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CLASSICAL SOLUTIONS OF THE PERTURBED WAVE EQUATION WITH SINGULAR POTENTIAL

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ABSTRACT. This paper discusses the solutions to the perturbed wave equation containing a singular potential term in the Lorentzian metric. We present the classical solution to the problem using the separation of variables method for any dimension, n. Special solutions are obtained for even n's and properties of these solutions are discussed. Finally, we also consider the solution to the Cauchy problem for the case $n = 2$.

1. INTRODUCTION

The primary aim of this paper is to discuss the classical (in the sense of classical mechanics) structure of the solutions to the perturbed, massless, wave equation

$$\Box\phi + \frac{n(n+2)}{(1+x^2)^2}\phi = 0 \tag{1}$$

where $\Box$ represents the D'Alembertian operator, ϕ is a scalar field and $x = (t, \mathbf{x}) \in \mathbf{M}^n$, n dimensional Minkowski space. The interesting feature of this problem is the external potential term, $\frac{n(n+2)}{(1+x^2)^2}$ with the metric signature (-,+,+,+,...) which gives rise to a singularity as $t^2 \to 1 + x_1^2 + x_2^2 + ... + x_{n-1}^2$. The equation (1) emerges from a first order perturbation of the wave equation where the field ϕ can be thought of as a background fluctuation to the wave equation.

The existing literature on the subject of semilinear wave equations, abounds in abstract analytic studies of these equations. However, there is little in terms of concrete examples of equations with singularity. Our focus in this paper will be to study the specific semilinear equation (1) and analyze its properties. Furthermore, since the solution to the full problem is a rather difficult if not an impossible task, we try and understand the structure of the first order perturbation problem. We do realize that equation (1) provides perhaps a simpler case of a partial differential equation with a singular potential term where calculations can be made in a fairly rigorous manner. However, having said this, it must be pointed out that equation (1) is not arbitrarily chosen. The primary motivation for this study comes from

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our attempt to quantize the field, ϕ, which has not been dealt with here and will be the subject of a later, follow-up paper (see [**4**] for some preliminary results). The classical results, in particular, the Cauchy Problem, which is studied here, are essential to the quantization procedure [**1**]. We adopt the geometric scheme proposed by Irving Segal for the quantization of nonlinear hyperbolic equations [**1**]. Though a description of this program is outside the scope of this paper, it will suffice to say that it involves the construction of a distribution function $\mathcal{D}(x, x')$, such that it solves equation (1) along with the initial conditions $\mathcal{D}(x, x')|_{t=a} = 0$ and $\partial_t \mathcal{D}(x, x')|_{t=a} = \delta(x - b)$ where we write $x' = (a, b)$. This function $\mathcal{D}(x, x')$ directly allows us to establish a symplectic structure upon the underlying classical manifold which is a starting point for the quantization procedure. Though the ultimate objective of this study lies in the quantization, the solution and properties of the classical problem are interesting in themselves and, we feel, merit independent attention. Also note that though our problem is motivated primarily by the physics behind the equations, we choose to study the problem in arbitrary n dimension since there is considerable interest in the physics community in some dimensions greater than four.

In section 2 we discuss the origins of the perturbed wave equation. We argue here that the powers of the field variables emerge in such a way as to make the field conformally invariant. This invariance property is discussed in section 3. In the following section 4, we discuss the classical solution to equation (1) using a simple separation of variables argument and special properties of solutions are discussed in the case of even dimensions. Though a closed form solution to this problem can be obtained without difficulty, the properties of the equation in different dimensions are nontrivial. The main contribution of this paper lies in providing a solution to the central problem by a recursive formula, for any even dimension. This allows us to generate the solution to the perturbed problem in terms of the solution to the wave equation. The final section is devoted to the Cauchy problem for $n = 2$. The discussion here is restricted to presenting the final result of the initial value problem; details are omitted due to the tedious nature of the calculations. The more interesting, and perhaps complicated aspect of the problem, namely, the quantization of the field and its behavior in the neighborhood of the singularity is reserved for future study. At this stage, however, the paper remains more of mathematical interest. The above mentioned aspects of the problem must be looked at before we can make comments on the physics behind the equations.

