NON-ORIENTABLE BIEMBEDDINGS OF STEINER TRIPLE SYSTEMS OF ORDER 15

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ABSTRACT. It is shown that each possible pair of the 80 isomorphism classes of Steiner triple systems of order 15 may be realized as the colour classes of a face 2-colourable triangulation of the complete graph in a non-orientable surface. This supports the conjecture that every pair of STS(n)s, $n \ge 9$, can be biembedded in a non-orientable surface.

1. Introduction

It is well known that the complete graph K_n has a triangulation in an orientable surface if and only if $n \equiv 0, 3, 4$ or 7 (mod 12) and in a non-orientable surface if and only if $n \equiv 0$ or 1 (mod 3), $n \neq 3, 4, 7, [11]$. In either case the set of faces forms a twofold triple system of order n, TTS(n) for short, i.e. a collection of triples (the triangles) which have the property that every pair (the edges) is contained in precisely two triples. We say that the twofold triple system is embedded in the surface. From a design theoretic perspective a natural question to ask is which TTS(n)s can be so embedded? The answer is straightforward. Let V be the base set upon which the TTS(n) is defined. For each $x \in V$, define a neighbourhood graph G_x . The vertex set of G_x is $V \setminus \{x\}$ and two vertices y, zare joined by an edge if $\{x, y, z\}$ is a triple of the system. Clearly G_x is a union of disjoint cycles. A TTS(n) occurs as a triangulation of a surface if and only if every neighbourhood graph $G_x, x \in V$, consists of a single cycle, [5]. When this condition is not satisfied, sewing together the triangles of the TTS(n) results in a pseudo-surface. If the triples of the TTS(n) can be cyclically ordered so that every ordered pair of distinct elements of V is contained in precisely one cyclically ordered triple then the surface is orientable.

In any triangulation of K_n , the number of faces around each vertex is n-1. Hence if n-1 is even, i.e. if $n \equiv 3$ or 7 (mod 12) in the orientable case and if $n \equiv 1$ or 3 (mod 6) in the non-orientable case, it may be possible to colour each face using one of two colours, say black or white, so that no two faces of the same colour are adjacent. We say that the triangulation is (properly) face 2-colourable. Such triangulations in an orientable surface are known to exist for all $n \equiv 3$ or 7 (mod 12), [11], [12]. Surprisingly, for non-orientable surfaces the spectrum of

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n for which there exists a face 2-colourable triangulation of K_n has only recently been determined. Additional constructions given in [11] show that these exist for $n \equiv 3 \pmod{6}$, $n \geq 9$. A recent paper, [7], proves the corresponding existence result for $n \equiv 1 \pmod{6}$, $n \geq 13$.

Given a face 2-colourable triangulation of K_n , the set of faces of each colour class forms a *Steiner triple system of order* n, STS(n) for short, i.e. collection of triples which have the property that every pair is contained in precisely one triple. We say that each STS(n) is embedded, and that the pair of STS(n)s is biembedded in the surface. It has been known for over 150 years, [9], that an STS(n) exists if and only if $n \equiv 1$ or 3 (mod 6), see also [4]. We are led to the following questions.

- 1. Which STS(n)s can be so embedded in an orientable (respectively non-orientable) surface? In particular, in the non-orientable case can every STS(n) be embedded?
- 2. Which pairs of STS(n)s can be embedded in an orientable (respectively non-orientable) surface?

The latter question needs clarification. Clearly an arbitrary pair of labelled STS(n)s will not, in general, be biembeddable; they may for example have a common triple. But this is not the spirit of the question. The triples of one of the Steiner triple systems can be thought of as being fixed and forming the black triangles of a possible biembedding. The question is then whether there exists a permutation of the points of the other STS(n) so that the resulting triples form the white triangles.

Both questions appear to be very difficult to answer; they may in fact be well beyond current methods. Further, it seems difficult even to make a reasonable conjecture. In this paper we will be interested mainly in the second question and the non-orientable case. However we first review the relevant results about Steiner triple systems and what is known in the orientable case.

2. Orientable biembeddings

The numbers of non-isomorphic STS(n)s for n=3,7,9,13 and 15 are known; there are 1,1,1,2 and 80 respectively, [10]. Indeed the number of non-isomorphic STS(19)s has also recently been determined; there are altogether 11,084,874,829 of them, [8]. The case n=3 is trivial; there is a unique biembedding of the system with itself in the sphere, with automorphism group S_3 of order 6. The case n=7 is less trivial, but well-known; there is a unique biembedding of the system with itself in the torus, with automorphism group AGL(1,7) of order 42. We include as automorphisms all mappings that either exchange the colour classes or reverse the orientation. The next case to consider is n=15. Using the standard numbering of the STS(15)s as in [10], it is known that there exist orientable biembeddings of the systems \$\pm\$1, \$\$\pm\$76 and \$\$\pm\$80 with themselves, [2]. These are the only three of the 80 STS(15)s to have an automorphism of order 5 and the biembeddings can be obtained from index 3 current graphs. System \$\$\pm\$1 is the point-line design of the projective geometry PG(3,2) and it was shown in [1] that, up to isomorphism, there is precisely one orientable face 2-colourable triangular embedding of K_{15} in

