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COMPACTIFICATIONS OF FRACTAL STRUCTURES

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Abstract. In this paper we introduce GF-compactifications, which are compactifications of GF-spaces (a new notion introduced by the authors). We study properties of this new kind of compactification and prove that every GF-compactification is of Wallman type. We also prove that every metrizable compactification of a metric space X is a GF-compactification and, as a corollary, that every metric compactification is of Wallman type, giving a new proof of a result that dates back to Aarts. Finally, we prove some extension theorems.

1. Introduction

Wallman compactification are the most successful attempt to obtain the whole lattice of compactifications of any topological space by means of a unique definition. In fact, until [**18**] was published, it was believed that every compactification was of Wallman type and it was known since [**1**] that every metric compactification is of Wallman type.

In this paper we introduce GF-compactifications, which are compactifications of GF-spaces (introduced in [**4**] by the authors). They are an analogue for GF-spaces of the spirit underlying the definition of Wallman compactification. In fact, they are nothing but half-completions of totally bounded point-symmetric non-archimedean quasimetrics. We are going to study properties of this compactifications (for example their relations with subspaces and products), and we are going to prove that every metrizable compactification of a topological space X, can be obtained as the GF-compactification of X, for some fractal structure over X. Afterwards, we are going to prove some extension theorems for mappings between GF-spaces. Finally, we are going to see how to get GF-compactifications by means of Wallman compactifications (associated with a lattice constructed from the fractal structure), obtaining as a corollary a classical result that dates back to Aarts in the sixties.

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2. GF-spaces

Now, we recall from [**4**] some definitions and introduce some notations that will be useful in this paper.

Let $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a countable family of coverings. Recall that $\mathrm{St}(x, \Gamma_n) = \bigcup_{x \in A_n, A_n \in \Gamma_n} A_n$; we also define $U^{\mathbf{\Gamma}}_{xn} = \mathrm{St}(x, \Gamma_n) \setminus \bigcup_{x \notin A_n, A_n \in \Gamma_n} A_n$ which will be noted also by U_{xn} if there is no doubt about the family. We also denote by $\mathrm{St}(x, \mathbf{\Gamma}) = \{\mathrm{St}(x, \Gamma_n) : n \in \mathbb{N}\}$ and $\mathcal{U}_x = \{U_{xn} : n \in \mathbb{N}\}$.

A (base $\mathcal{B}$ of a) quasiuniformity $\mathcal{U}$ on a set X is a (base $\mathcal{B}$ of a) filter $\mathcal{U}$ of binary relations (called entourages) on X such that (a) each element of $\mathcal{U}$ contains the diagonal Δ_X of $X \times X$ and (b) for any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ satisfying $V \circ V \subseteq U$. A base $\mathcal{B}$ of a quasiuniformity is called transitive if $B \circ B = B$ for all $B \in \mathcal{B}$. The theory of quasiuniform spaces is covered in [**12**].

If $\mathcal{U}$ is a quasiuniformity on X, then so is $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$, where $U^{-1} = \{(y, x) : (x, y) \in U\}$. The generated uniformity on X is denoted by $\mathcal{U}^*$. A base is given by the entourages $U^* = U \cap U^{-1}$. The topology $\tau(\mathcal{U})$ induced by the quasiuniformity $\mathcal{U}$ is that in which the sets $U(x) = \{y \in X : (x, y) \in U\}$, where $U \in \mathcal{U}$, form a neighbourhood base for each $x \in X$. There is also the topology $\tau(\mathcal{U}^{-1})$ induced by the inverse quasiuniformity. In this paper, we consider only spaces where $\tau(\mathcal{U})$ is T_0.

A quasipseudometric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:(i) $d(x, x) = 0$, and (ii) $d(x, y) \leq d(x, z) + d(z, y)$. If in addition d satisfies the condition (iii) $d(x, y) = 0$ iff $x = y$, then d is called a quasimetric. A non-archimedean quasipseudometric is a quasipseudometric that verifies $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

Each quasipseudometric d on X generates a quasiuniformity $\mathcal{U}_d$ on X which has as a base the family of sets of the form $\{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$, $n \in \mathbb{N}$. Then the topology $\tau(\mathcal{U}_d)$ induced by $\mathcal{U}_d$, will be denoted simply by $\tau(d)$.

A space (X, τ) is said to be (non-archimedeanly) quasipseudometrizable if there is a (non-archimedean) quasipseudometric d on X such that $\tau = \tau(d)$.

A relation $\leq$ on a set G is called a partial order on G if it is a transitive antisymmetric reflexive relation on G. If $\leq$ is a partial order on a set G, then $(G, \leq)$ is called a partially ordered set.

$(G, \leq, \tau)$ will be called a poset (partially ordered set) or T_0-Alexandroff space if $(G, \leq)$ is a partially ordered set and τ is that in which the sets $[g, \rightarrow[= \{h \in G : g \leq h\}$ form a neighborhood base for each $g \in G$ (we say that the topology τ is induced by $\leq$). Note that then $\overline{\{g\}} =]\leftarrow, g]$ for all $g \in G$.

Let us remark that a map $f : G \to H$ between two posets G and H is continuous if and only if it is order preserving, i.e. $g_1 \leq g_2$ implies $f(g_1) \leq f(g_2)$.

Let Γ be a covering of X. Γ is said to be locally finite if for all $x \in X$ there exists a neighborhood of x which meets only a finite number of elements of Γ. Γ is said to be a tiling, if all elements of Γ are regularly closed and they have disjoint interiors (see [**3**]). We say that Γ is quasi-disjoint if $A^\circ \cap B = \emptyset$ or $A \cap B^\circ = \emptyset$ holds for all $A \neq B \in \Gamma$. Note that if Γ is a tiling, then it is quasi-disjoint.

Definition 2.1. Let X be a topological space. A pre-fractal structure over X is a family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that $\mathcal{U}_x$ is an open neighborhood base of x for all $x \in X$.

Furthermore, if Γ_n is a closed covering and for all n, Γ_{n+1} is a refinement of Γ_n, such that for all $x \in A_n$, with $A_n \in \Gamma_n$, there is $A_{n+1} \in \Gamma_{n+1} : x \in A_{n+1} \subseteq A_n$, we will say that $\mathbf{\Gamma}$ is a fractal structure over X.

If $\mathbf{\Gamma}$ is a (pre-) fractal structure over X, we will say that $(X, \mathbf{\Gamma})$ is a generalized (pre-) fractal space or simply a (pre-) GF-space. If there is no doubt about $\mathbf{\Gamma}$, then we will say that X is a (pre-) GF-space.

If $\mathbf{\Gamma}$ is a fractal structure over X, and $\mathrm{St}(x, \mathbf{\Gamma})$ is a neighborhood base of x for all $x \in X$, we will call $(X, \mathbf{\Gamma})$ a starbase GF-space.

If Γ_n has the property P for all $n \in \mathbb{N}$, and $\mathbf{\Gamma}$ is a fractal structure over X, we will say that $\mathbf{\Gamma}$ is a fractal structure over X with the property P, and that X is a GF-space with the property P. For example, if Γ_n is locally finite for every natural number n, and $\mathbf{\Gamma}$ is a fractal structure over X, we will say that $\mathbf{\Gamma}$ is a locally finite fractal structure over X, and that $(X, \mathbf{\Gamma})$ is a locally finite GF-space.

Call $U_n = \{(x,y) \in X \times X : y \in U_{xn}\}$, $U_{xn}^{-1} = U_n^{-1}(x)$ and $\mathcal{U}_x^{-1} = \{U_{xn}^{-1} : n \in \mathbb{N}\}$.

It is proved in [**4**, Prop. 3.2] that $U_{xn}^{-1} = \bigcap_{x \in A_n} A_n$ for all $x \in X$ and all $n \in \mathbb{N}$.

Let $(X, \mathbf{\Gamma})$ be a (pre-)GF-space and A be a subspace of X. The fractal structure induced to A (see [**4**]) is defined as $(A, \mathbf{\Gamma}_A)$, where $\mathbf{\Gamma}_A = \{\Gamma'_n : n \in \mathbb{N}\}$ and $\Gamma'_n = \{A_n \cap A : A_n \in \Gamma_n\}$.

Let $(X_i, \mathbf{\Gamma}^i)$ be a countable family of (pre-) GF-spaces. The fractal structure induced to the countable product (see [**4**]) is defined as $(\prod_{i\in\mathbb{N}} X_i, \prod_{i\in\mathbb{N}} \mathbf{\Gamma}^i)$, where $\prod_{i\in\mathbb{N}} \mathbf{\Gamma}^i = \{\Gamma_n : n \in \mathbb{N}\}$ and $\Gamma_n = \{\bigcap_{i\le n} p_i^{-1}(A_n^i) : A_n^i \in \Gamma_n^i\}$ (where p_i is the projection from the product space to X_i).

In [**4**], the authors introduced the following construction. Let $\mathbf{\Gamma}$ be a fractal structure, and let define $G_n = \{U_{xn}^* : x \in X\}$, and define in G_n the following order relation $U_{xn}^* \le_n U_{yn}^*$ if $y \in U_{xn}$. It holds that G_n is a poset with this order relation and its associated topology.

Let ρ_n be the quotient map from X onto G_n which carries x in X to $U_n^*(x)$ in G_n. It also holds that ρ_n is continuous.

We also consider the map $\phi_n : \mathrm{G}_n \longrightarrow \mathrm{G}_{n-1}$ defined by $\phi_n(\rho_n(x)) = \rho_{n-1}(x)$. It holds that ϕ_n is continuous.

Let ρ be the map from X to $\varprojlim G_n$ which carries x in X to $(\rho_n(x))_n$ in $\varprojlim G_n$. Note that ρ is well defined and continuous (by definition of ϕ_n and the continuity of ρ_n and ϕ_n for all n). It holds that ρ is an embedding of X into $\varprojlim G_n$. We shall identify X with $\rho(X)$ whenever we need it.

Definition 2.2. Let $(X, \mathbf{\Gamma})$ be a GF-space, and $G_n = G_n(\mathbf{\Gamma})$. We define $G(X) = G(X, \mathbf{\Gamma})$ as the subset of all closed points of $\varprojlim G_n$.

Note that $G(X)$ is also the set of minimal points of $\varprojlim G_n$, with the order $g \le h$ if and only if $g_n \le_n h_n$ for all $n \in \mathbb{N}$ and that $G(X)$ is T_1.

The fractal structure, noted by $G(\mathbf{\Gamma})$, associated with $\varprojlim G_n$ is defined as $G(\Gamma_n) = \{A_n(g_n) : g_n \in G_n\}$, where for each $g_n \in G_n$, we define $A_n(g_n) = \{h \in \varprojlim G_n : h_n \leq_n g_n\}$.

