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0-CONDITIONS AND TOLERANCE SCHEMES

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ABSTRACT. We study the relation between tolerance schemes and certain 0-conditions on the tolerance lattice of an algebra. We prove new properties of the tolerance lattices of algebras with majority terms characterized by 0-conditions which are related with distributivity at 0.

1. INTRODUCTION

The congruence and tolerance schemes are the sources of important results in the study of congruence and tolerance lattices of algebras. As examples we mention [**9**], [**4**], [**3**] or [**5**]. Generalizing the result of H.-J. Bandelt [**1**], it was proved in [**7**] that the tolerance lattice Tol$A$ of an algebra A with a majority term is 0-modular and pseudocomplemented. As Tol$A$ is an algebraic lattice, the latter property is equivalent to another 0-condition, namely to 0-distributivity of Tol$A$.

In this paper we study the interrelation between two tolerance schemes and certain 0-conditions on Tol$A$. We show that whenever the first scheme is valid, Tol$A$ cannot contain some splitting sublattices. We prove that for any algebra A having the so called tolerance intersection property (see Definition 2.2), 0-distributivity of Tol$A$ is equivalent to another 0-condition and implies 0-modularity. Generalizing [**7**], Theorem 3.9, we also show that the tolerance lattice of an algebra with a majority term satisfies a 0-condition which is not a consequence of its above mentioned known properties.

2. PRELIMINARIES

A lattice L with 0 is said to be **0-distributive** if, for $a, b, c \in L$,

(D_0) $$a \wedge c = 0 \text{ and } b \wedge c = 0 \text{ imply } (a \vee b) \wedge c = 0.$$

L is called **0-modular**, if for $a, b, c \in L$,

(M_0) $$a \wedge c = 0 \text{ and } b \leq c \text{ imply } (a \vee b) \wedge c = b.$$

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A lattice L with 0 is called **pseudocomplemented** if for every element $x \in L$, there exists an $x^* \in L$ such that for any $y \in L$, $y \wedge x = 0 \Leftrightarrow y \leq x^*$. It is known that an algebraic lattice is 0-distributive if and only if it is pseudocomplemented [**11**]. In view of J. C. Varlet's result [**11**, Theorem 5] a lattice with 0 is 0-modular if and only if it does not contain an N_5 sublattice including 0. It is important to note that there is no possible characterization using sublattices for 0-distributivity (see e.g. [**8**] or [**10**]). Now, we introduce two other 0-conditions:
We shall call a lattice L with 0 **pseudo-0-distributive** if for $a, b, c \in L$,

(D_0^*) $\qquad a \wedge b = 0$ and $a \wedge c = 0$ imply $(a \vee b) \wedge c = b \wedge c$.

L will be called **super-0-distributive** if, for $a, b, c \in L$,

(SD_0) $\qquad a \wedge b = 0$ implies $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.

In other words, L is super-0-distributive if for any $a, b \in L$ with $a \wedge b = 0$, (a, b) is a distributive pair of L. Clearly, we have $(SD_0) \Rightarrow (D_0^*) \Rightarrow (M_0)$. There is no interrelation between (D_0) and (D_0^*): The lattice N_5 including 0 satisfies (D_0), however it does not satisfy (D_0^*). The lattice in Figure 1 does not satisfy (D_0), however it satisfies (D_0^*).

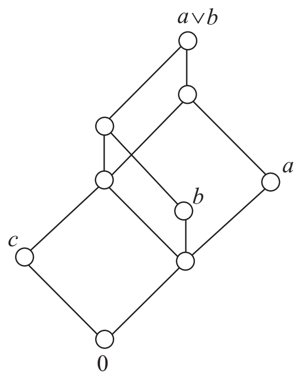

Figure 1

Lemma 2.1. *Any pseudocomplemented* 0*-modular lattice is pseudo*-0*-distributive.*