2. The Perturbed Wave Equation

The perturbed wave equation in n dimensions can be obtained from the Lagrangian density function of the form

$$\mathcal{L} = \int (\frac{1}{2} g^{\mu\nu} \frac{\partial \psi}{\partial x^\mu} \frac{\partial \psi}{\partial x^\nu} - \frac{k(n-2)^2}{2} \psi^{\frac{2n}{n-2}}) \sqrt{\det(g)} d^n x \tag{2}$$

where $k \in \mathbf{R}^+$ and $g^{\mu\nu}$ is the metric tensor with the Lorentzian signature.

Applying the Euler-Lagrange condition to $\mathcal{L}$ yields the n-dimensional wave equation, namely

$$\Box\psi + kn(n-2)\psi^{\frac{n+2}{n-2}} = 0. \tag{3}$$

The power of ψ here is chosen so that the field equation remains *conformally invariant* (see section 3). Now, we choose λ to be a scalar and suppose that $\frac{\psi}{\lambda}$ also satisfies the wave equation

$$\Box(\frac{\psi}{\lambda}) + kn(n-2)(\frac{\psi}{\lambda})^{\frac{n+2}{n-2}} = 0. \tag{4}$$

Then, on multiplying by λ and choosing $\lambda = k^{\frac{n-2}{4}}$ equation (4) reduces to

$$\Box\psi + n(n-2)\psi^{\frac{n+2}{n-2}} = 0 \tag{5}$$

where we have managed to eliminate the constant k. It can be seen without difficulty that $\psi_0 = (1+x^2)^{\frac{2-n}{2}}$ solves equation (5).

If we now let $\psi = \psi_0 + \epsilon\phi$, then the first order perturbation about this solution yields

$$\Box\phi + \frac{n(n+2)}{(1+x^2)^2}\phi = 0 \tag{6}$$

which is the equation we shall attempt to solve in the following sections. Note that the dimensional dependence of the wave equation appears in the potential term only in the form $n(n+2)$.

It can be seen from the form of the Lagrangian that the problem is not well defined, as presented above, when $n = 2$. The situation can be remedied by making a simple transformation to the Lagrangian density function, by letting $\psi = \eta^{\frac{n-2}{n+2}}$ in equation (2) and dividing throughout by $(n-2)^2$. The variation of the resulting form of $\mathcal{L}$ for $n = 2$ becomes

$$\Box\chi + 8\frac{k}{a}e^{a\chi} = 0 \tag{7}$$

where we set $\eta = e^{a\chi}$. It is readily verified that the solution to equation (7) is given by $\chi_0 = -\frac{2}{a}\log(k + x^2)$. Therefore perturbing χ in the form $\chi = \chi_0 + \epsilon\phi$ yields at $o(\epsilon)$,

$$\Box\phi + \frac{8k}{(k+x^2)^2}\phi = 0 \tag{8}$$

which, as can be seen is identical to equation (1) for $n = 2$ with $k = 1$.

3. Conformal Invariance

In this section, we investigate the conformal invariance property of the equation (1) in curved space. It is a well established result that $\Box\psi + \frac{(n-2)\mathcal{R}}{4(n-1)}\psi$ is conformally invariant [**5**],[**6**], where $\mathcal{R}$ represents the curvature term. Therefore it suffices, in our problem, to show that the term $\psi^{\frac{2n}{n-2}}$ is also invariant under conformal transformations. A relatively simple argument for the invariance property of this term is presented. Consider the Lagrangian function (for $n \neq 1$) in curved space given by

$$\mathcal{L} = \int \left(\frac{1}{2}\partial^{\mu}\psi\partial_{\mu}\psi - \frac{k(n-2)^2}{2}\psi^{\frac{2n}{n-2}} - \frac{(n-2)\mathcal{R}}{8(n-1)}\psi^2 \right) \sqrt{\det(g)}d^n x. \tag{9}$$