which both the black and the white systems are isomorphic to system $\sharp 1$. The only other result that appears to be known is that there is no orientable biembedding of system $\sharp 1$ with system $\sharp 2$ (the STS(15) obtained from system $\sharp 1$ by a Pasch switch i.e. replacing any Pasch configuration: $\{a,b,c\},\{a,y,z\},\{x,b,z\},\{x,y,c\}$ with its "opposite": $\{x,y,z\},\{x,b,c\},\{a,y,c\},\{a,b,z\}\}$, [3]. Hence, in answer to question 2, not every pair of STS(n)s, $n \equiv 3$ or 7 (mod 12), can be biembedded in an orientable surface although much further investigation is needed before any reasonable conjecture can be framed. In the next section we turn our attention to the non-orientable case.

3. Non-orientable biembeddings

As with the orientable situation, the case n=3 is trivial. There is no nonorientable biembedding of the system with itself. It is also well-known that there is no biembedding of the STS(7) with itself in the Klein bottle (the surface with non-orientable genus = 2). For n = 9, there is a unique biembedding of the system with itself in a non-orientable surface of genus 5. The automorphism group is $C_3 \times S_3$ of order 18. Permutations of odd order stabilize the colour classes and those of even order exchange the colour classes, [6]. Recently, two of the present authors and M. Knor have enumerated the face two-colourable triangulations of K_{13} , [6]. One of the two STS(13)s has a cyclic automorphism and we denote this system by C. The other STS(13) is non-cyclic and may be obtained from C by a Pasch switch as described above; we denote this system by N. Summarizing the results from [6] there are 615 non-isomorphic biembeddings of C with C, 8,539 non-isomorphic biembeddings of C with N and 29,454 biembeddings of Nwith N. However the only known non-orientable results for STS(15)s are that there are three non-isomorphic biembeddings of system #1 with system #1, and five non-isomorphic biembeddings of system \$1 with system \$2, [1], [3]. Clearly therefore, a systematic investigation of the biembeddability of all 80 systems would be of considerable interest. In this paper we report our result that every pair of STS(15)s can be biembedded in a non-orientable surface. As a consequence of this we believe that there is now sufficient evidence to state the following conjecture. Conjecture Every pair of STS(n)s, $n \ge 9$, can be biembedded in a non-orientable surface.

4. Methodology

The algorithm for determining whether a pair of STS(15)s, say A and B, can be biembedded in a non-orientable surface is essentially straightforward. First, representations of systems A and B on the same base set V are chosen; in practice the listings given in [10] on the set $\{1, 2, ..., 15\}$ were used. System A is now held fixed with its triples forming the black triangles of a possible biembedding. Permutations of the base set V are then considered in turn and applied to system B. If Π is any such permutation, we test whether the triples of system $\Pi(B)$ can

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Table 1. Representative permutations that yield a biembedding of each STS(15) with itself

form the white triangles. This is easily done as described in the Introduction. The pair of systems A and $\Pi(B)$ can be biembedded if and only if, when they are considered as a TTS(15), every neighbourhood graph $G_x, x \in V$, consists of a single cycle.

Each permutation is recorded as a linear array $\Pi(1), \Pi(2), \ldots, \Pi(15)$ and in this representation the permutations are considered in lexicographical order. An elementary observation is that if system A can be biembedded with system $\Pi(B)$ then their sets of triples are disjoint. Consequently, any permutation which results in a common triple can be rejected. In practice, the first 50,000 permutations without a common triple were stored and then tested as described above. In all but one case a biembedding was found and the search terminated. In the remaining case,(system \$1\$ with system \$4\$), it was necessary to take a second and a third batch of 50,000 permutations before a biembedding was found. However, in comparison with 15! possible permutations these are small batches and the implication is that there are many biembeddings of each pair.

5. Results

It is both inappropriate and infeasible to list here $(80\times81)/2=3240$ biembeddings, representing the pairs of STS(15)s. These are available from the authors, and will appear in the first author's Ph.D. thesis. However, as a representative sample we give the biembeddings of each of the 80 systems with itself. In order to do this we take the representation of each system as given in [10] as the black system and specify the permutation which is applied to this to give the white system. The permutations $\Pi_i, 1 \leq i \leq 80$ are recorded as linear arrays $\Pi_i(1), \Pi_i(2), \ldots, \Pi_i(15)$ and are given in the Table 1 below. For succinctness, we write 10 = A, 11 = B, 12 = C, 13 = D, 14 = E, 15 = F.

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