Proof. Let L be a pseudocomplemented and 0-modular lattice and let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$. Then $b, c \leq a^*$. Hence we get $(a \vee b) \wedge c \leq (a \vee b) \wedge a^*$. Since $a \wedge a^* = 0$ and $b \leq a^*$, using (M_0) we get $(a \vee b) \wedge a^* = b$. Thus $(a \vee b) \wedge c \leq b \wedge c$. As $b \wedge c \leq (a \vee b) \wedge c$ is straightforward, we conclude $(a \vee b) \wedge c = b \wedge c$, proving (D_0^*). $\square$

Let $(\mathrm{Con}A, \wedge, \vee)$ and $(\mathrm{Tol}A, \wedge, \sqcup)$ stand for the congruence lattice and the tolerance lattice of an algebra A, respectively. $\triangle$ denotes the identity relation on A. The relational product of two binary relations $\rho, \sigma \subseteq A^2$ is denoted by $\rho \circ \sigma$. $\overline{\varphi}$ stands for the transitive closure of a $\varphi \in \mathrm{Tol}A$. As $\overline{\varphi} \in \mathrm{Con}A$, it is easy to see that

$\overline{(\alpha \sqcup \beta)} = \overline{\alpha} \vee \overline{\beta}$, for all $\alpha, \beta \in \mathrm{Tol}A$. The inclusions $\alpha \sqcup \beta \subseteq (\alpha \circ \beta) \cap (\beta \circ \alpha)$ and $\overline{\alpha} \circ \overline{\beta} \subseteq \overline{\alpha} \vee \overline{\beta}$ (see e.g. [**7**, Lemma 2.1] and [**2**]) will be used also in our proofs. In view of [**6**] and [**7**], algebras in congruence modular varieties satisfy the so called tolerance intersection property:

Definition 2.2. An algebra A is said to satisfy the **tolerance intersection property, TIP** for short, if for any $\alpha, \beta \in \mathrm{Tol}A$ we have $\overline{\alpha} \wedge \overline{\beta} = \overline{(\alpha \wedge \beta)}$.

In this paper we shall investigate two **tolerance schemes** and their relations with 0-conditions on the tolerance lattice of an algebra.

SCHEME

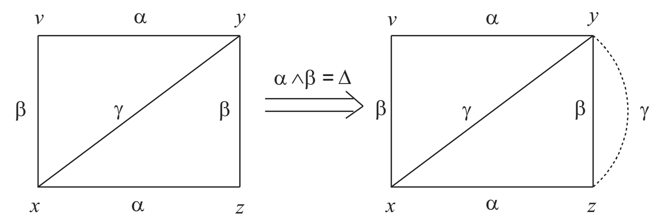

Figure 2

R-SCHEME

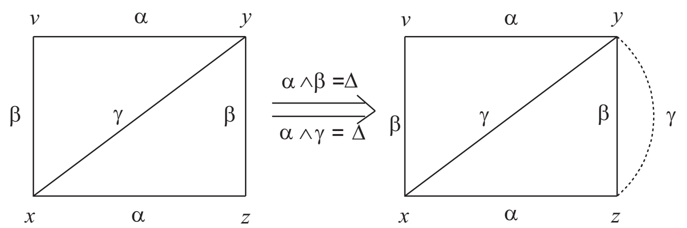

Figure 3

Since the latter scheme is the restriction of the first, if an algebra A satisfies the SCHEME, then it satisfies the R-SCHEME too.

3. 0-CONDITIONS DERIVED FROM SCHEMES

Theorem 3.1. *Let A be an algebra satisfying the SCHEME. Then* Tol$A$ *does not contain M_3,N_5, or $M_{2,3}$ as a sublattice including $\triangle$.*

Proof. Assume that Tol$A$ contains as a sublattice one of the following below:

