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NON-ORIENTABLE BIEMBEDDINGS OF STEINER TRIPLE SYSTEMS OF ORDER 15

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Abstract. It is shown that each possible pair of the 80 isomorphism classes of Steiner triple systems of order 15 may be realized as the colour classes of a face 2-colourable triangulation of the complete graph in a non-orientable surface. This supports the conjecture that every pair of STS(n)s, $n \geq 9$, can be biembedded in a non-orientable surface.

1. Introduction

It is well known that the complete graph K_n has a triangulation in an orientable surface if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ and in a non-orientable surface if and only if $n \equiv 0$ or $1 \pmod 3$, $n \neq 3, 4, 7$, [**11**]. In either case the set of faces forms a *twofold triple system of order* n, TTS(n) for short, i.e. a collection of triples (the triangles) which have the property that every pair (the edges) is contained in precisely two triples. We say that the twofold triple system is embedded in the surface. From a design theoretic perspective a natural question to ask is which TTS(n)s can be so embedded? The answer is straightforward. Let V be the base set upon which the TTS(n) is defined. For each $x \in V$, define a *neighbourhood graph* G_x. The vertex set of G_x is $V \setminus \{x\}$ and two vertices y, z are joined by an edge if $\{x, y, z\}$ is a triple of the system. Clearly G_x is a union of disjoint cycles. A TTS(n) occurs as a triangulation of a surface if and only if every neighbourhood graph $G_x, x \in V$, consists of a single cycle, [**5**]. When this condition is not satisfied, sewing together the triangles of the TTS(n) results in a pseudo-surface. If the triples of the TTS(n) can be cyclically ordered so that every ordered pair of distinct elements of V is contained in precisely one cyclically ordered triple then the surface is orientable.

In any triangulation of K_n, the number of faces around each vertex is $n-1$. Hence if $n-1$ is even, i.e. if $n \equiv 3$ or $7 \pmod{12}$ in the orientable case and if $n \equiv 1$ or $3 \pmod 6$ in the non-orientable case, it may be possible to colour each face using one of two colours, say black or white, so that no two faces of the same colour are adjacent. We say that the triangulation is *(properly) face 2-colourable.* Such triangulations in an orientable surface are known to exist for all $n \equiv 3$ or $7 \pmod{12}$, [**11**], [**12**]. Surprisingly, for non-orientable surfaces the spectrum of

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n for which there exists a face 2-colourable triangulation of K_n has only recently been determined. Additional constructions given in [**11**] show that these exist for $n \equiv 3 \pmod 6$, $n \geq 9$. A recent paper, [**7**], proves the corresponding existence result for $n \equiv 1 \pmod 6$, $n \geq 13$.

Given a face 2-colourable triangulation of K_n, the set of faces of each colour class forms a *Steiner triple system of order* n, STS(n) for short, i.e. collection of triples which have the property that every pair is contained in precisely one triple. We say that each STS(n) is embedded, and that the pair of STS(n)s is biembedded in the surface. It has been known for over 150 years, [**9**], that an STS(n) exists if and only if $n \equiv 1$ or $3 \pmod 6$, see also [**4**]. We are led to the following questions.

1. Which STS(n)s can be so embedded in an orientable (respectively non-orientable) surface? In particular, in the non-orientable case can every STS(n) be embedded?
2. Which pairs of STS(n)s can be embedded in an orientable (respectively non-orientable) surface?

The latter question needs clarification. Clearly an arbitrary pair of labelled STS(n)s will not, in general, be biembeddable; they may for example have a common triple. But this is not the spirit of the question. The triples of one of the Steiner triple systems can be thought of as being fixed and forming the black triangles of a possible biembedding. The question is then whether there exists a permutation of the points of the other STS(n) so that the resulting triples form the white triangles.

Both questions appear to be very difficult to answer; they may in fact be well beyond current methods. Further, it seems difficult even to make a reasonable conjecture. In this paper we will be interested mainly in the second question and the non-orientable case. However we first review the relevant results about Steiner triple systems and what is known in the orientable case.

