *On the proof of characterization of the exponential dichotomy*, Proc. Am. Math. Society **127** (1999), 779–782.
6. van Minh N., Räbiger and R Schnaubelt F., *Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on half-line*, Integral Equations Operator Theory, **32** (1998) 332–353.
7. van Neerven J., *The Asymptotic Behaviour of Semigroups of linear operators,* Operator Theory Advances and Applications, vol. **88**, Birkhauser, Basel, 1996.
8. Pazy A., *Semigroups of operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.
9. Perron O., *Die Stabilitatsfrage bei Differential gleichungen* Math. Z. **32** (1930), 703–728.
10. Pruss J., *On the spectrum of C_0-semigroups*, Trans. Am. Math. Society **284** (1984), 847–857.
11. Vu Quoc Phong. *On the exponential stability and dichotomy of C_0-semigroups*, Studia Mathematica **132** (2) 1999, 141–149.

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