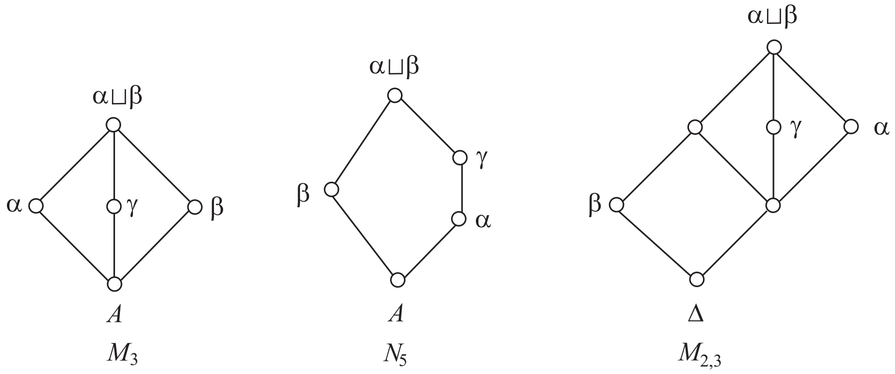

Figure 4

Suppose $(x,y)\in\gamma$. Then $(x,y)\in\alpha\sqcup\beta\subseteq(\alpha\circ\beta)\cap(\beta\circ\alpha)$, and hence in all of the cases there are $z,v\in A$ such as shown in Figure 5.

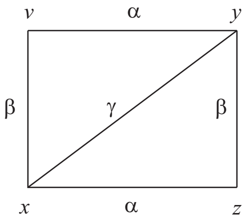

Figure 5

Applying the SCHEME, we get $(y,z)\in\gamma$. Thus we have $(y,z)\in\beta\wedge\gamma=\triangle$. Since $(x,z)\in\alpha$, we obtain $(x,y)\in\alpha\circ(\beta\wedge\gamma)=\alpha\circ\triangle=\alpha$, a contradiction with $\gamma\nleq\alpha$. □

Theorem 3.2. *Let A be an algebra with TIP. Then the following assertions are equivalent:*

(i) Tol$A$ *is* 0-*distributive.*
(ii) *A satisfies the R-SCHEME.*
(iii) Tol$A$ *is pseudo*-0-*distributive.*

Proof. (i)⇒(ii): Assume Tol$A$ satisfies (D_0) and take α, β, γ $\in$Tol$A$ with $\alpha\wedge\beta=\triangle$, $\alpha\wedge\gamma=\triangle$, and $x,y,z,v\in A$ as in Figure 5. Then (D_0) implies $\alpha\wedge(\beta\sqcup\gamma)=\triangle$ and this gives $\alpha\wedge(\overline{\beta}\vee\overline{\gamma})=\alpha\wedge\overline{(\beta\sqcup\gamma)}\le\overline{\alpha}\wedge\overline{(\beta\sqcup\gamma)}=\overline{\alpha\wedge(\beta\sqcup\gamma)}=\overline{\triangle}=\triangle$.

Therefore we get $(x,z)\in\alpha\cap(\beta\circ\gamma)\subseteq\alpha\cap(\overline{\beta}\circ\overline{\gamma})\subseteq\alpha\wedge(\overline{\beta}\vee\overline{\gamma})=\triangle$, i.e. $x=z$. Hence we obtain $(y,z)=(y,x)\in\gamma$. Thus A satisfies the R-SCHEME.

(ii)⇒(iii): Take α,β,γ $\in$Tol$A$ with $\alpha\wedge\beta=\triangle$, $\alpha\wedge\gamma=\triangle$ and let $(x,y)\in(\alpha\sqcup\beta)\wedge\gamma$. As $(\alpha\sqcup\beta)\subseteq(\alpha\circ\beta)\cap(\beta\circ\alpha)$, we have $(x,y)\in(\alpha\circ\beta)\cap(\beta\circ\alpha)\cap\gamma$, therefore there exist elements $v,z\in A$ that are shown in Figure 5. Now, applying the R-SCHEME we obtain $(y,z) \in \beta \wedge \gamma$, whence $(x,z) \in (\gamma \circ (\beta \wedge \gamma)) \cap \alpha \subseteq (\gamma \circ \gamma) \cap \alpha \subseteq \overline{\gamma} \wedge \overline{\alpha} = \overline{(\gamma \wedge \alpha)} = \triangle$. Hence we obtain $(x,z) \in \triangle$, i.e. $x = z$. Thus we have $(x,y) = (z,y) \in \beta \wedge \gamma$. As a consequence we get

$$(\alpha \sqcup \beta) \wedge \gamma \leq \beta \wedge \gamma.$$

Since the reversed inequality is obvious, we obtain $(\alpha \sqcup \beta) \wedge \gamma = \beta \wedge \gamma$, i.e. Tol$A$ satisfies (D_0^*).