2. Orientable biembeddings

The numbers of non-isomorphic STS(n)s for $n = 3, 7, 9, 13$ and 15 are known; there are $1, 1, 1, 2$ and 80 respectively, [**10**]. Indeed the number of non-isomorphic STS(19)s has also recently been determined; there are altogether 11,084,874,829 of them, [**8**]. The case $n = 3$ is trivial; there is a unique biembedding of the system with itself in the sphere, with automorphism group S_3 of order 6. The case $n = 7$ is less trivial, but well-known; there is a unique biembedding of the system with itself in the torus, with automorphism group $\mathrm{AGL}(1, 7)$ of order 42. We include as automorphisms all mappings that either exchange the colour classes or reverse the orientation. The next case to consider is $n = 15$. Using the standard numbering of the STS(15)s as in [**10**], it is known that there exist orientable biembeddings of the systems $\sharp 1$, $\sharp 76$ and $\sharp 80$ with themselves, [**2**]. These are the only three of the 80 STS(15)s to have an automorphism of order 5 and the biembeddings can be obtained from index 3 current graphs. System $\sharp 1$ is the point-line design of the projective geometry $\mathrm{PG}(3, 2)$ and it was shown in [**1**] that, up to isomorphism, there is precisely one orientable face 2-colourable triangular embedding of K_{15} in which both the black and the white systems are isomorphic to system ♯1. The only other result that appears to be known is that there is no orientable biembedding of system ♯1 with system ♯2 (the STS(15) obtained from system ♯1 by a *Pasch switch* i.e. replacing any Pasch configuration: $\{a,b,c\},\{a,y,z\},\{x,b,z\},\{x,y,c\}$ with its "opposite": $\{x,y,z\},\{x,b,c\},\{a,y,c\},\{a,b,z\}$), [**3**]. Hence, in answer to question 2, not every pair of STS(n)s, $n \equiv 3$ or 7 (mod 12), can be biembedded in an orientable surface although much further investigation is needed before any reasonable conjecture can be framed. In the next section we turn our attention to the non-orientable case.

3. Non-orientable biembeddings

As with the orientable situation, the case $n = 3$ is trivial. There is no non-orientable biembedding of the system with itself. It is also well-known that there is no biembedding of the STS(7) with itself in the Klein bottle (the surface with non-orientable genus = 2). For $n = 9$, there is a unique biembedding of the system with itself in a non-orientable surface of genus 5. The automorphism group is $C_3 \times S_3$ of order 18. Permutations of odd order stabilize the colour classes and those of even order exchange the colour classes, [**6**]. Recently, two of the present authors and M. Knor have enumerated the face two-colourable triangulations of K_{13}, [**6**]. One of the two STS(13)s has a cyclic automorphism and we denote this system by C. The other STS(13) is non-cyclic and may be obtained from C by a Pasch switch as described above; we denote this system by N. Summarizing the results from [**6**] there are 615 non-isomorphic biembeddings of C with C, 8,539 non-isomorphic biembeddings of C with N and 29,454 biembeddings of N with N. However the only known non-orientable results for STS(15)s are that there are three non-isomorphic biembeddings of system ♯1 with system ♯1, and five non-isomorphic biembeddings of system ♯1 with system ♯2, [**1**], [**3**]. Clearly therefore, a systematic investigation of the biembeddability of all 80 systems would be of considerable interest. In this paper we report our result that every pair of STS(15)s can be biembedded in a non-orientable surface. As a consequence of this we believe that there is now sufficient evidence to state the following conjecture.

Conjecture Every pair of STS(n)s, $n \geq 9$, can be biembedded in a non-orientable surface.

4. Methodology

The algorithm for determining whether a pair of STS(15)s, say A and B, can be biembedded in a non-orientable surface is essentially straightforward. First, representations of systems A and B on the same base set V are chosen; in practice the listings given in [**10**] on the set $\{1,2,\ldots,15\}$ were used. System A is now held fixed with its triples forming the black triangles of a possible biembedding. Permutations of the base set V are then considered in turn and applied to system B. If Π is any such permutation, we test whether the triples of system $\Pi(B)$ can

System	Permutation	System	Permutation
1	12436785CBDE9AF	41	124365BADE978CF
2	1243678C5BE9ADF	42	124367E5CDB9A8F
3	12436785CBDE9AF	43	1243685ADE7B9CF
4	1243678D59EACBF	44	124367A9D5E8BCF
5	1243678BCA9E5DF	45	124367D5A8BEC9F
6	124368C7EAD5B9F	46	1243659DBE8C7AF
7	1243678CEA9DB5F	47	124365DACE98B7F
8	1243678EDACB95F	48	124365B9DCA8E7F
9	1243678EDACB95F	49	1243678ADE5C9BF
10	1243679EDACB58F	50	12436785D9BECAF
11	124367CBDE8A95F	51	124367BADC9E85F
12	1243679A5CED8BF	52	124365CDA9E87BF
13	1243678BCED9A5F	53	12436785ADBEC9F
14	1243678AEB5D9CF	54	1243658DA9E7CBF
15	124367A59ED8CBF	55	12436587BEA9CDF
16	124368D5AE9BC7F	56	124365BCA9E87DF
17	124368B5DE9AC7F	57	124368D5CB9E7AF
18	124367EC9B8D5AF	58	1243658DA9E7CBF
19	124367C85EDA9BF	59	1243658CBAED79F
20	1243679DA5E8CBF	60	1243659C7BDEA8F
21	124367ABD8E59CF	61	12436789B5DCEAF
22	1243678B5C9EADF	62	1243659EBDA8C7F
23	1243678DAECB95F	63	1243679D5BEA8CF
24	124369DCE78A5BF	64	12436589DEC7BAF
25	124369DCEA578BF	65	1243659C7AEDB8F
26	124369A8E5DB7CF	66	124367CD95AE8BF
27	124368CBD5E97AF	67	1243658E97ABCDF
28	12436789D5ECABF	68	1243658E7AD9CBF
29	124367B8D9ECA5F	69	1243658E7CA9BDF
30	124365CB8DEA79F	70	124365AD9EB78CF
31	124367A9DE5BC8F	71	1243659EBADC78F
32	1243659BED78ACF	72	124367EDB5C9A8F
33	1243659EBAD7C8F	73	124365AC9ED78BF
34	12436789ED5CABF	74	124365CDA9EB87F
35	124367C9DA8BE5F	75	124367C5A8DE9BF
36	124367BE8ADC95F	76	1243659ADB8E7CF
37	124365CDABE978F	77	124365C9EAD78BF
38	124365BDA97EC8F	78	1243685EAC9B7DF
39	1243678D9B5ECAF	79	1243658E7CAB9DF
40	12436785D9ECABF	80	12436589DCBEA7F