Associated with each fractal structure $\mathbf{\Gamma}$, we can construct (see [**4**]) a non-archimedean quasipseudometric $d_{\mathbf{\Gamma}}$ defined by $2^{-(n+1)}$ if $y \in U_{xn} \setminus U_{x(n+1)}$, by 1 if $y \notin U_{x1}$ and by 0 if $y \in U_{xn}$ for all $n \in \mathbb{N}$. It holds that $U_{xn} = B(x, \frac{1}{2^n})$. If d is a non-archimedean quasi-pseudometric and we define $\Gamma_n = \{B_{d^{-1}}(x, \frac{1}{2^n}) : x \in X\}$, then $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ is a fractal structure, which we call fractal structure associated with d. If $\mathbf{\Gamma}$ is a fractal structure, it follows that $G(\mathbf{\Gamma})$ coincides with the fractal structure associated with $d_{\mathbf{\Gamma}}$.

3. GF-maps

In this section we study the concept of GF-map and GF-isomorphism (whose equivalent for quasi-pseudometrics is the concept of surjective isometry, and whose equivalent for inverse limit of a sequence of posets is the concept of map between inverse limits), and we also study a weaker concept, which will correspond to the concept of quasi-uniformly continuous map.

Definition 3.1. Let $(X, \mathbf{\Gamma})$ and $(Y, \mathbf{\Delta})$ be GF-spaces, and let $f\colon X \to Y$ be a map such that $f(U_{xn}) \subseteq U_{f(x)n}$ for all $x \in X$ and for all $n \in \mathbb{N}$. We call that map a GF-map.

If there exists a bijective GF-map between X and Y, such that the inverse is also a GF-map, we say that X and Y are GF-isomorphic.

The above definition can be rephrased as follows.

Proposition 3.2. *Let $(X, \mathbf{\Gamma})$ and $(Y, \mathbf{\Delta})$ be GF-spaces, and let $f\colon X \to Y$ be a map. Then the following statements are equivalent:*

1. *f is a GF-map.*
2. *f is nonexpansive, that is, $d_{\mathbf{\Gamma}}(f(x), f(y)) \leq d_{\mathbf{\Delta}}(x, y)$ for all $x, y \in X$.*
3. *$\rho_n \circ f = f_n \circ \rho_n$, where $f_n : G_n(X) \to G_n(Y)$ is an order-preserving map, for all $n \in \mathbb{N}$.*

Proof. The equivalence between 1) and 2) is clear.

1) implies 3). It is clear that if f is a GF-map, then $f(U^*_{xn}) \subseteq U^*_{f(x)n}$. Define $f_n : G_n(X) \to G_n(Y)$ by $f_n(\rho_n(x)) = \rho_n(f(x))$. It is clear from what we have proved that f_n is well defined, and since f is a GF-map, if $y \in U_{xn}$ then $f(y) \in U_{f(x)n}$, and then f_n is order-preserving.

3) implies 1). Let $y \in U_{xn}$, then $\rho_n(x) \leq_n \rho_n(y)$, and since f_n is order-preserving then $f_n(\rho_n(x)) \leq f_n(\rho_n(y))$, and since $\rho_n \circ f = f_n \circ \rho_n$ then $\rho_n(f(x)) \leq_n \rho_n(f(y))$ and then $f(y) \in U_{f(x)n}$, what proves that f is a GF-map. □

A sufficient condition to get a GF-map is the following.

Proposition 3.3. *Let $(X, \mathbf{\Gamma})$ and $(Y, \mathbf{\Delta})$ be GF-spaces, and let $f\colon X \to Y$ be a map. Suppose that for all $x \in X$ and all $B_n \in \Delta_n$ such that $f(x) \in B_n$ there exists $A_n \in \Gamma_n$ such that $x \in A_n$ and $f(A_n) \subseteq B_n$. Then f is a GF-map.*

Proof. Let $y \in U_{xn}^{-1}$, and let $B_n \in \Delta_n$ be such that $f(x) \in B_n$, then there exists $A_n \in \Gamma_n$ such that $x \in A_n$ and $f(A_n) \subseteq B_n$. Since $y \in U_{xn}^{-1} = \bigcap_{x \in C_n} C_n$, then $y \in A_n$ and hence $f(y) \in f(A_n) \subseteq B_n$. Then $f(y) \in \bigcap_{f(x) \in B_n} B_n = U_{f(x)n}^{-1}$, and therefore $f(U_{xn}) \subseteq U_{f(x)n}$, so f is a GF-map. □

We recall the following notion.

Definition 3.4. Let $(X, \mathbf{\Gamma})$ be a GF-space. We say that $(X, \mathbf{\Gamma})$ is half-complete if $d_{\mathbf{\Gamma}}$ is a half-complete quasipseudometric, that is, every $(d_{\mathbf{\Gamma}})^*$-Cauchy sequence is d-convergent.

The next two lemmas are interesting by themselves and give a sufficient condition to obtain the equality between $\rho(X)$ and $G(X)$.

Lemma 3.5. *Let $\mathbf{\Gamma}$ be a fractal structure over X, and suppose that the associated quasi-pseudometric d verifies that every d^*-Cauchy sequence which is d^{-1} convergent to x is d-convergent to x (this happens, for example, either if $\mathbf{\Gamma}$ is starbase or d is point-symmetric). Then $\rho(X) \subseteq G(X)$.*

Proof. Let $x \in X$ and suppose that there exists $g = (\rho_n(x_n)) \in \varprojlim G_n$ such that $(\rho_n(x_n)) < \rho(x)$, that is, there exists $l \in \mathbb{N}$ such that $\rho_l(x_l) <_l \rho_l(x)$. Then it is clear that (x_n) is a d^*-Cauchy sequence, since $U^*_{x_{n+1}n+1} \subseteq U^*_{x_n n}$ for all $n \in \mathbb{N}$ (since $(\rho_n(x_n)) \in \varprojlim G_n$) and since (x_n) is a sequence which d^{-1}-converges to x, so by hypothesis we have that (x_n) d-converges to x. Hence for all $n \in \mathbb{N}$ there exists $m \geq n$ such that $x_k \in U_{xn}$ for all $k \geq m$, whence $\rho_n(x) \leq_n \rho_n(x_k) = \rho_n(x_n)$ for all $n \in \mathbb{N}$, what contradicts that $\rho_l(x_l) <_l \rho_l(x)$. The contradiction shows that $\rho(x)$ is minimal, o equivalently, $\rho(x) \in G(X)$. □

Lemma 3.6. *Let $(X, \mathbf{\Gamma})$ be a half complete GF-space T_1. Then $\rho(X) = G(X)$.*

Proof. Let $x \in X$ and suppose that there exists $(\rho_n(y_n)) \in \varprojlim G_n$ such that $\rho_n(y_n) \leq_n \rho_n(x)$ for all $n \in \mathbb{N}$. Since X is half complete then there exists $y \in X$ such that $\rho_n(y) \leq_n \rho_n(y_n) \leq_n \rho_n(x)$, and hence we have that $x \in U_{yn}$ for all $n \in \mathbb{N}$, and since X is T_1, it follows that $x = y$, and hence $\rho_n(y_n) = \rho_n(x)$ for all $n \in \mathbb{N}$. Therefore $\rho(x) \in G(X)$.

Conversely, let $(\rho_n(x_n)) \in G(X)$. Since X is half complete then there exists $x \in X$ such that $\rho_n(x) \leq_n \rho_n(x_n)$ for all $n \in \mathbb{N}$, and since $(\rho_n(x_n)) \in G(X)$ it follows that $\rho_n(x_n) = \rho_n(x)$ for all $n \in \mathbb{N}$, whence $(\rho_n(x_n)) \in \rho(X)$. □

From the point of view of GF-isomorphisms, $\mathbf{\Gamma}$ and $G(\mathbf{\Gamma})$ are the same.

Proposition 3.7. *Let $\mathbf{\Gamma}$ be a fractal structure over X. Then $(X, \mathbf{\Gamma})$ and $(X, G(\mathbf{\Gamma}))$ are GF-isomorphic. In fact, the identity map is a GF-isomorphism.*

Proof. Let $y \in (U_{xn}^{G(\mathbf{\Gamma})})^{-1} = \bigcap_{x \in U_{zn}^{-1}} U_{zn}^{-1}$, then, since $x \in U_{xn}^{-1}$ we have that $y \in U_{xn}^{-1}$. Conversely, suppose that $y \in U_{xn}^{-1}$, and let $z \in X$ such that $x \in U_{zn}^{-1}$, then by transitivity, $y \in U_{zn}^{-1}$, and then $y \in \bigcap_{x \in U_{zn}^{-1}} U_{zn}^{-1} = (U_{xn}^{G(\mathbf{\Gamma})})^{-1}$. Therefore the identity map is a GF-isomorphism. □

The concept of isomorphism in this category can be characterized by means of the associated posets.

Proposition 3.8. *Let X, Y be half complete T_1 GF-spaces. Then X and Y are GF-isomorphic if and only if there exist $f_n : G_n(X) \to G_n(Y)$ poset isomorphisms such that $\phi_{n+1} \circ f_{n+1} = f_n \circ \phi_{n+1}$ for all $n \in \mathbb{N}$.*

Proof. By the previous lemma we have that $\rho(X) = G(X)$, since $\mathbf{\Gamma}$ is half complete and X is T_1. Suppose that X and Y are GF-isomorphic, and let f be a GF-isomorphism, then it is clear that $f(U^*_{xn}) = U^*_{f(x)n}$ and that $\rho_n(x) \leq_n \rho_n(y)$ if and only if $\rho_n(f(x)) \leq_n \rho_n(f(y))$. On the other hand, if we define f_n by $f_n(\rho_n(x)) = \rho_n(f(x))$, it follows that $\phi_{n+1} \circ f_{n+1} \circ \rho_{n+1}(x) = \phi_{n+1} \circ \rho_{n+1} \circ f(x) = \rho_n \circ f(x) = f_n \circ \rho_n(x) = f_n \circ \phi_{n+1} \circ \rho_{n+1}(x)$, and hence $\phi_{n+1} \circ f_{n+1} = f_n \circ \phi_{n+1}$.