(iii)⇒(i): Assume that Tol$A$ satisfies (D_0^*) and let $\alpha, \beta, \gamma \in$Tol$A$ such that $\alpha \wedge \gamma = \triangle$ and $\beta \wedge \gamma = \triangle$ and take $(x,y) \in (\alpha \sqcup \beta) \wedge \gamma$.

Since $(\alpha \sqcup \beta) \wedge \gamma \subseteq (\alpha \circ \beta) \cap (\beta \circ \alpha) \cap \gamma$, there exist $v, z \in A$ such as shown in Figure 5. Then we have $(y,z) \in (\gamma \circ \alpha) \cap \beta \subseteq (\overline{\gamma} \circ \overline{\alpha}) \cap \overline{\beta} \subseteq (\overline{\alpha} \vee \overline{\gamma}) \wedge \beta = \overline{(\alpha \sqcup \gamma)} \wedge \overline{\beta} = \overline{(\alpha \sqcup \gamma) \wedge \beta}$. Applying (D_0^*) (with $a = \gamma$, $b = \alpha$, $c = \beta$) we get $(\gamma \sqcup \alpha) \wedge \beta = \alpha \wedge \beta \leq \alpha$. Hence $(z,y) \in \overline{(\alpha \sqcup \gamma) \wedge \beta}. \leq \overline{\alpha}$. Thus we obtain $(x,y) \in (\alpha \circ \overline{\alpha}) \cap \gamma \subseteq \overline{\alpha} \wedge \overline{\gamma} = \overline{(\alpha \wedge \gamma)} = \triangle$ proving $(\alpha \sqcup \beta) \wedge \gamma = \triangle$, i.e. (D_0) is satisfied. ☐

Corollary 3.3. (i) *If an algebra A with TIP satisfies the SCHEME, then* Tol$A$ *is a pseudocomplemented and* 0*-modular lattice.*

(ii) *Let A be an algebra with TIP. If* Tol$A$*is a* 0*-distributive lattice, then it is* 0*-modular too.*

Proof. (i) As SCHEME implies R-SCHEME, the assumptions of (i) imply, in view of Theorem 3.2, that Tol$A$ satisfies both (D_0) and (D_0^*). Since Tol$A$ is an algebraic lattice, (D_0) implies that Tol$A$ is pseudocomplemented. As (D_0^*) implies (M_0), Tol$A$ is a 0-modular lattice.
(ii) We apply Theorem 3.2 and the fact that (D_0^*) implies (M_0). ☐

4. Super-0-distributive tolerance lattices

Let A be an algebra. A term function $m : A^3 \to A$ is called a **majority term** if $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$ holds for all $x, y \in A$. For instance, any lattice $(L, \wedge, \vee)$ admits a majority term.

Lemma 4.1. *Let A be an algebra with a majority term* $m(x,y,z)$ *and let* $\alpha, \beta \in$ Tol$A$ *(and* $a, b, c \in A$*). Then:*
(i) *If* $(a,b) \in \alpha$ *and* $(b,c) \in \beta$ *then* $(m(a,b,c), b) \in \alpha \wedge \beta$.
(ii) *We have* $\alpha \sqcup \beta \subseteq (\alpha \circ \beta) \cap (\beta \circ \alpha) \subseteq (\alpha \wedge \beta) \circ (\alpha \sqcup \beta)$.

Proof. (i) We have $(m(a,b,c), b) = (m(a,b,c), m(b,b,c)) \in \alpha$ and
$m((a,b,c), b) = (m(a,b,c), m(a,b,b)) \in \beta$, and hence we get
$(m(a,b,c), b) \in \alpha \wedge \beta$**.**
(ii) We have $\alpha \sqcup \beta \subseteq (\alpha \circ \beta) \cap (\beta \circ \alpha)$ by Lemma 2.1 of [**7**]. In order to prove the second inclusion, assume that $(a,c) \in (\alpha \circ \beta) \cap (\beta \circ \alpha)$. Then there exist $b, d \in A$ such that $(a,b) \in \alpha$, $(b,c) \in \beta$ and $(a,d) \in \beta$, $(d,c) \in \alpha$. Now, using (i) we get
$(m(a,b,c), b) \in \alpha \wedge \beta$.