If we let $g \to \Omega^2 g$ and $\psi \to \Omega^{\alpha}\psi$, where Ω is a smooth positive scalar field, then the relevant term

$$\begin{aligned} \psi^{\frac{2n}{n-2}}\sqrt{\det(g)} &\to \Omega^{\frac{2n\alpha}{n-2}}\psi^{\frac{2n}{n-2}}\Omega^n\sqrt{\det(g)} \\ &= \Omega^{\frac{2n\alpha}{n-2}+n}\psi^{\frac{2n}{n-2}}\sqrt{\det(g)} \\ &= \psi^{\frac{2n}{n-2}}\sqrt{\det(g)} \end{aligned}$$

for $\alpha = \frac{2-n}{2}$. Therefore for the appropriate choice of weight, α, the perturbed wave equation, originating from equation (9), is conformally invariant. Hence for $n = 4$, for instance, $\alpha = -1$.

4. Classical Solutions

4.1. Solution in n dimensions

Using separation of variables technique we can write

$$\phi = \Sigma_k f_k \left(-\frac{1}{R^2} \right) y_k(x) \tag{10}$$

where $\sqrt{x \cdot x} = R$ is the radial term and $y_k(x)$ (the harmonic term) is a polynomial homogeneous in x of degree k and which obeys the wave equation i.e. $\Box y_k(x) = 0$. Then y_k must obey the following equation,

$$t\partial_t y_k + x_1\partial_{x_1}y_k + x_2\partial_{x_2}y_k + ... + x_{n-1}\partial_{x_{n-1}}y_k = ky_k$$

for each k (see property iii of the homogeneous operator in section 4.2). We impose the physical boundary condition

$$lim_{R\to\infty} f_k(-\frac{1}{R^2}) = 0 \tag{11}$$

for each k. We also require that each term $f_k\left(-\frac{1}{R^2}\right) y_k(x)$ in the expression above solve the differential equation (1).

Substituting for ϕ in the perturbed equation we obtain the differential equation

$$4uf_k''(u) - (2n - 8 + 4k)f_k'(u) - \frac{n(n+2)}{(1-u)^2}f_k(u) = 0 \tag{12}$$

where the prime here is used to indicate the derivative with respect to the variable u. We then let $f_k(u) = (1-u)^{\frac{n+2}{2}}g_k(u)$. The result of this transformation is the differential equation,

$$g_k'' + \left(\frac{-n+4-2k}{2u(1-u)} - \frac{n+8-2k}{2(1-u)} \right) g_k' - \frac{(n+2)(2-k)}{2u(1-u)}g_k = 0 \tag{13}$$

after factoring out $(1-u)^{1+\frac{n}{2}}$. We recognize this to be *Gauss' differential equation* [**2**]

$$y'' + \left(\frac{c}{z(1-z)} - \frac{1+a+b}{(1-z)}\right) y' - \frac{ab}{z(1-z)} y = 0, \tag{14}$$

where $a = 2-k$, $b = \frac{n+2}{2}$, $c = 2-k-\frac{n}{2}$. Gauss' differential equation is known to have twenty-four different solution on the real line. These are also referred to as Kummer's solutions and arise from various transformations of equation (14). In attempting to provide an integral representation of the solution, we note that [**2**]

$$g_{12}(u) \quad = \quad (-u)^{\frac{n}{2}}(1-u)^{-1-n} F[-\frac{n}{2}, k-1, k+\frac{n}{2}, \frac{1}{u}] \tag{15}$$

provides the appropriate solution to the equation (1). Hence retracing our steps we have

$$\phi = \int_0^1 p(tx) \frac{(1-t)^{\frac{n}{2}}}{t^2} (t + (\frac{t-1}{1+x^2}))^{\frac{n}{2}} dt, \tag{16}$$

where $p(x) = \sum_k y_k(x)$. That ϕ is indeed a solution to the original equation (1) is verified in [**4**].