Conversely, let f_n be a poset isomorphism between $G_n(X)$ and $G_n(Y)$ such that $\phi_{n+1} \circ f_{n+1} = f_n \circ \phi_{n+1}$. We define $f\colon X \to Y$ by $\rho_n(f(x)) = f_n(\rho_n(x))$. It is clear that f is a map from X to $\varprojlim G_n(Y)$. Let us prove that f is well defined, that is, $f(x) \in G(Y)$ for all $x \in X$ (note that since Y is half complete and T_1 then, by the previous lemma, it follows that $\rho(Y) = G(Y)$). To see this, let $x \in X$ and suppose that there exists $y \in Y$ such that $\rho_n(y) \leq_n \rho_n(f(x)) = f_n(\rho_n(x))$. It is easy to see that $(f_n^{-1}(\rho_n(y))) \in \varprojlim G_n(X)$. Since X is half complete, there exists $z \in X$ with $\rho_n(z) \leq_n f_n^{-1}(\rho_n(y))$ for all $n \in \mathbb{N}$. Then $f_n(\rho_n(z)) \leq_n \rho_n(y) \leq_n f_n(\rho_n(x))$ for all $n \in \mathbb{N}$, whence $\rho_n(z) \leq_n \rho_n(x)$ for all $n \in \mathbb{N}$, and since X is T_1, then $z = x$, and hence $\rho_n(y) = f_n(\rho_n(x))$ for all $n \in \mathbb{N}$. Therefore $f(x) \in G(Y)$. Moreover, it is clear that f is a GF-map and analogously it can be proved that $f^{-1} : Y \to X$ defined by $\rho_n(f^{-1}(y)) = f_n^{-1}(\rho_n(y))$ for all $n \in \mathbb{N}$ and for all $y \in Y$ is a GF-map, which is the inverse map of f, and hence f is a GF-isomorphism. □

It seems that the concept of GF-map is related with that of map between inverse sequences. In [**10**], it is proposed another definition for map between inverse sequences. For such definition, the related concept in our context is the weaker one of quasi-uniformly continuous map, as we see in the following proposition.

Proposition 3.9. *Let $(X, \mathbf{\Gamma})$ and $(Y, \mathbf{\Delta})$ be GF-spaces, and let $f\colon X \to Y$ be a map. Then the following statements are equivalent:*

1. *For all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $f(U_{xm}) \subseteq U_{f(x)n}$ for all $x \in X$.*
2. *f is quasi-uniformly continuous.*
3. *There exists a cofinal sequence (i_n) such that $\rho_n \circ f = f_n \circ \rho_{i_n}$, where $f_n : G_{i_n}(X) \to G_n(Y)$ is an order-preserving map, for all $n \in \mathbb{N}$.*

Proof. The equivalence between 1) and 2) is clear.

1) implies 3). By hypothesis (with n=1), there exists $i_1 \in \mathbb{N}$ such that $f(U_{xi_1}) \subseteq U_{f(x)1}$ for all $x \in X$. If we have defined i_k such that $f(U_{xi_k}) \subseteq U_{f(x)k}$, then we define $i_{k+1} > i_k$ such that $f(U_{xi_{k+1}}) \subseteq U_{f(x)k+1}$. It is clear that (i_k) is a cofinal sequence, since it is strictly increasing.

On the other hand, it is clear that $f(U^*_{xi_n}) \subseteq U^*_{f(x)n}$, so we define $f_n : G_{i_n}(X) \to G_n(Y)$ by $f_n(\rho_{i_n}(x)) = \rho_n(f(x))$. It is clear from what we have proved that f_n

is well defined, and by hypothesis, if $y \in U_{xi_n}$ then $f(y) \in U_{f(x)n}$, and then f_n is order-preserving.

3) implies 1). Let $y \in U_{xi_n}$, then $\rho_{i_n}(x) \leq_{i_n} \rho_{i_n}(y)$, and since f_n is order-preserving then $f_n(\rho_{i_n}(x)) \leq_n f_n(\rho_{i_n}(y))$, and since $\rho_n \circ f = f_n \circ \rho_{i_n}$ then $\rho_n(f(x)) \leq_n \rho_n(f(y))$ and then $f(y) \in U_{f(x)n}$, what proves 1). □

4. GF-compactifications

The beginning of this section is devoted to collect the main properties of GF-compactification, that are introduced in Definition 4.3, including behavior under subspaces, products or dense subspaces.

The second half of the section characterizes the GF-compactifications of rationals and naturals with their usual topologies as a tool to prove Theorem 4.20 where every metric compactification is proved to be a GF-compactification.

We start deducing properties for $G(\mathbf{\Gamma})$ from the ones owned by $\mathbf{\Gamma}$.

Theorem 4.1. *Let $\mathbf{\Gamma}$ be a finite fractal structure over X such that $d_{\mathbf{\Gamma}}$ is point symmetric. Then $G(\mathbf{\Gamma})$ is a finite half complete fractal structure over $G(X)$.*

Proof. $G(\mathbf{\Gamma})$ is clearly finite. Let us prove that it is half complete.

Let $(g_n) \in \varprojlim G_n$ and let $\mathcal{F} = \{(h_n) \in \varprojlim G_n : (h_n) \leq (g_n)\}$. Since $(g_n) \in \mathcal{F}$, it is nonempty, and if $(h_n^i) \in \varprojlim G_n$ is a decreasing chain, then if $h_n = \min\{h_n^i :\in I\}$ (note that the minimum exists, since Γ_n is finite and $\{h_n^i :\in I\}$ is a chain) we have that (h_n) is a bound for the chain. Then by Zorn's lemma, $\mathcal{F}$ has a minimal element (h_n). Hence $(h_n) \in G(X)$ and $(h_n) \leq (g_n)$. Therefore $G(X)$ is half complete. □

The following result is going to be used in next corollary. The proof can be found in [**6**].

Lemma 4.2. *Let X be a topological space. X is second countable if and only if it admits a finite fractal structure.*

The following result associates a compactification to certain GF-spaces.

Corollary 4.3. *Let $\mathbf{\Gamma}$ be a finite fractal structure over X such that $d_{\mathbf{\Gamma}}$ is point symmetric. Then $(G(X), G(\mathbf{\Gamma}))$ is a second countable T_1 compactification of X.*

Proof. By Lemma 3.5, X is a subset of $G(X)$, and if $g = (\rho_n(x_n)) \in G(X)$, then (x_n) converges to g in $G(X)$ (since $g_n = \rho_n(x_n) = \rho_n(x_k)$ for all $k \geq n$, and hence $x_k \in U_{gn}$ for all $k \geq n$). Therefore X is dense in $G(X)$.

On the other hand, since $G(\mathbf{\Gamma})$ is finite, by Lemma 4.2 it follows that $G(X)$ is second countable. $G(X)$ is T_1, since it the subset of closed points of $\varprojlim G_n$. $G(X)$ is compact because it is a left K-complete (since $G(\mathbf{\Gamma})$ is half complete by the previous theorem) totally bounded space and then we can apply [**15**, Prop. 4]. □

We will denote the GF-space $(G(X), G(\mathbf{\Gamma}))$ by $G(X, \mathbf{\Gamma})$.

The next example shows that there exists a finite tiling starbase GF-space $(X, \mathbf{\Gamma})$ such that $G(X, \mathbf{\Gamma})$ is not Hausdorff, so the T_1 axiom cannot be improved in the above Corollary.

Example 4.4. *Let* $[0,1]$ *with the finite tiling starbase fractal structure defined as* $\Gamma_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}] : 0 \leq k \leq 2^n - 1\}$. *Let* $X = [0,1] \times [0,1] \setminus \{(\frac{1}{2}, \frac{1}{2})\}$, *with the fractal structure* $\mathbf{\Gamma}' = (\mathbf{\Gamma} \times \mathbf{\Gamma})_X$ *by* $([0,1] \times [0,1], \mathbf{\Gamma} \times \mathbf{\Gamma})$. *Then it is clear that it is a finite tiling starbase fractal structure over* X.

Let us prove that $G(X, \mathbf{\Gamma}')$ *is not a Hausdorff space.*

Let $g_n^1 = \rho_n(\frac{1}{2}, \frac{1}{2} + \frac{1}{2^{n+1}})$, $g_n^2 = \rho_n(\frac{1}{2}, \frac{1}{2} - \frac{1}{2^{n+1}})$, $g_n^3 = \rho_n(\frac{1}{2} + \frac{1}{2^{n+1}}, \frac{1}{2})$ *and* $g_n^4 = \rho_n(\frac{1}{2} - \frac{1}{2^{n+1}}, \frac{1}{2})$. *Then it is clear that* $(g_n^i)_n \in G(X)$ *for* $i = 1, 2, 3, 4$. *Moreover* $U_{g^1 n} \cap U_{g^3 n} =]\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}[\times]\frac{1}{2}, \frac{1}{2} + \frac{1}{2^n}[\neq \emptyset$ *for all* $n \in \mathbb{N}$. *Therefore* $G(X, \mathbf{\Gamma}')$ *is not Hausdorff.*

The notion of GF-compactification is now formally introduced.

Definition 4.5. Let $\mathbf{\Gamma}$ be a finite fractal structure over X with $d_{\mathbf{\Gamma}}$ point symmetric. We will say that $G(X)$ is the GF-compactification of X relative to $\mathbf{\Gamma}$, and we will note it by $G(X, \mathbf{\Gamma})$.

Now we recall the concept of direct product of posets.

Definition 4.6. Let G, H be posets. We define the poset $G \times H$ with the order $(g, h) \leq (g', h')$ if and only if $g \leq g'$ and $h \leq h'$.

With that definition, we relate the GF-compactification of a product with the product of the GF-compactifications.

Proposition 4.7. *Let* $(X, \mathbf{\Gamma}^1)$, $(Y, \mathbf{\Gamma}^2)$ *be GF-spaces. Then* $G_n(X \times Y)$ *is poset isomorphic to* $G_n(X) \times G_n(Y)$ *and* $G(X \times Y, \mathbf{\Gamma}^1 \times \mathbf{\Gamma}^2)$ *is GF-isomorphic to* $G(X, \mathbf{\Gamma}^1) \times G(Y, \mathbf{\Gamma}^2)$.

Proof. It is clear that $(g, h) \in G(X \times Y)$ if and only if $g \in G(X)$ and $h \in G(Y)$, and it is also clear that $(g_n, h_n) \leq (g'_n, h'_n)$ if and only if $g_n \leq g'_n$ and $h_n \leq h'_n$ and hence the image of $U_{(g,h)n}$ is equal to $U_{gn} \times U_{hn}$ (and hence that map is a GF-isomorphism). □

The same result is valid for countable products.

Proposition 4.8. *Let* $(X_i, \mathbf{\Gamma}^i)$ *be a countable family of GF-spaces. Then* $G_n(\prod_{i \in \mathbb{N}} X_i)$ *is poset isomorphic to* $\prod_{i=1}^n G_n(X_i)$, *and* $G(\prod_{i \in \mathbb{N}} X_i, \prod_{i \in \mathbb{N}} \mathbf{\Gamma}^i)$ *is GF-isomorphic to* $\prod_{i \in \mathbb{N}} G(X_i, \mathbf{\Gamma}^i)$.

Proof. Let $\mathbf{\Gamma} = \prod_{i \in \mathbb{N}} \mathbf{\Gamma}^i$, $X = \prod_{i \in \mathbb{N}} X_i$ and $G_n = G_n(\prod_{i \in \mathbb{N}} X_i)$. First note that since $U^*_{xn} = (U^1_{x_1 n})^* \times \cdots \times (U^n_{x_n n})^* \times \prod_{i > n} X_i$, we have that G_n is poset isomorphic to $G_n(X_1) \times \cdots \times G_n(X_n)$, using the map which sends $\rho_n(x)$ to $(\rho_n(x_1), \ldots, \rho_n(x_n))$.