Because of symmetry, we obtain also:
$(m(b,c,d),c) \in \alpha \wedge \beta$,
$(m(c,d,a),d) \in \alpha \wedge \beta$,
$(m(d,a,b),a) \in \alpha \wedge \beta$.
Let $\delta \in$Tol$A$ with $\alpha \leq \delta$ and $\beta \leq \delta$. Then $(d,c) \in \delta$, $(b,c) \in \delta$, hence
$(m(d,a,b),c) = (m(d,a,b), m(c,a,c)) \in \delta$.
Together with $(a, m(d,a,b)) \in \alpha \wedge \beta$ it yields $(a,c) \in (\alpha \wedge \beta) \circ \delta$. Taking now $\delta = (\alpha \sqcup \beta)$, we obtain the required inclusion. □

Remark 4.2. *For $\alpha, \beta \in$ TolA with $\alpha \wedge \beta = \triangle$, Lemma 4.1* (ii) *gives $\alpha \sqcup \beta = (\alpha \circ \beta) \cap (\beta \circ \alpha)$ the relation which was proved in* [**7**], *Lemma 3.8.*

Lemma 4.3. *If A is an algebra with a majority term m, then A satisfies the SCHEME.*

Proof. Take $\alpha, \beta, \gamma \in$ Tol$A$ with $\alpha \wedge \beta = \triangle$ and $a, b, c \in A$ as it is shown in Figure 6. In view of Lemma 4.1(i) we have $(m(a,b,c),b) \in \alpha \wedge \beta = \triangle$, i.e. $m(a,b,c) = b$. Then we obtain

$$(b,c) = (m(a,b,c), m(c,b,c)) \in \gamma,$$

proving that A satisfies the SCHEME. □

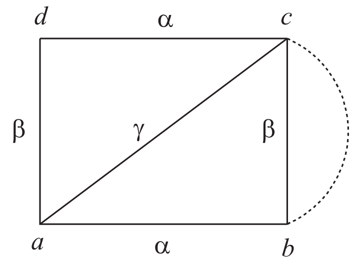

Figure 6

Theorem 4.4. *Let A be an algebra with TIP and satisfying the condition:*

$(*)$ $\quad \alpha \sqcup \beta = (\alpha \circ \beta) \cap (\beta \circ \alpha)$, *for all* $\alpha, \beta \in$ Tol$A$ *with* $\alpha \wedge \beta = \triangle$.

Then the following assertions are equivalent:
(i) *A satisfies the SCHEME.*
(ii) Tol$A$ *is super-0-distributive.*

Proof. (i)⇒(ii): Assume that A satisfies the SCHEME and take any $x, y \in A$ with $(x,y) \in (\alpha \sqcup \beta) \wedge \gamma \subseteq (\alpha \circ \beta) \cap (\beta \circ \alpha) \cap \gamma$. Then there exist $v, z \in A$ that are shown in Figure 5. Then, using SCHEME we obtain $(y,z) \in \beta \wedge \gamma$ and $(x,z) \in \alpha \wedge \gamma$. By symmetry we must have also $(x,v) \in \beta \wedge \gamma$ and $(v,y) \in \alpha \wedge \gamma$. Hence $(x,y) \in ((\alpha \wedge \gamma) \circ (\beta \wedge \gamma)) \cap ((\beta \wedge \gamma) \circ (\alpha \wedge \gamma))$.