4.2. Solution in Even Dimensions

We observe here that the problem simplifies and reveals more interesting features when n is even. The special results we obtain for the case of even dimensions is based on recognition of the fact that the solution to the perturbed equation can be written in the form $\phi = \sum_{r=0}^{\frac{n}{2}} P_r(x) \frac{1}{(1+x^2)^r}$. This is proved in the Lemma below.

Lemma 1. *The solution to equation (1) can be written in the form*

$$\phi = \sum_{r=0}^{\frac{n}{2}} P_r(x) \frac{1}{(1+x^2)^r} \tag{17}$$

where $\Box P_r(x) = 0$ *for every* $r \in [0, \frac{n}{2}]$.

Proof. The solution to the perturbed equation is given by

$$\phi = \int_0^1 p(tx) \alpha^{\frac{n}{2}} \frac{(1-t)^{\frac{n}{2}}}{t^2} dt \tag{18}$$

where $p(tx) = \sum_k p_k(tx)$, $\alpha = (t + \frac{1-t}{1+x^2})$. Hence for n even, α is a polynomial. So using the Binomial Series expansion we have

$$\alpha^{\frac{n}{2}} = t^{\frac{n}{2}} (1 + \frac{1-t}{t(1+x^2)})^{\frac{n}{2}} = t^{\frac{n}{2}} \sum_{r=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{r} \frac{(1-t)^r}{t^r (1+x^2)^r} \tag{19}$$

where $\binom{\frac{n}{2}}{r} = \frac{\frac{n}{2}!}{r!(\frac{n}{2}-r)!}$, the binomial coefficient. Therefore

$$\phi \quad = \quad \sum_{r=0}^{\frac{n}{2}} \left(\binom{\frac{n}{2}}{r} \int_0^1 p(tx)(1-t)^{\frac{n}{2}+r} t^{\frac{n}{2}-2-r} dt \right) \frac{1}{(1+x^2)^r} = \sum_{r=0}^{\frac{n}{2}} P_r(x) \frac{1}{(1+x^2)^r}$$

where $p(tx)$ satisfies the wave equation and therefore, so does P_r. $\Box$

Therefore, for $n = 2$, $\phi = P_0 + P_1\rho$, where $\rho = \frac{1}{(1+x^2)}$, and for $n = 4$, $\phi = P_0 + P_1\rho + P_2\rho^2$ etc. This representation is also valid for odd n, although in that case the above representation yields an infinite series. This is not conducive to the kind of calculations that we perform below. In addition, even n suffices for our purposes since our ultimate interest lies in the physically relevant dimension $n = 4$.

We will now show that the equation (1) can be solved directly by taking equation (17) as an ansatz and where the coefficients P_r's are, in fact, not independent but determined by a certain recursion relation and in addition, also solve the wave equation. For sake of convenience we shall define the homogeneous operator $H := x \cdot \nabla$ which has the following well known properties [**3**]:
(i) $Hp_n(x) = np_n(x)$
(ii) The commutation relation, $[\Box, H] = 2\Box$, from which it follows that
(iii) $\Box H y(x) = (H + 2)\Box y(x)$.