Let $f : G(X, \mathbf{\Gamma}) \to \prod_{i \in \mathbb{N}} G(X_i, \mathbf{\Gamma}^i)$ be defined as $f(g) = (g^i)_i$ where if $g = (\rho_n(x^n))_n$ with $x^n = (x_i^n)$, then g^i is defined as

$$\begin{aligned} g^1 &= (\rho_1(x_1^n))_n, \\ g^2 &= (\rho_1(x_2^2), \rho_2(x_2^2), \ldots, \rho_n(x_2^n), \ldots), \\ &\vdots \\ g^i &= (\rho_1(x_i^i), \rho_2(x_i^i), \ldots, \rho_{i-1}(x_i^i), \rho_i(x_i^i), \rho_{i+1}(x_i^{i+1}), \ldots, \rho_n(x_i^n), \ldots) \\ &\quad \text{for all } i \in \mathbb{N}. \end{aligned}$$

Now, $(\rho_n(x^n))_n \in G(X, \mathbf{\Gamma})$ if and only if there not exists $(\rho_n(y^n))_n$ with $(\rho_n(y^n))_n < (\rho_n(x^n))_n$ or equivalently there exists no n_0 such that $\rho_{n_0}(y^{n_0}) < \rho_{n_0}(x^{n_0})$, or what is the same, there exist no n_0 and $i_0 \leq n_0$ with $\rho_{n_0}(y_{i_0}^{n_0}) < \rho_{n_0}(x_{i_0}^{n_0})$, that is, there exists no i_0 with $h^{i_0} < g^{i_0}$ for some h^{i_0}, what is true when and only when, $g^i \in G(X_i, \mathbf{\Gamma}^i)$ for all $i \in \mathbb{N}$ what means that $(g^i)_i \in \prod_{i \in \mathbb{N}} G(X_i, \mathbf{\Gamma}^i)$.

Therefore f is well defined and bijective.

It is clear that $g_n \leq h_n$ if and only if $g_n^i \leq h_n^i$ for all $i \leq n$, and hence $f(U_{xn}) = U_{x_1 n}^1 \times \ldots \times U_{x_n n}^n \times \prod_{i > n} G(X_i, \mathbf{\Gamma}^i)$. Therefore f is a GF-isomorphism. □

If A is a subset of X, we will denote by A' the set of minimal point of $\rho(A)$ (i.e., for $g \in \varprojlim G_n$, $g \in A'$ if and only if there exists no $h \in \varprojlim G_n$ with $h \leq g$ and $h_n \in \rho_n(A)$ for all $n \in \mathbb{N}$).

In the next results, we are going to see that A' is essentially $G(A)$.

Proposition 4.9. *Let $\mathbf{\Gamma}$ be a fractal structure over X, and $A \subseteq X$. Then $G_n(\mathbf{\Gamma}_A)$ is poset isomorphic to $G_n(\mathbf{\Gamma}) \cap \rho_n(A)$.*

Proof. It is clear, since $(U_{xn}^{\mathbf{\Gamma}_A})^* = U_{xn}^* \cap A$ for all $x \in A$. □

Corollary 4.10. *Let $\mathbf{\Gamma}$ be a finite starbase fractal structure over X, and $A \subseteq X$. Then A' is GF-isomorphic to $G(A, \mathbf{\Gamma}_A)$.*

Proof. Let $f : A' \to G(A, \mathbf{\Gamma}_A)$ be defined by $f(g) = h$ where if $g_n = U_{x_n n}^*$ with $x_n \in A$ then $h_n = (U_{x_n n}^{\mathbf{\Gamma}_A})^*$. It is clear that if $g, h \in A'$ with $g_n \leq_n h_n$, then $(f(g))_n \leq (f(h))_n$ (since $(U_A)_{xn}^* = U_{xn}^* \cap A$ for all $x \in A$). Therefore f is continuous and open (since $f(U_{gn}) = U_{f(g)n}$ for all $g \in A'$ and $n \in \mathbb{N}$), and it is clear that f is bijective. □

Proposition 4.11. *Let $\mathbf{\Gamma}$ be a finite fractal structure over X such that $d_{\mathbf{\Gamma}}$ is point-symmetric, and let A be a subspace of X, such that $A' \subseteq G(X)$. Then $Cl_{G(X,\mathbf{\Gamma})}(A) = A'$ and it is GF-isomorphic to $G(A, \mathbf{\Gamma}_A)$.*

Proof. It is clear by Proposition 4.9 that $G_n(A) = G_n(X) \cap \rho_n(A)$. Let $g \in A'$, and let n be a natural number. Then $g_n = \rho_n(a)$, with $a \in A$, and hence $a \in U_{gn}$. Therefore $g \in Cl_{G(X)}(A)$ (since $g \in G(X)$, because $A' \subseteq G(X)$), and then $\rho(A) \subseteq A' \subseteq Cl_{G(X)}(A)$, and since A' is compact (since it is homeomorphic to $G(A, \mathbf{\Gamma}_A)$ which is compact), we obtain that $A' = Cl_{G(X)}(A)$, and we get the desired result. □

Corollary 4.12. *Let $\mathbf{\Gamma}$ be a finite fractal structure over X such that $d_{\mathbf{\Gamma}}$ is point-symmetric, and let A be a subspace of X such that $U^*_{xn} \cap A \neq \emptyset$ for all $x \in X$ (that is, A is a $d^*_{\mathbf{\Gamma}}$-dense subspace of X). Then $G(A, \mathbf{\Gamma}_A)$ is GF-isomorphic to $G(X, \mathbf{\Gamma})$.*

Proof. It is clear that $A' \subseteq G(X)$, and then $G(A, \mathbf{\Gamma}_A)$ is GF-isomorphic to $Cl_{G(X,\mathbf{\Gamma})} A$ by the previous proposition. Now, since it is also clear that A is dense in X and X is dense in $G(X)$, then A is dense in $G(X, \mathbf{\Gamma})$, and therefore $G(A, \mathbf{\Gamma}_A)$ is GF-isomorphic to $G(X, \mathbf{\Gamma})$. □

The following example shows that the condition $A' \subseteq G(X)$ cannot be avoided in the previous proposition.

Example 4.13. Let $X = \mathbb{R}$ be the set of real numbers with the following finite starbase fractal structure: for each n, let $\Gamma_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}] : -n2^n \leq k \leq n2^n - 1\} \cup \{] \leftarrow, -n], [n, \rightarrow [\}$, and let Q be the set constructed by picking an irrational number up from each $A_n \in \Gamma_n$. Let us show that $G(X) = [0,1]$. If $g \in G(X)$, let $g_n = \rho_n(x_n)$. If there exists $n_0 \in \mathbb{N}$ such that $-n_0 \leq x_n \leq n_0$ for all $n \in \mathbb{N}$, then $x_n \in [-n_0, n_0]$ for all $n \geq n_0$, and then it is easy to see that $g \in X$. Otherwise, there are two options: first, $x_n \in]n, \rightarrow [$ for all $n \in \mathbb{N}$; second, $x_n \in] \leftarrow, -n[$ for all $n \in \mathbb{N}$. Let us denote the point g by a in the first case and by b in the second one, then $G(X)$ is a two-point Hausdorff compactification of $\mathbb{R}$, and hence it is homeomorphic to $[0,1]$ (see [**13**] where it is proved that all Hausdorff compactifications of $\mathbb{R}$ by two points are topologically equivalent). Therefore $Cl_{G(X)}(Q) = G(X) = [0,1]$ (since Q is dense in X and hence in G(X)).

On the other hand $G(Q) = \mathbf{C}$, the Cantor set. To see that, note that $G_n \cap Q$ is discrete and hence $G(Q)$ is zero-dimensional. Then $G(Q)$ is a perfect (since Q also is) zero-dimensional compact metrizable (since $\mathbf{\Gamma}_Q$ is zero-dimensional, then it is starbase, and then $G(Q)$ is a metrizable (see [**5**])) space, and then it is the Cantor set.

The following result is a characterization of perfect metrizable compact spaces by means of GF-compactifications. Note that the equivalence between (1) and (3) does not involve GF-spaces and is new (as far as the authors know).

Theorem 4.14. *Let X be a topological space and let $\mathbb{Q}$ be the set of rational numbers. The following statements are equivalent:*

1) *X is a perfect metrizable compact space.*
2) *X is Hausdorff and $X = G(\mathbb{Q}, \mathbf{\Gamma})$, for some finite starbase fractal structure $\mathbf{\Gamma}$ over $\mathbb{Q}$.*
3) *X is a metrizable compactification of $\mathbb{Q}$.*

Proof. 2) implies 3) by Corollary 4.3.

3) implies 1) is clear.

Let us prove 1) implies 2).

Let X be a perfect metrizable compact space, and let $\mathbf{\Gamma}'$ be a finite starbase fractal structure over X. Let Q be constructed by taking for each $g_n \in G_n$ a point $x \in X$ with $\rho_n(x) = g_n$. Since each G_n is finite, it is obvious that Q is countable and it is also clear that Q is dense in X, and since X is perfect then Q is perfect too, see for example [**7**]. Therefore Q is a countable perfect metrizable space, and then it is homeomorphic to the set of rational numbers by Sierpinski's Theorem (see [**16**], and see [**7**] for a different proof). Also note that Q verifies the hypotheses of Corollary 4.12, and then by proposition 4.11 we have that $X = G(X, \mathbf{\Gamma}')$ is GF-isomorphic to $G(Q, \mathbf{\Gamma})$, where $\mathbf{\Gamma}$ is the restriction of $\mathbf{\Gamma}'$ to Q. □

Corollary 4.15. *Every metrizable compactifications of $\mathbb{Q}$, the set of rational numbers, is of the form $G(\mathbb{Q}, \mathbf{\Gamma})$ for some finite starbase fractal structure $\mathbf{\Gamma}$ over $\mathbb{Q}$.*

As a consequence we characterize the metrizable compactifications of a perfect space.

Corollary 4.16. *Every metrizable compactifications of a separable metrizable perfect space X, is of the form $G(X, \mathbf{\Gamma})$ for some finite starbase fractal structure $\mathbf{\Gamma}$ over X.*

Proof. Let Y be a metrizable compactification of X, and let Q be a countable (perfect) dense subspace of X. Then Q is homeomorphic to the set of rational numbers by Sierpinski's Theorem and Y is a metrizable compactification of Q. By Corollary 4.16, there exists a finite starbase fractal structure $\mathbf{\Gamma}'$ over Q, such that $Y = G(Q, \mathbf{\Gamma}')$. Let $\mathbf{\Gamma}'' = G(\mathbf{\Gamma}')$ be the induced fractal structure over Y, and let $\mathbf{\Gamma}$ be the restriction of $\mathbf{\Gamma}''$ to X. Since $Y = Q'$ and $Q \subseteq X$, then $Y = X'$, so $Y = G(X, \mathbf{\Gamma})$. □

The same result that we got for perfect spaces, we get for $\mathbb{N}$, which can be considered as "the opposite" of perfectness (in topological sense).