As we have $(\alpha \wedge \gamma) \wedge (\beta \wedge \gamma) = \alpha \wedge \beta \wedge \gamma = \triangle$, by applying condition $(*)$ to tolerances $\alpha \wedge \gamma$ and $\beta \wedge \gamma$ we get $(x,y) \in (\alpha \wedge \gamma) \sqcup (\beta \wedge \gamma)$. Thus $(\alpha \sqcup \beta) \wedge \gamma \leq$

$(\alpha \wedge \gamma) \sqcup (\beta \wedge \gamma)$. Since the reversed inequality is obviously satisfied, we conclude that

$$(\alpha \sqcup \beta) \wedge \gamma = (\alpha \wedge \gamma) \sqcup (\beta \wedge \gamma).$$

(ii)⇒(i): Assume that Tol$A$ satisfies (ii) and take $\alpha, \beta, \gamma \in$Tol$A$ with $\alpha \wedge \beta = \triangle$ and $x, y, z, v \in A$ as it is shown in Figure 7.

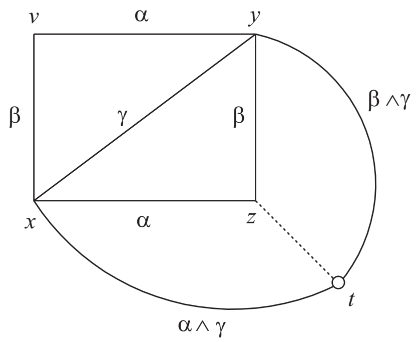

Figure 7

Then $(x, y) \in (\alpha \circ \beta) \cap (\beta \circ \alpha) \cap \gamma = (\alpha \sqcup \beta) \wedge \gamma = (\alpha \wedge \gamma) \sqcup (\beta \wedge \gamma) \subseteq (\alpha \wedge \gamma) \circ (\beta \wedge \gamma)$. Therefore, there exists an element $t \in A$ such that $(x, t) \in \alpha \wedge \gamma$ and $(t, y) \in \beta \wedge \gamma$. Then $(z, t) \in (\alpha \circ \alpha) \cap (\beta \circ \beta) \subseteq \overline{\alpha} \wedge \overline{\beta} = \overline{(\alpha \wedge \beta)} = \triangle$. Hence $z = t$, and this implies $(y, z) \in \beta \wedge \gamma \leq \gamma$, proving the SCHEME. □

Corollary 4.5. *If A is an algebra with a majority term, then* Tol$A$ *is super*-0-*distributive.*

Proof. Since the variety generated by an algebra with a majority term is congruence-distributive, A is an algebra with TIP (see [**7**]). In view of Remark 4.2 and Lemma 4.3, A satisfies the condition $(*)$ and the SCHEME, therefore we can apply Theorem 4.4 and this gives that Tol$A$ is super-0-distributive. □

Remark 4.6. Since lattices are algebras with majority term, their tolerance lattices are also super-0-distributive, i.e. for any lattice L and any $\alpha, \beta \in$ Tol$L$ with $\alpha \wedge \beta = \triangle$, (α, β) is a distributive pair in Tol$L$.

Corollary 4.7. *There exists a finite pseudocomplemented* 0-*modular lattice which is not isomorphic to the tolerance lattice of any algebra with a majority term.*

Proof. Let us consider the lattice L shown in Figure 8. Clearly, L is a pseudocomplemented lattice (we have $\alpha^* = (\alpha \wedge \gamma)^* = \beta$, $\beta^* = (\beta \wedge \gamma)^* = \alpha$, $0^* = 1$, and $x^* = 0$ for any other $x \in L$). It is easy to see that L satisfies (M_0). However for $\alpha \wedge \beta = 0$, we have $(\alpha \vee \beta) \wedge \gamma > (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$, hence L does not satisfy (SD_0).

As by Corollary 4.5, the tolerance lattice of any algebra with a majority term satisfies (SD_0), there is no such algebra A with Tol$A \cong L$. □

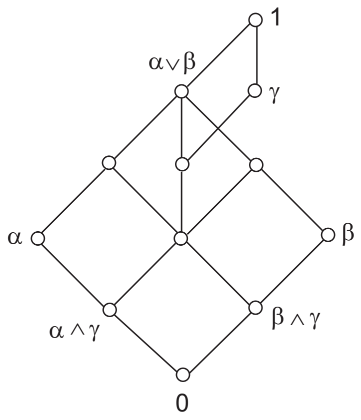

Figure 8

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