Theorem 1. *The function P_r introduced in equation (17) obeys the following recursion relation:*

$$P_r = \frac{2(r+1)[2H + (n-2r-4)]P_{r+1}}{(n-2r)(n+2r+2)} \tag{20}$$

Proof. Let us take ϕ as in equation (17). Then applying the box operator to both sides of the equation, with the assumption that $\Box P_r = 0$, yields

$$\begin{aligned}\Box\phi &= \sum_{r=0}^{\frac{n}{2}}[P_r\,(\Box\rho^r) + 2\nabla P_r \cdot \nabla\rho^r] \\ &= \sum_{r=0}^{\frac{n}{2}}[((4r^2 + 4r - 2nr)P_r - 4rHP_r)\rho^{r+1} - (4r^2 + 4r)P_r\rho^{r+2}].\end{aligned}$$

Hence the wave equation becomes

$$\begin{aligned}0 &= \sum_{r=0}^{\frac{n}{2}}[((4r^2 + 4r - 2nr)P_r - 4rHP_r)\rho^{r+1} - (4r^2 + 4r - n^2 - 2n)P_r\rho^{r+2}] \\ &= \sum_{0}^{\frac{n}{2}-1}[2(r+1)(2r + 4 - n - 2H)P_{r+1} - (4r^2 + 4r - n^2 - 2n)P_r]\rho^{r+2}.\end{aligned}$$

So it is sufficient that we have

$$\begin{aligned}0 &= 2(r+1)[(2r + 4 - n - 2H)P_{r+1} - (4r^2 + 4r - n^2 - 2n)P_r] \\ \Rightarrow & \frac{(n-2r)(n+2r+2)}{2(r+1)}P_r = [2H + n - 2r - 4]P_{r+1}\end{aligned}$$

from which the Theorem follows. □

We will now verify by an induction argument that the equation (20) is consistent with the assumption that P_r satisfies the wave equation.

Proposition 1. *If $P_{\frac{n}{2}}$ obeys $\Box P_{\frac{n}{2}} = 0$ then $\Box P_r = 0$ for any $0 \le r \le \frac{n}{2}$.*

Proof. We prove the proposition by a reverse induction argument. It can be seen by concrete calculations that $\Box P_1 = 0, \Box P_2 = 0$, etc. Let us assume that $\Box P_{\frac{n}{2}} = 0$. Then it can easily be shown that all P_r's satisfy $\Box P_r = 0$. Starting from the above theorem, we write

$$P_r = a_1[a_2 H + a_3]P_{r+1}$$

where $a_1 = \frac{2(r+1)}{(n-2r)(n+2r+2)}$, $a_2 = 2$ and $a_3 = n - 2r - 4$. Therefore applying the box operator to both sides of the equation gives us

$$\begin{aligned} \Box P_r &= a_1 \Box [a_2 H + a_3] P_{r+1} \\ &= a_1 a_2 \Box H P_{r+1} + a_1 a_3 \Box P_{r+1} \\ &= [a_1 a_2 (H+2) + a_1 a_3] \Box P_{r+1} \end{aligned}$$

where we have employed the second property of the homogeneous operator in the last equality. Using this last recursion relation in r, it is easily seen that $\Box P_r$ must be zero for every $0 \le r \le \frac{n}{2}$. $\Box$

Table 1 provides examples, in different dimensions, of the recursion relationship obtained in equation (20).

Dimension, n	**Relations**			
2	$P_0 = \frac{(H-1)P_1}{2}$			
4	$P_0 = \frac{HP_1}{6}$	$P_1 = \frac{(H-1)P_2}{2}$		
6	$P_0 = \frac{(H+1)P_1}{12}$	$P_1 = \frac{HP_2}{5}$	$P_2 = \frac{(H-1)P_3}{2}$	
8	$P_0 = \frac{(H+2)P_1}{20}$	$P_1 = \frac{(H+1)P_2}{9}$	$P_2 = \frac{3HP_3}{14}$	$P_3 = \frac{(H-1)P_4}{2}$

Table 1. The recursion relations for P_r.

4.3. Inverse Relations

To establish completeness, we must, in addition to the result of the previous section, also show that the mapping $\mathcal{F} : P_{\frac{n}{2}} \to \phi$ is surjective. This can be affirmed by checking to see if we can invert the expression in equation (17) above i.e. if we can write each of the P_r's as a function of ϕ. In this section we shall discuss this inverse relationship for certain specific dimensions. Table 2 summarizes the findings. At the outset, it must be stated that calculations pertaining to inversion become extremely complex with increasing n. For this purpose results are shown only up to $n = 6$, although, in principle, we can do so for any even dimension. We shall present our calculations below for dimensions $n = 2$ and $n = 4$. However, unlike in section 4.2, we are unable to obtain a general recursion formula in this case.