Theorem 4.17. *Let X be a metrizable compactification of $\mathbb{N}$. Then there exists a finite starbase fractal structure $\mathbf{\Gamma}$ over $\mathbb{N}$ such that $X = G(\mathbb{N}, \mathbf{\Gamma})$.*

Proof. In [**17**] is shown that $X = \mu(K)$ for some compact metrizable space $K = X \setminus \mathbb{N}$. In this proof, we use the construction of $\mu(K)$ [**17**, Proof of Theorem 1]).

For each $A'_n \in \Gamma'_n$, let $A_n = A'_n \cup \{(d_i, k) \in M : d_i \in A_n; k \geq n\}$. Now let $\Gamma_n = \{A_n : A'_n \in \Gamma'_n\} \cup \{\{(d_i, k)\} : (d_i, k) \in M; i, k < n\}$

Let us prove that $\mathbf{\Gamma}$ is a finite starbase fractal structure over M.

It is clear that Γ_n is finite for all $n \in \mathbb{N}$, and that A_n is closed for all $A_n \in \Gamma_n$. Let $n \in \mathbb{N}$ and $A_n \in \Gamma_n$ and let (x_i) be a sequence of elements of A_n which converges to $x \in M$. Let $A_n = A'_n \cup \{(d_i, k) \in M : d_i \in A'_n; k \geq n\}$. We can suppose that $x \in K$, because in other case, $\{x\}$ would be open and then there would be an element in the sequence equal to x, and we would have that $x \in A_n$. Since (x_i) converges to x, then there exists $m \in \mathbb{N}$ such that $x_i \in H_{x,n,n}$ for all $i \geq m$. If $x_i \in K$ for all $i \in \mathbb{N}$, then $x_i \in U'_{xn}$, and hence $x \in (U'_{x_i n})^{-1} = \bigcap_{x_i \in B'_n} B'_n \subseteq A'_n \subseteq A_n$, so $x \in A_n$. If $x_i \in \mathbb{N}$ for all $i \in \mathbb{N}$, then $x_i = (d_i, j)$ with $d_i \in U'_{xn}$ and $j > n$. Then $x \in (U'_{d_i n})^{-1} = \bigcap_{d_i \in B'_i} B'_i \subseteq A'_n \subseteq A_n$ and hence $x \in A_n$. Anyway, there always exists a subsequence of (x_i) in any of the two previous cases, and hence $x \in A_n$, so A_n is closed.

It is also clear that $\mathrm{St}((d_n,m),\Gamma_{m+1})=\{(d_n,m)\}$. Let us show that $\mathrm{St}(x,\Gamma_n)=\mathrm{St}(x,\Gamma'_n)\cup\{(d_i,j):d_i\in\mathrm{St}(x,\Gamma'_n);j\geq n\}$ for all $x\in K$. Let $x\in K$ (note that then $x\in A_n$ if and only if $x\in A'_n$), then $\mathrm{St}(x,\Gamma_n)=\bigcup_{x\in A_n}A_n=\bigcup_{x\in A'_n}(A'_n\cup\{(d_i,k)\in M:d_i\in A'_n;k\geq n\})=\mathrm{St}(x,\Gamma'_n)\cup\{(d_i,k)\in M:d_i\in\mathrm{St}(x,\Gamma'_n);k\geq n\}$. Now, let $n,m\in\mathbb{N}$ and $x\in K$, then there exists $p\in\mathbb{N}$ such that $\mathrm{St}(x,\Gamma'_p)\subseteq U'_{xl}$, where $l=max\{n,m+1\}$. Then $\mathrm{St}(x,\Gamma_p)=\mathrm{St}(x,\Gamma'_p)\cup\{(d_i,j):d_i\in\mathrm{St}(x,\Gamma'_p);j\geq p\}\subseteq H_{x,l,l-1}\subseteq H_{x,n,m}$. Therefore $\mathbf{\Gamma}$ is starbase.

It is clear that $A_{n+1}\subseteq A_n$ if $A'_{n+1}\subseteq A'_n$, and then Γ_{n+1} is a refinement of Γ_n. Let $x\in A_n$.

If $x\notin K$ then $x=(d_i,j)$ with $d_i\in A'_n$ and $j\geq n$. If $j=n$ then let $A_{n+1}=\{(d_i,j)\}\in\Gamma_{n+1}$, and $x\in A_{n+1}\subseteq A_n$. If $j>n$, then $A_n=A'_n\cup\{(d_l,k)\in M:d_l\in A'_n;k\geq n\}$, and then $d_i\in A'_n$. Then there exists $A'_{n+1}\in\Gamma'_{n+1}$ such that $d_i\in A'_{n+1}\subseteq A'_n$. Hence $(d_i,j)\in A_{n+1}\subseteq A_n$.

If $x\in K$, then $x\in A'_n$, and then there exists $A'_{n+1}\in\Gamma'_{n+1}$ such that $x\in A'_{n+1}\subseteq A'_n$, and hence $x\in A_{n+1}\subseteq A_n$.

Therefore $\mathbf{\Gamma}$ is a finite starbase fractal structure over M.

Analogously to the preceding paragraphs we can see that $U^*_{xn}=(U'_{xn})^*\cup\{(d_i,j):d_i\in(U'_{xn})^*;j\geq n\}$ for all $x\in K$, and since $(U'_{xn})^*\cap D\neq\emptyset$ (by construction of D) for all $x\in K$ then $U^*_{xn}\cap\mathbb{N}=\{(d_i,j):d_i\in(U'_{xn})^*;j\geq n\}\neq\emptyset$ for all $x\in K$. On the other hand, if $x\in\mathbb{N}$ it is obvious that $(U'_{xn})^*\cap\mathbb{N}\neq\emptyset$, and then $(U'_{xn})^*\cap\mathbb{N}\neq\emptyset$ for all $x\in X$, and $n\in\mathbb{N}$. Therefore by Corollary 4.12, we have that $X=G(\mathbb{N},\mathbf{\Gamma}_{\mathbb{N}})$. □

The next result decomposes separable metrizable spaces into two pieces with certain properties. It is analogous to a classical result, but it is not exactly the same.

Lemma 4.18. *Let X be a separable metrizable space, and let N be the subspace of isolated points of X. Then $X=P\cup C$, with P perfect in X (dense in itself and closed in X), C is scattered and $P\cap C=\emptyset$. Moreover $C\subseteq\overline{N}$.*

Proof. The first part is known. Note that C can be constructed from N in the following way, that also allows to prove the second part.

Denote by N^d the derived set of N. If α is an ordinal, denote $N^\alpha=(\bigcup_{\beta<\alpha}N^\beta)^d$. It is also known (see [**14**]) that there exists an ordinal α such that $C=N^\alpha$ (since it is scattered). Then, since $A^d\subseteq\overline{A}$ for each subset A of X, we can deduce (using transfinite induction) that $A^\alpha\subseteq\overline{A}$ for every subset A of X and every ordinal α. Therefore $C\subseteq\overline{N}$. □

The following result is going to be used in next theorem. The proof can be found in [**8**].

Lemma 4.19. *Let X be a topological space. X is a separable metrizable space if and only if it admits a finite starbase fractal structure.*

The next result allows to get **every** metrizable compactification by mean of fractal structures. In its proof, we use the previous results proved for perfect spaces and for the naturals.

Theorem 4.20. *Let X be a separable metrizable space. If Y is a metrizable compactification of X then $Y = G(X, \mathbf{\Gamma})$ for some finite starbase fractal structure $\mathbf{\Gamma}$ over X.*

Proof. Let $Y = P \cup C$ with P perfect in Y and C countable and scattered, and $P \cap C = \emptyset$. Let N be the subspace of isolated points of Y. Then $C \subseteq \overline{N}$ by the previous lemma.

If $P = \emptyset$, then Y is a compactification of N. If N is finite then Y is compact and the result is trivial. If N is countable then it is homeomorphic to $\mathbb{N}$ the set of natural numbers, and the result follows from Theorem 4.17.

If $N = \emptyset$, then Y and X are perfect, and the result follows from Corollary 4.16.

If N is finite, then $Y = P \cup N$, and we can apply Corollary 4.16 to P, and construct a finite starbase fractal structure over X from that of P with the desired property, adding the points of N.

So, the only case we have to consider is to suppose that P is nonempty and N is countable but not finite. Then it is clear that N is homeomorphic to $\mathbb{N}$, the set of natural numbers.

Let Q' be a dense and countable subspace of $X \setminus \overline{N}$, let Q'' be a dense and countable subspace of $P \cap \overline{N}$, and let $Q = Q' \cup Q''$. Then Q is a dense and countable subspace of P (and hence dense in itself), since $\overline{Q} = \overline{Q'} \cup \overline{Q''} = \overline{X \setminus \overline{N}} \cup (P \cap \overline{N}) = P$ (note that since X is dense in Y, then $N \subseteq X$, and hence $Y = \overline{X} = \overline{X \setminus \overline{N}} \cup \overline{N}$, whence $P \setminus \overline{N} \subseteq \overline{X \setminus \overline{N}}$). Therefore Q is homeomorphic to the rational numbers $\mathbb{Q}$, by Sierpinski's Theorem.

Since P is compact, it is a metrizable compactification of Q, and by Corollary 4.16 there exists a finite starbase fractal structure $\mathbf{\Gamma}^1$ over Q such that $P = G(Q, \mathbf{\Gamma}^1)$, and let $\mathbf{\Gamma}^2 = I(\mathbf{\Gamma}^1)$ be the induced (finite starbase) fractal structure over P. Let $x \in P$, then there exists $x_n \in Q$ such that $\rho_n(x) = \rho_n(x_n)$, and then $(U^2_{xn})^* \cap Q = (U^1_{x_n n})^* \neq \emptyset$.

Let $\mathbf{\Gamma}^3$ be a finite starbase fractal structure over $\overline{N} \setminus N$ (which is compact, since N is open in Y, and then we can get a finite starbase fractal structure over it by Lemma 4.19), and let $\Gamma^4_n = \Gamma^3_n \cup \{A^2_n \cap A^3_n : A^2_n \in \Gamma^2_n; A^3_n \in \Gamma^3_n\}$. Let us show that $\mathbf{\Gamma}^4$ is a finite starbase fractal structure over $\overline{N} \setminus N$.