Table 1. Representative permutations that yield a biembedding of each STS(15) with itself

form the white triangles. This is easily done as described in the Introduction. The pair of systems A and $\Pi(B)$ can be biembedded if and only if, when they are considered as a TTS(15), every neighbourhood graph $G_x, x \in V$, consists of a single cycle.

Each permutation is recorded as a linear array $\Pi(1), \Pi(2), \ldots, \Pi(15)$ and in this representation the permutations are considered in lexicographical order. An elementary observation is that if system A can be biembedded with system $\Pi(B)$ then their sets of triples are disjoint. Consequently, any permutation which results in a common triple can be rejected. In practice, the first 50,000 permutations without a common triple were stored and then tested as described above. In all but one case a biembedding was found and the search terminated. In the remaining case,(system ♯1 with system ♯4), it was necessary to take a second and a third batch of 50,000 permutations before a biembedding was found. However, in comparison with 15! possible permutations these are small batches and the implication is that there are many biembeddings of each pair.

5. Results

It is both inappropriate and infeasible to list here $(80\times 81)/2 = 3240$ biembeddings, representing the pairs of STS(15)s. These are available from the authors, and will appear in the first author's Ph.D. thesis. However, as a representative sample we give the biembeddings of each of the 80 systems with itself. In order to do this we take the representation of each system as given in [**10**] as the black system and specify the permutation which is applied to this to give the white system. The permutations $\Pi_i, 1 \le i \le 80$ are recorded as linear arrays $\Pi_i(1), \Pi_i(2), \ldots, \Pi_i(15)$ and are given in the Table 1 below. For succinctness, we write 10 = A, 11 = B, 12 = C, 13 = D, 14 = E, 15 = F.

References

1. Bennett G. K., Grannell M. J. and Griggs T. S., *Bi-embeddings of the projective space* PG(3,2), J. Statist. Plann. Inference **86** (2000), 321–329.

2. Bennett G. K., M.J. Grannell and Griggs T. S., *Bi-embeddings of Steiner triple systems of order* 15, Graphs Combin. **17** (2001), 193–197.

3. Bennett G. K., M.J. Grannell and Griggs T. S., *On the bi-embeddability of certain Steiner triple systems of order* 15, European J. Combin. **23** (2002), 499–505.

4. Colbourn C. J. and Rosa A., *Triple systems*, Oxford University Press, London, 2001.

5. Ducrocq P. M. and Sterboul F., *Les G-systèmes triples*, Ann. Discrete Math. **9** (1980), 141–145.

6. Grannell M. J., Griggs T. S. and Knor M., *Face two-colourable triangulations of* K_{13}, J. Combin. Math. Combin. Comput., (to appear).

7. Grannell M. J. and Korzhik V. P., *Non-orientabl biembeddings of Steiner triple systems*, (submitted).

8. Kaski P. and Östergärd P. R. C., *The Steiner triple systems of order* 19, (submitted).

9. Kirkman T. P., *On a problem in combinations*, Cambridge and Dublin Math. J. **2** (1847), 191–204.

10. Mathon R. A., Phelps K. T. and Rosa A., *Small Steiner triple systems and their properties*, Ars Combin. **15** (1983), 3–110.

11. Ringel G., *Map color theorem*, Springer-Verlag, New York and Berlin, 1974.

12. Youngs J. W. T., *The mystery of the Heawood conjecture* in Graph Theory and its Applications, (B. Harris ed.), Academic Press, 1970, 17–50.

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