1. Case $n = 2$

Lemma 2. *The operator* $(H+1)^{-1}$ *exists.*

Proof. We define

$$\phi_k = \int_0^1 t^k \phi(tx) dt$$

where $k \geq 0$ ensures convergence of the integral. Then,

$$H\phi_k = \int_0^1 t^k x \cdot \nabla(\phi(tx)) dt = \int_0^1 t^{k+1} \frac{d}{dt}(\phi(tx)) dt. \tag{21}$$

Now integrating by parts we have

$$\begin{aligned} H\phi_k &= \int_0^1 \frac{d}{dt}\left[t^{k+1}\phi(tx)\right] dt - \int_0^1 (k+1) t^k \phi(tx) dt \\ &= \int_0^1 \frac{d}{dt}\left[t^{k+1}\phi(tx)\right] dt - (k+1)\phi_k \\ \Rightarrow (H+k+1)\phi_k &= \phi(x) \\ \Rightarrow \phi_k &= (H+k+1)^{-1}\phi(x). \end{aligned}$$

The Lemma follows, since ϕ is well defined and from the definition of ϕ_k (by setting $k = 0$ in our case). $\square$

Proposition 2. P_0 *and* P_1 *are invertible and are given by*

$$\begin{aligned} P_0 &= \phi + (H+1)^{-1}(-2\rho\phi) \\ P_1 &= \rho^{-1}(H+1)^{-1}(2\rho\phi) \end{aligned}$$

Proof. We write the equation $\phi = P_0 + \rho P_1$ in the form

$$\rho^{-1}\phi = \rho^{-1}P_0 + P_1. \tag{22}$$

Applying the box operator to both sides of the above equation yields

$$\Box\left(\rho^{-1}\phi\right) = \Box\left(\rho^{-1}P_0 + P_1\right) = \Box\left(\rho^{-1}P_0\right)$$

Carrying the box operator through we have

$$\begin{aligned} LHS &= \Box(\rho^{-1})\phi + \rho^{-1}\Box(\phi) + 2\nabla\rho^{-1}\nabla\phi = \rho^{-1}\Box\phi + 4(H+1)\phi \\ RHS &= \Box(\rho^{-1})P_0 + \rho^{-1}\Box P_0 + 2\nabla\rho^{-1}\nabla P_0 = 4(H+1)P_0. \end{aligned}$$

Hence equating the two sides and simplifying we get

$$P_0 = \phi + (H+1)^{-1}(-2\rho\phi) \tag{23}$$

and hence it follows that

$$P_1 = \rho^{-1}(\phi - P_0) = \rho^{-1}(H+1)^{-1}(2\rho\phi).$$

$\square$

2. Case $n = 4$

We follow the same line of arguments as above for this case too.

Lemma 3. *The operator* $(H+2)^{-1}(H+3)^{-1}$ *exists.*

Proof. Let us define

$$\begin{aligned}\phi_k &= \int_0^1 t^k \phi(tx)dt\\ \phi_{k,m} &= \int_0^1\int_0^1 t^k s^m\left(\phi(stx)\right)dtds.\end{aligned}$$