It is clear that Γ^4_n is a closed covering of $\overline{N} \setminus N$, and that Γ^4_{n+1} is a refinement of Γ^4_n. Let $x \in A^4_n$ with $A^4_n \in \Gamma^4_n$. If $A^4_n = A^3_n$ with $A^3_n \in \Gamma^3_n$ then there exists $A^3_{n+1} \in \Gamma^3_n \subseteq \Gamma^4_n$ such that $x \in A^3_{n+1} \subseteq A^4_n$. If $A^4_n = A^2_n \cap A^3_n$ with $A^2_n \in \Gamma^2_n$ and $A^3_n \in \Gamma^3_n$, then there exists $A^2_{n+1} \in \Gamma^2_{n+1}$ and $A^3_{n+1} \in \Gamma^3_{n+1}$ such that $x \in A^2_{n+1} \subseteq A^2_n$ and $x \in A^3_{n+1} \subseteq A^3_n$, and then if $A^4_{n+1} = A^2_{n+1} \cap A^3_{n+1} \in \Gamma^4_{n+1}$, we have that $x \in A^4_{n+1} \subseteq A^4_n$. It is also clear that $\mathrm{St}(x, \Gamma^4_n) = \mathrm{St}(x, \Gamma^3_n)$ for all $x \in \overline{N} \setminus N$ (note that if $x \in A^2_n \cap A^3_n$, then $x \in A^3_n \subseteq \mathrm{St}(x, \Gamma^3_n)$). Therefore $\mathbf{\Gamma}^4$ is a finite starbase fractal structure over $\overline{N} \setminus N$.

Let $\mathbf{\Gamma}^5$ be the finite starbase fractal structure over $\overline{N}$ constructed in Theorem 4.17 (using $\mathbf{\Gamma}^4$ as the fractal structure over $\overline{N} \setminus N$). Then $\overline{N} = G(N, \mathbf{\Gamma}^5_N)$ and $(U^5_{xn})^* \cap N \neq \emptyset$ for all $x \in \overline{N}$ and $n \in \mathbb{N}$.

Now let $\Gamma_n = \{A^2_n : A^2_n \in \Gamma^2_n; A^2_n \cap \overline{N} = \emptyset\} \cup \Gamma^5_n \cup \{A^2_n \cup \{(d_i, j) : d_i \in A^2_n \cap D, j \geq n, j > i\} : A^2_n \in \Gamma^2_n, A^2_n \cap \overline{N} \neq \emptyset\}$.

Let us show that $\mathbf{\Gamma}$ is a finite starbase fractal structure over Y.

It is clear that Γ_n is a closed covering. It is easy to see that Γ_{n+1} is a refinement of Γ_n and that for all $x \in A_n$, there exists $A_{n+1} \in \Gamma_{n+1}$ such that $x \in A_{n+1} \subseteq A_n$.

Let $x \in X$ and let U be an open neighborhood of x in Y. If $x \notin \overline{N}$, then there exists $n \in \mathbb{N}$ such that $\mathrm{St}(x, \Gamma_n^2) \subseteq U \setminus \overline{N}$, and then $\mathrm{St}(x, \Gamma_n) = \mathrm{St}(x, \Gamma_n^2) \subseteq U$. If $x \in N$, then there exists $n \in \mathbb{N}$ such that $\mathrm{St}(x, \Gamma_n) = \{x\}$. If $x \in \overline{N} \setminus N$, then there exist $n, m \in \mathbb{N}$, with $m \geq n$, such that $\mathrm{St}(x, \Gamma_n^5) \subseteq U$ and $\mathrm{St}(x, \Gamma_m^2) \subseteq U \cap \mathrm{St}(x, \Gamma_n^3)$, $\mathrm{St}(x, \Gamma_n^3) \subseteq U$. Let $A_m \in \Gamma_m$ such that $x \in A_m$. If $A_m \in \Gamma_m^5$, then $A_m \subseteq U$, otherwise $A_m = A_m^2 \cup \{(d_i, j) : d_i \in A_m^2 \cap D, j \geq m, j > i\}$ with $A_m^2 \in \Gamma_m^2$. Since $x \notin \overline{N}$, then $x \in A_m^2$, and then $A_m^2 \subseteq U$. Furthermore, $A_m^2 \subseteq \mathrm{St}(x, \Gamma_n^3)$, and then $A_m = A_m^2 \cup \{(d_i, j) : d_i \in A_m^2 \cap D, j \geq m\} \subseteq \mathrm{St}(x, \Gamma_n^3) \cup \{(d_i, j) : d_i \in \mathrm{St}(x, \Gamma_n^3) \cap D, j \geq n\} \subseteq \mathrm{St}(x, \Gamma_n^5) \subseteq U$. Therefore $\mathrm{St}(x, \Gamma_m) \subseteq U$, and $\mathbf{\Gamma}$ is a finite starbase fractal structure.

If $x \in Y \setminus \overline{N}$, and U is an open neighborhood of x contained in $Y \setminus \overline{N}$, then there exists a natural number n such that $\mathrm{St}(x, \Gamma_n^2) \subseteq U$, but then $U_{xn}^* = (U_{xn}^2)^*$ and $U_{xn}^* \cap Q \neq \emptyset$. Therefore $U_{xn}^* \cap Q' \neq \emptyset$ (since $U_{xn}^* \subseteq Y \setminus \overline{N}$), and hence $U_{xn}^* \cap X \neq \emptyset$ (since $Q' \subseteq X$).

If $x \in \overline{N} \setminus N$, then $U_{xn}^{-1} = \bigcap_{x \in A_n; A_n \in \Gamma_n} A_n = \bigcap_{x \in B_n; B_n \in \Gamma_n^2 \cup \Gamma_n^4} B_n \cup \{(d_i, j) : d_i \in A_n^2 \cap D, j \geq n\} = \bigcap_{x \in B_n; B_n \in \Gamma_n^5} B_n = (U_{xn}^5)^{-1}$ and hence $U_{xn} = U_{xn}^5$ and then $U_{xn}^* = (U_{xn}^5)^*$ and so $U_{xn}^* \cap N \neq \emptyset$. Therefore $U_{xn}^* \cap X \neq \emptyset$.

Therefore $U_{xn}^* \cap X \neq \emptyset$ for all $x \in Y$ and for all $n \in \mathbb{N}$ (note that $N \subseteq X$) and by Corollary 4.12, $Y = G(X, \mathbf{\Gamma}_X)$, what proves the theorem. $\square$

5. Characterization of metric continua

We start our study of the connectivity with the case of a poset.

Definition 5.1. Let G be a poset. We say that G is connected if for $g, h \in G$ there exist $n \in \mathbb{N}$ and $\{g_0, g_2, \ldots, g_{n+1}\} \subseteq G$ (called a chain joining g and h) such that $g_0 = g$, $g_{n+1} = h$ and g_i is related by $\leq$ with g_{i-1} and g_{i+1} for all $i = 1, \ldots, n$.

The following definition is in the spirit of Definition 5.1.

Definition 5.2. Let $\mathbf{\Gamma}$ be a pre-fractal structure over X. We say that Γ_n is connected, if for all $x, y \in X$, there exists a finite subfamily $\{A_n^i : 0 \leq i \leq k+1\}$ of Γ_n with $x \in A_n^0$, $y \in A_n^{k+1}$ and $A_n^i \cap A_n^j \neq \emptyset$ for all $|i - j| \leq 1$ (we call it a chain in Γ_n joining x and y). We say that $\mathbf{\Gamma}$ is connected if so is Γ_n for all $n \in \mathbb{N}$.

The proof of the following proposition is straightforward.

Proposition 5.3. *Let $\mathbf{\Gamma}$ be a fractal structure over a topological space X. Then $G(\Gamma_n)$ is connected if and only if the associated poset $G_n(\mathbf{\Gamma})$ is connected.*

If X is connected, then every fractal structure over it is connected.

Proposition 5.4. *Let $\mathbf{\Gamma}$ be a fractal structure over a connected space X. Then $\mathbf{\Gamma}$ is connected.*

Proof. Suppose that there exists $n \in \mathbb{N}$ and $x, y \in X$ such that x and y cannot be joined by a chain in Γ_n. Let $C_x = \{z \in X : \text{there exists a chain joining } x \text{ and } z\}$. It is clear that if $z \in C_x$ then $\mathrm{St}(z, \Gamma_n) \subseteq C_x$, and hence $U_{zn} \subseteq C_x$. Therefore C_x is open. On the other hand, $C_x = \mathrm{St}(C_x, \Gamma_n) = \bigcup_{A_n \cap C_x \neq \emptyset} A_n$ is closed, since Γ_n is closure-preserving. Therefore C_x is a proper clopen set (it is nonempty since $x \in C_x$ and it is not equal to X, since $y \notin C_x$), which is a contradiction with the fact that X is connected. □

Anyway, if X is compact and the fractal structure is starbase, then the converse of Proposition 5.4 holds.

Proposition 5.5. *Let $\mathbf{\Gamma}$ be a starbase fractal structure over a compact space X. Then X is connected if and only if $\mathbf{\Gamma}$ is connected.*

Proof. One implication is by proposition 5.4.

For the converse, suppose that $\mathbf{\Gamma}$ is connected, but X is not. Then there exists F_1 and F_2 nonempty clopen subspaces of X such that $X = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$. By [**5**, Lemma 3.4], there exists $n \in \mathbb{N}$ such that $\mathrm{St}(F_1, \Gamma_n) \cap F_2 = \emptyset$ and $\mathrm{St}(F_2, \Gamma_n) \cap F_1 = \emptyset$, and hence $\mathrm{St}(F_1, \Gamma_n) \cap \mathrm{St}(F_2, \Gamma_n) = \emptyset$.

Since F_1 and F_2 are nonempty, let $x \in F_1$ and $y \in F_2$, and since Γ_n is connected, there exists $\{A_n^i : i = 0, \ldots, k+1\}$ such that $x \in A_n^0$, $y \in A_n^{k+1}$, and $A_n^i \cap A_n^j \neq \emptyset$ for all $|i - j| \leq 1$. Since $x \in F_1$ and $x \in A_n^0$, then $A_n^0 \subseteq \mathrm{St}(F_1, \Gamma_n) \subseteq F_1$. Let $z_1 \in A_n^0 \cap A_n^1$, then $z_1 \in F_1$, and then $A_n^1 \subseteq \mathrm{St}(F_1, \Gamma_n) \subseteq F_1$. Inductively we get that $A_n^i \subseteq F_1$ for all $0 \leq i \leq k+1$, but then $y \in F_1 \cap F_2$, which is a contradiction with $F_1 \cap F_2 = \emptyset$. Therefore X is connected. □

Compactness allows good properties for $G(\mathbf{\Gamma})$.