Then

$$H\phi_{k,m} = \int_0^1\int_0^1 t^k s^m H(\phi(stx))dtds = \int_0^1\int_0^1 t^{k+1}s^m\frac{d}{dt}(\phi(stx))dtds$$

using equation (21). Now integrating the right hand side of the last integral by parts we have

$$\begin{aligned}H\phi_{k,m} &= \int_0^1 s^m\phi(sx)ds - (k+1)\int_0^1\int_0^1 t^k s^m\left(\phi(stx)\right)dtds\\ &= \phi_m - (k+1)\phi_{k,m}\end{aligned}$$

by the definition above. Therefore on simplification

$$\begin{aligned}(H+k+1)\phi_{k,m} = \phi_m\\ \Rightarrow (H+m+1)(H+k+1)\phi_{k,m} &= (H+m+1)\phi_m = \phi(x).\end{aligned}$$

The last equality follows from Lemma 2. Note that ϕ is well defined and using the definitions of ϕ_k, $\phi_{k,m}$ above it is clearly seen that the operator $(H+2)^{-1}(H+3)^{-1}$ exists (with $k=1$ and $m=2$). From the pattern of the two Lemma's above, it can be inferred that the Lemma can be generalized for any higher (even) dimension. □

Proposition 3. *P_0, P_1 and P_2 are invertible and are given by*

$$\begin{aligned}P_0 &= (H+2)^{-1}(H+3)^{-1}[(H+2)(H+3)\phi - 6\rho(H+1)\phi]\\ P_1 &= (4H+8)^{-1}[\rho^{-1}8(H+3)\phi - 32\phi - 8\rho^{-1}(H+3)P_0 + 8P_0]\\ P_2 &= \rho^{-2}\phi - \rho^{-2}(\phi - 6(H+2)^{-1}(H+3)^{-1}(\rho(H+1)\phi))\\ &- \rho^{-1}(4H+8)^{-1}[\rho^{-1}8(H+3)\phi - 32\phi - 8\rho^{-1}(H+3)P_0 + 8P_0)]\end{aligned}$$

Proof. We have

$$\rho^{-2}\phi = \rho^{-2}P_0 + \rho^{-2}P_1 + P_2. \tag{24}$$

So

$$\Box(\rho^{-2}\phi) = \Box(\rho^{-2}P_0 + \rho^{-2}P_1 + P_2) = \Box(\rho^{-2}P_0) + \Box(\rho^{-2}P_1).$$

Simplifying the left and right hand sides of the last equation above separately,

$$\begin{aligned}LHS &= (\Box\rho^{-2})\phi + \rho^{-2}\Box\phi + 2\nabla\rho^{-2}\nabla\phi = \rho^{-1}(8H+24)\phi - 32\phi\\ RHS &= (\Box\rho^{-2})P_0 + (\Box\rho^{-1})P_1 + 2\nabla\rho^{-2}\nabla P_0 + 2\nabla\rho^{-2}\nabla P_1\\ &= \rho^{-1}(8H+24)P_0 + (4H+8)P_1 - 8P_0.\end{aligned}$$

Therefore

$$\rho^{-1}(8H+24)\phi - 32\phi = \rho^{-1}(8H+24)P_0 + (4H+8)P_1 - 8P_0. \tag{25}$$

We now apply the box operator a second time on both sides to get

$$\begin{aligned}
\Box^2(\rho^{-2}\phi) &= \Box(\rho^{-1}(8H+24)\phi - 32\phi) \\
&= 24(\Box\rho^{-1})\phi + 24\rho^{-1}\Box\phi + 48\nabla\rho^{-1}\nabla\phi \\
&= 32(H+3)(H+2)\phi - 192\rho(H+1)\phi \\
\Box^2(\rho^{-2}P_0 + \rho^{-2}P_1 + P_2) &= \Box(\rho^{-1}(8H+24)P_0 + (4H+8)P_1 - 8P_0) \\
&= 24(\Box\rho^{-1})P_0 + 48\nabla\rho^{-1}\nabla P_0 + 8(\Box\rho^{-1})HP_0 \\
&+ 16\nabla\rho^{-1}\nabla(HP_0) = (32H^2 + 160H + 192)P_0.
\end{aligned}$$

So equating the left and right sides above we have

$$P_0 = (H+2)^{-1}(H+3)^{-1}[(H+2)(H+3)\phi - 6\rho(H+1)\phi] \tag{26}$$

and using equation (25) we obtain

$$P_1 = (4H+8)^{-1}(\rho^{-1}(8H+24)\phi - 32\phi - \rho^{-1}(8H+24)P_0 + 8P_0) \tag{27}$$

with P_0 given as above. Similarly P_2 can be obtained in terms of ϕ, P_0 and P_1 from equation (24). □

Dimension, n	Relations
2	$P_0 = \phi + (H+1)^{-1}(-2\rho\phi)$
	$P_1 = \rho^{-1}(H+1)^{-1}(2\rho\phi)$
4	$P_0 = (H+2)^{-1}(H+3)^{-1}[(H+2)(H+3)\phi - 6\rho(H+1)\phi]$
	$P_1 = (4H+8)^{-1}[\rho^{-1}8(H+3)\phi - 32\phi - 8\rho^{-1}(H+3)P_0 + 8P_0]$
	$P_2 = \rho^{-2}\phi - \rho^{-2}P_0 - \rho^{-1}P_1$
6	$P_0 = \phi + (H+4)^{-1}(H+3)^{-1}(H+2)^{-1}[12\rho(H+1)(H+2)\phi$ $+12\rho^2(H-2)\phi + 24\rho^3\phi]$
	$P_1 = 18(H+2)^{-1}(H+3)^{-1}P_0 + (H+2)^{-1}(H+3)^{-1}$ $[-3\rho^{-1}(H+3)(H+4)P_0 + 3\rho^{-1}(H+3)(H+4)\phi$ $-6(5H+11)\phi + 36\rho\phi]$
	$P_2 = (H+2)^{-1}[2P_1 + 6\rho^{-1}P_0 - 2\rho^{-1}(H+3)P_1 - 3\rho^{-2}(H+4)P_0$ $+3\rho^{-2}(H+4)\phi - 18\rho^{-1}\phi]$
	$P_3 = \rho^{-3}\phi - \rho^{-1}P_2 - \rho^{-2}P_1 - \rho^{-3}P_0$

Table 2. Inverse Relations for P_r for different values of n.

5. Cauchy Problem

In this section we discuss the solution to the Initial Value Problem for equation (1). The primary motivation for this attempt comes from Segal's outline for quantization [**1**] which first requires us to obtain a fundamental solution to the differential equation. However, the Cauchy problem is an interesting result in itself and an important aspect of the classical discussion. Also, the discussion is restricted to the case $n = 2$ since the computations can be quite complex in higher dimensions. However the technique used to solve the problem for $n = 2$ can be mimicked for higher dimensions. It must also be stated that we solved the problem by analytic continuation into the Euclidean space [**4**]. The solution to the initial value problem for equation (1) in two dimensions is given by

$$\begin{aligned}
\phi &= \frac{\phi(x-(t-a)) + \phi(x+(t-a))}{2} \\
&- 2\int_{x+(t-a)}^{x-(t-a)} \frac{(t(1+a^2+w^2) + a(x^2-2wx-t^2-1))}{(1+x^2-t^2)(1-a^2+w^2)^2}\phi(w)dw \\
&- \frac{1}{2}\int_{x+(t-a)}^{x-(t-a)} \frac{(1-x^2+t^2)(1+a^2-w^2) - 4at + 4wx}{(1+x^2-t^2)(1-a^2+w^2)}\varphi(w)dw \qquad (28)
\end{aligned}$$

where $\phi(w)$ and $\varphi(w)$ refer to the initial conditions, $\phi|_{t=a} = \phi(w)$ and $\partial_t\phi|_{t=a} = \varphi(w)$. It is easily verified that ϕ, as given above satisfies the perturbed wave equation. The reader is referred to [**4**] for the proof. Notice that the expression for ϕ above brings out the causal structure of the function, which is evaluated between the points $x - (t - a)$ and $x + (t - a)$.

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