Theorem 5.6. *Let $\mathbf{\Gamma}$ be a fractal structure over a compact Hausdorff space X. Then $G(\mathbf{\Gamma})$ is starbase.*

Proof. Suppose that $G(\mathbf{\Gamma})$ is not starbase, then there exist $x \in X$ and $l \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $x_n \in \mathrm{St}(x, G(\Gamma_n)) \setminus U_{xl}$. Since X is compact, then there exists $y \in X$ such that y is an adherent point of (x_n). By construction of the sequence it is clear that $y \neq x$, and then, since X is Hausdorff, there exists $m \in \mathbb{N}$ such that $U_{xm} \cap U_{ym} = \emptyset$. Let $k \geq m$ be such that $x_k \in U_{ym}$. Since $x_k \in \mathrm{St}(x, G(\Gamma_k))$, then there exists $z \in X$ such that $x, x_k \in U_{zk}^{-1}$. Then $z \in U_{xk} \cap U_{x_k k} \subseteq U_{xm} \cap U_{ym}$ (note that $x_k \in U_{ym}$, and then $U_{x_k k} \subseteq U_{x_k m} \subseteq U_{ym}$), and this contradicts that $U_{xm} \cap U_{ym} = \emptyset$. Therefore $G(\mathbf{\Gamma})$ is starbase. □

We characterize metric continua.

Theorem 5.7. *Let X be a topological space and let $\mathbb{Q}$ be the set of rational numbers. The following statements are equivalent:*

1) *X is a metrizable continuum.*
2) *X is Hausdorff and $X = G(\mathbb{Q}, \mathbf{\Gamma})$ for some finite starbase fractal structure $\mathbf{\Gamma}$ over $\mathbb{Q}$ with $G(\mathbf{\Gamma})$ connected.*
3) *X is Hausdorff and can be represented as the set of closed points of an inverse limit of a sequence of finite connected spaces.*

Proof. 1) implies 2). By Theorem 4.14, there exists $\mathbf{\Gamma}$, a finite starbase fractal structure over $\mathbb{Q}$ such that $X = G(\mathbb{Q}, \mathbf{\Gamma})$. Since X is connected, then $G(\mathbf{\Gamma})$ (considered over X) is connected by Proposición 5.4, and hence $G_n(\mathbb{Q}) = G_n(X)$ is also connected for all $n \in \mathbb{N}$ by Proposition 5.3, and then it follows that $G(\mathbf{\Gamma})$ is connected (considered over $\mathbb{Q}$) by Proposition 5.3 again, and it is clear that $G(\mathbb{Q}, G(\mathbf{\Gamma})) = G(\mathbb{Q}, \mathbf{\Gamma})$.

2) implies 3). It is clear by the relation between fractal structures and subsets of a inverse limit of a sequence of posets (see Lemma 3.6).

3) implies 1). X is compact metrizable by Theorem 4.14 and since $G(\mathbf{\Gamma})$ is connected by Proposition 5.3 and it is starbase by Theorem 5.6, then X is connected by Theorem 5.5. □

6. Extension theorems for GF-compactifications

One of the many equivalent ways to define Stone-Čech compactification is by means of the extension property. This extensions of functions from a space to its compactification have become so important that every time one defines a new compactification notion, one has to ask which extension theorems does this notion verify. Moreover, since there are extension theorems for completions and quasi-uniformly continuous mappings and GF-compactifications are a kind of completion, we must ask if quasi-uniformly continuous mappings may be extended to our GF-compactification.

The next two results show how to extend a map between two GF-spaces to the GF-compactification of the first one.

Theorem 6.1. *Let $\mathbf{\Gamma}$ be a finite starbase fractal structure over X such that $d_{\mathbf{\Gamma}}$ is point-symmetric, let $(Y, \mathbf{\Delta})$ be a half complete Hausdorff GF-space, and let $f\colon X \to Y$ be a quasi-uniformly continuous map. Then there exists a continuous map $F : G(X, \mathbf{\Gamma}) \to Y$ such that $F|_X = f$.*

Proof. Let (i_n) and f_n be as in Proposition 3.9.

Define F as follows. Let $g = (\rho_n(x_n)) \in G(X, \mathbf{\Gamma})$, then, since f is quasi-uniformly continuous, by Proposition 3.9 we have that $\phi_{n+1}(\rho_{n+1}(f(x_{i_{n+1}}))) = \rho_n(f(x_{i_{n+1}})) = f_n(\rho_{i_n}(x_{i_{n+1}})) = f_n(\rho_{i_n}(x_{i_n})) = \rho_n(f(x_{i_n}))$, and hence it follows that $(\rho_n(f(x_{i_n}))) \in \varprojlim G_n(Y)$. Since Y is half complete there exists $z = z(g) \in Y$ such that $\rho_n(z) \leq_n \rho_n(f(x_n))$. We define $F(g) = z$.

Let us prove that F is well defined.

Let $y \in Y$ with $y \neq z$, and such that $\rho_n(y) \leq_n \rho_n(f(x_{i_n}))$. Then $f(x_{i_n}) \in U_{zn} \cap U_{yn}$ for all $n \in \mathbb{N}$, which contradicts that Y is a Hausdorff space. Therefore F is well defined.

Let us check that $F(U_{gi_n}) \subseteq \mathrm{St}(F(g), G(\Delta_n))$ for all $g \in G(X, \mathbf{\Gamma})$ and hence, since $G(\mathbf{\Gamma})$ is starbase (by Theorem 5.6), F is continuous.

Let $h \in U_{gi_n}$, with $g = (\rho_k(x_k))$ and $h = (\rho_k(y_k))$, then $\rho_{i_n}(x_{i_n}) \leq_{i_n} \rho_{i_n}(y_{i_n})$, and hence, since f is quasi-uniformly continuous and Proposition 3.9, we have that $\rho_n(F(g)) \leq_n \rho_n(f(x_{i_n})) \leq_n \rho_n(f(y_{i_n}))$ and since $\rho_n(F(h)) \leq_n \rho_n(f(y_{i_n}))$, then $F(h) \in \mathrm{St}(F(g), G(\Delta_n))$.

On the other hand, if $x \in X$, then $f(x) \in Y$ and then $F(\rho(x)) = f(x)$ (note that $F(\rho(x)) \leq \rho(f(x)))$, and hence F is an extension of f. □

Corollary 6.2. *Let $\mathbf{\Gamma}$ be a finite fractal structure over X such that $d_{\mathbf{\Gamma}}$ is point-symmetric, let $(Y, \mathbf{\Delta})$ be a compact Hausdorff GF-space, and let $f\colon X \to Y$ be a quasiuniformly continuous map. Then there exists a continuous map $F : G(X, \mathbf{\Gamma}) \to Y$ such that $F|_X = f$.*

Proof. It is clear that if Y is a compact Hausdorff space then $G(\mathbf{\Delta})$ is starbase (by Theorem 5.6) and Y is half complete, so we can apply the previous theorem. □

In the following result the map between two GF-spaces is extended to a map between both GF-compactifications.

Corollary 6.3. *Let $(X, \mathbf{\Gamma})$ and $(Y, \mathbf{\Delta})$ be finite GF-spaces such that $d_{\mathbf{\Gamma}}$ and $d_{\mathbf{\Delta}}$ are point-symmetric, with $G(Y, \mathbf{\Delta})$ Hausdorff, and let $f\colon X \to Y$ be a quasiuniformly continuous map. Then there exists a continuous map $F : G(X, \mathbf{\Gamma}) \to G(Y, \mathbf{\Delta})$ such that $F|_X = f$.*

Proof. We apply the above corollary to the map $f : X \to G(Y, \mathbf{\Delta})$. □

The next proposition shows that any continuous map can be made a GF-map for some fractal structures, which will have additional properties depending on the properties of the spaces.

Proposition 6.4. *Let $(X, \mathbf{\Gamma}')$ and $(Y, \mathbf{\Delta})$ be GF-spaces, and let $f\colon X \to Y$ be a continuous function. Then there exists a fractal structure $\mathbf{\Gamma}$ over X such that f is a GF-map. Moreover if $\mathbf{\Gamma}'$ is starbase then $\mathbf{\Gamma}$ is starbase and if $\mathbf{\Gamma}'$ and $\mathbf{\Delta}$ are finite then $\mathbf{\Gamma}$ is finite too.*

Proof. Let $\Gamma_n = \{A'_n \cap f^{-1}(B_n) : A'_n \in \Gamma'_n; B_n \in \Delta_n\}$. Let us prove that $\mathbf{\Gamma}$ is a fractal structure over X.

It is clear that Γ_n is a closed covering, that Γ_{n+1} is a refinement of Γ_n and that for all $x \in A_n$ with $A_n \in \Gamma_n$ there exists $A_{n+1} \in \Gamma_{n+1}$ such that $x \in A_{n+1} \subseteq A_n$. Let us show that $U_{xn} \subseteq U'_{xn}$ for all $x \in X$, or what is the same, that $U_{xn}^{-1} \subseteq (U'_{xn})^{-1}$ for all $x \in X$. Let $y \in U_{xn}^{-1}$, and let $A'_n \in \Gamma'_n$ be such that $x \in A'_n$. Let $B_n \in \Delta_n$ be such that $f(x) \in B_n$. Then it is clear that $x \in A'_n \cap f^{-1}(B_n)$, and since $y \in U_{xn}^{-1} = \bigcap_{x \in A_n} A_n$, then $y \in A'_n \cap f^{-1}(B_n) \subseteq A'_n$. Then $y \in \bigcap_{x \in A'_n} A'_n = (U'_{xn})^{-1}$. Therefore $\mathbf{\Gamma}$ is a fractal structure over X.

Let us prove that $U_{xn} = U'_{xn} \cap f^{-1}(U_{f(x)n})$, and hence open.

Let $y \in U_{xn}$, then $y \in U'_{xn}$ by the previous paragraph. Let $B_n \in \Delta_n$ be such that $f(y) \in B_n$. Then $y \in f^{-1}(B_n)$. Let $A'_n \in \Gamma'_n$ be such that $y \in A'_n$ and let $A_n = A'_n \cap f^{-1}(B_n)$. Then it is clear that $y \in A_n$, and since $x \in U_{yn}^{-1}$ we have that $x \in A_n \subseteq f^{-1}(B_n)$, and hence $f(x) \in B_n$. Therefore $f(x) \in \bigcap_{f(y) \in B_n} B_n = U_{f(y)n}^{-1}$, and then $f(y) \in U_{f(x)n}$, and hence $y \in f^{-1}(U_{f(x)n})$.

Conversely, let $y \in U'_{xn} \cap f^{-1}(U_{f(x)n})$. Let $A_n = A'_n \cap f^{-1}(B_n)$ be such that $y \in A'_n$, then $y \in A'_n$, and since $x \in (U'_{yn})^{-1} = \bigcap_{y \in C'_n} C'_n$ we have that $x \in A'_n$.

On the other hand, $y \in f^{-1}(B_n)$ and then $f(y) \in B_n$. Since $f(x) \in U_{f(y)n}^{-1} = \bigcap_{f(y)\in D_n} D_n$ we have that $f(x) \in B_n$, and then $x \in f^{-1}(B_n)$, and hence $x \in A'_n \cap f^{-1}(B_n) = A_n$. Therefore $x \in \bigcap_{y\in A_n} A_n = U_{yn}^{-1}$, and hence $y \in U_{xn}$.

Let us prove that f is a GF-map.

Let $y \in U_{xn}^{-1}$, and let $B_n \in \Delta_n$ be such that $f(x) \in B_n$; then $x \in f^{-1}(B_n)$. Let $A'_n \in \Gamma'_n$ be such that $x \in A'_n$. Then $x \in A'_n \cap f^{-1}(B_n)$, and since $y \in U_{xn}^{-1} = \bigcap_{x\in A_n} A_n$, we have that $y \in A'_n \cap f^{-1}(B_n) \subseteq f^{-1}(B_n)$ and hence $f(y) \in B_n$. Then $f(y) \in \bigcap_{f(x)\in B_n} B_n = U_{f(x)n}^{-1}$. Therefore $f(U_{xn}) \subseteq U_{f(x)n}$ and f is a GF-map.

Finally note that $\mathrm{St}(x, \Gamma_n) \subseteq \mathrm{St}(x, \Gamma'_n)$, and hence $\mathbf{\Gamma}$ is starbase if $\mathbf{\Gamma}'$ is. It is also clear that if $\mathbf{\Gamma}'$ and $\mathbf{\Delta}$ are finite, so is $\mathbf{\Gamma}$. $\square$

If both spaces have better properties, we can get the following two improvements.

Corollary 6.5. *Let X be a separable metrizable space, let Y be a second countable space and let $f\colon X \to Y$ be a continuous map. Then there exist a finite starbase fractal structure over X and a finite fractal structure over Y such that f is a GF-map.*

Proof. Since X is a separable metrizable space, then it admits a finite starbase fractal structure (by Lemma 4.19), and since Y is a second countable space, then it admits a finite fractal structure (by Lemma 4.2), and hence we can apply the previous proposition. $\square$

Corollary 6.6. *Let X be a separable metrizable space, let Y be a compact metrizable space and let $f\colon X \to Y$ be a continuous map. Then there exists a second countable T_1 compactification $K(X)$ of X and a continuous map $F : K(X) \to Y$ such that $F|X = f$.*

Proof. Let $\mathbf{\Gamma}'$ and $\mathbf{\Delta}$ be finite starbase fractal structures over X and Y respectively. Then, by the previous proposition there exists a finite starbase fractal structure $\mathbf{\Gamma}$ over X, such that f is a GF-map, and then, by Proposition 6.1, there exists $F : K(X) \to Y$ an extension of f, where $K(X) = G(X, \mathbf{\Gamma})$ is a second countable T_1 compactification of X. $\square$

7. GF-compactifications as Wallman's compactifications

Let $\mathbf{\Gamma}$ be a finite fractal structure over X with $d_{\mathbf{\Gamma}}$ being point symmetric (for example if it is starbase). We define $\mathcal{L} = \mathcal{L}(\mathbf{\Gamma}) = \{\bigcup_{i\in I}\bigcap_{j\in J} A^{ij}_{n_{ij}} : A^{ij}_{n_{ij}} \in \Gamma_{n_{ij}}; I, J \text{ finite sets}\}$. It is clear that it is a lattice. On the other hand, since $\{U_{xn} : x \in X; n \in \mathbb{N}\}$ is an open base for X, then $\{X \setminus U_{xn} : x \in X; n \in \mathbb{N}\}$ is a closed base for X, and since $X \setminus U_{xn} = \bigcup_{x\notin A_n} A_n \in \mathcal{L}$, then $\mathcal{L}$ is a closed base for X. Since X is T_0, then $\mathcal{L}$ is a β-lattice. To check that it is an α-lattice, let $x \in X$ and let $L \in \mathcal{L}$ such that $x \notin L$. Since L is closed, and since $d_{\mathbf{\Gamma}}$ is point symmetric, then there exists $n \in \mathbb{N}$ such that $U_{xn}^{-1} \subseteq X \setminus L$. Since $U_{xn}^{-1} = \bigcap_{x\in A_n} A_n \in \mathcal{L}$, then $\mathcal{L}$ is an α-lattice.

Therefore $W(X,\mathcal{L})$, the Wallman's compactification associated with $\mathcal{L}$, is a T_1 compactification of X. Let $\Delta_n = \{B_{A_n} : A_n \in \Gamma_n\}$.

Theorem 7.1. *$\mathbf{\Delta}$ is a finite fractal structure over $W(X,\mathcal{L})$.*

Proof. It is clear that Δ_n is a finite closed covering for all $n \in \mathbb{N}$, and since $B_{L_1 \cup L_2} = B_{L_1} \cup B_{L_2}$ we have that $B_{A_n} = \bigcup_{A_{n+1} \subseteq A_n} B_{A_{n+1}}$. Let $\mathcal{F}$ be a $\mathcal{L}$-ultrafilter such that $\mathcal{F} \notin B_L$. Let $L = \bigcup_{i\in I} \bigcap_{j \in J} A^{ij}_{n_{ij}}$, and let $n = \max\{n_{ij} : i \in I;\ j \in J\}$.

Let us prove that $U_{\mathcal{F}n} \subseteq W(X,\mathcal{L}) \setminus B_L$ (to avoid confusion, note that $U_{\mathcal{F}n}$ is nothing new; it is the usual U_{xn}, only that here x is $\mathcal{F}$). Let $\mathcal{G}$ be an ultrafilter such that $\mathcal{G} \in U_{\mathcal{F}n}$ and suppose that $\mathcal{G} \in B_L$. Since $B_L = \bigcap_{j\in J} \bigcup_{i \in I} B_{A^{ij}_{n_{ij}}}$, then for all $j \in J$, there exists $i = i(j) \in I$ such that $\mathcal{G} \in B_{A^{ij}_{n_{ij}}}$. Then, since $\mathcal{F} \in U^{-1}_{\mathcal{G}n} = \bigcap_{\mathcal{G} \in B_{A_n}} B_{A_n}$, we have that $\mathcal{F} \in B_{A^{ij}_{n_{ij}}}$ for all $j \in J$, and then $\mathcal{F} \in \bigcap_{j \in J} \bigcup_{i\in I} B_{A^{ij}_{n_{ij}}} = B_L$. The contradiction proves the desired result.

Therefore $\mathbf{\Delta}$ is a finite fractal structure over $W(X,\mathcal{L})$. □

We will denote $W(X,\mathcal{L})$ by $W(X,\mathbf{\Gamma})$ hereafter. Note that $W(X,\mathbf{\Gamma})$ is a T_1 second countable (since it is a finite GF-space) Wallman compactification of X (see [**9**]).

Remark 7.2. *Note that $\mathcal{B}_{A_n} = Cl_{W(X,\mathbf{\Gamma})} A_n$ and $\mathcal{B}_{A_n} \cap X = A_n$, and hence $\mathbf{\Delta}_{|X} = \mathbf{\Gamma}$.*

Theorem 7.3. *Let $\mathbf{\Gamma}$ be a finite fractal structure over X such that the associated quasiuniformity is point symmetric. Then $G(X,\mathbf{\Gamma})$ is GF-isomorphic to $W(X,\mathbf{\Gamma})$.*

Proof. Let us prove that $U^*_{\mathcal{F}n} \cap X \neq \emptyset$.

$U^*_{\mathcal{F}n} = \bigcap_{A_n \in \mathcal{F}} \mathcal{B}_{A_n} \setminus \bigcup_{A_n \notin \mathcal{F}} \mathcal{B}_{A_n}$, and hence $\bigcap_{A_n\in\mathcal{F}} A_n \in \mathcal{F}$ and $\bigcup_{A_n \notin \mathcal{F}} A_n \notin \mathcal{F}$, and then $\bigcap_{A_n\in\mathcal{F}} A_n \setminus \bigcup_{A_n\notin\mathcal{F}} A_n \neq \emptyset$ (if $\bigcap_{A_n\in\mathcal{F}} A_n \setminus \bigcup_{A_n\notin\mathcal{F}} A_n = \emptyset$, then $\bigcap_{A_n\in\mathcal{F}} A_n \subseteq \bigcup_{A_n\notin\mathcal{F}} A_n$ and hence $\bigcup_{A_n\notin\mathcal{F}} A_n \in \mathcal{F}$). Let $x \in \bigcap_{A_n\in\mathcal{F}} A_n \setminus \bigcup_{A_n\notin\mathcal{F}} A_n$. Then $\mathcal{F}_x \in U^*_{\mathcal{F}n}$ (where $\mathcal{F}_x$ is the $\mathcal{L}$-ultrafilter generated by x).

Let $g_n : G_n(X) \to G_n(W(X,\mathbf{\Gamma}))$ be defined by $g_n(U^*_{xn}) = U^*_{\mathcal{F}_x n}$. Let us show that g_n is a poset isomorphism.

Note first that $\mathcal{F}_y \in B_{A_n}$ if and only if $A_n \in \mathcal{F}_y$, or equivalently $y \in A_n$. Then $y \in U_{xn}$ if and only if $x \in U^{-1}_{yn} = \bigcap_{y \in A_n} A_n$, or what is the same, $\mathcal{F}_x \in \bigcap_{\mathcal{F}_y \in B_{A_n}} B_{A_n}$, that is, $\mathcal{F}_y \in U_{\mathcal{F}_x n}$. From this equivalence we can deduce that g_n is well defined, injective and order-preserving. Since we have proved that $U^*_{\mathcal{F}n} \cap X \neq \emptyset$, then it is clear that g_n is surjective. Therefore g_n is a poset isomorphism. Moreover, it holds that $\phi_{n+1} \circ g_{n+1}(U^*_{xn+1}) = \phi_{n+1}(U_{\mathcal{F}_x n+1}) = U_{\mathcal{F}_x n} = g_n(U^*_{xn}) = g_n \circ \phi_{n+1}(U^*_{xn+1})$, and hence $\phi_{n+1} \circ g_{n+1} = g_n \circ \phi_{n+1}$.

Therefore, by Proposition 3.8, $G(X,\mathbf{\Gamma})$ is GF-isomorphic to $W(X,\mathbf{\Gamma})$. □

Our final result is the classical theorem due to Aarts (see [**1**]), and can be obtained by applying our techniques. This was a major breakthrough in the problem of finding if all Hausdorff compactification are of Wallman type, posed by Frink in [**11**] and finally solved in the negative by Uljanov in [**18**].

Corollary 7.4. *All metrizable compactifications of any (separable metrizable) space are of Wallman type.*

Proof. It follows from Theorem 4.20 and the previous theorem. □

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