Acta Math. Univ. Comenianae
Vol. LXXIII, 1(2004), pp. 21–29

ON THE HEREDITARY k-BUCHSBAUM PROPERTY FOR IDEALS I AND $\mathbf{in}(I)$

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1. Introduction

For undefined subsequent terminology, we refer to [**5**]. Throughout $I \subseteq K[x_0,\dots,x_n] = R_{n+1}$ will be a homogeneous polynomial ideal in the polynomial ring R_{n+1} over an infinite field K. Let $\mathfrak{m} = (x_0,\dots,x_n)$. $Y = \{y_0,\dots,y_d\}$ is a system of parameters (s.o.p) for I if $\dim(I) = \text{Krull-dim}(I) = d+1$ and (I,Y) is $\mathfrak{m}$-primary. For $k \geq 0$, Y is said to be an $\mathfrak{m}^k$-weak sequence for I if

(i) $I : y_0 \subseteq I : \mathfrak{m}^k$,
(ii) $(I, y_0,\dots,y_{i-1}) : y_i \subseteq (I, y_0,\dots,y_{i-1}) : \mathfrak{m}^k, 1 \leq i \leq d$.
(For $k = 0, \mathfrak{m}^0 = R_{n+1}$.)

Definition 1.1. I is said to be k-Buchsbaum (k-Bbm), if for every s.o.p $Y = \{y_0,\dots,y_d\} \subseteq \mathfrak{m}^{2k}$ for I, the system Y is an $\mathfrak{m}^k$-weak sequence for I. If $k = 0$ then I is also said to be Cohen-Macaulay or perfect.

Remark 1.2. It suffices for a single s.o.p to be as in Definition 1.1. For this and other equivalent definitions see [**6**] and the fundamental paper by Trung [**11**].

Definition 1.3. Let $T_{n+1} \subseteq R_{n+1}$ be the set of terms (i.e. monomials with coefficient 1). An admissible term order $<$ on T_{n+1} satisfies:

(i) $1 \leq t,\ t \in T_{n+1}$,
(ii) $t_1 < t_2$ implies $tt_1 < tt_2$, $t \in T_{n+1}$.

From now on all term orders will be admissible. For $0 \neq p(x) \in R_{n+1}$, $\text{in}(p(x))$ is the largest nonzero term of $p(x)$. For the ideal $I \subseteq R_{n+1}$, $\text{in}(I)$ is the ideal generated by all $\text{in}(p(x)), p(x) \in I$.

Definition 1.4. A Gröbner basis $G = \{G_1,\dots,G_s\} \subseteq I$ for I is a generating set for I such that $(\text{in}(G_1),\dots,\text{in}(G_s)) = \text{in}(I)$.

Remark 1.5. For an algorithm to obtain G from a generating set of I see [**4**] or [**2**].

Received October 10, 2002.
2000 *Mathematics Subject Classification.* Primary 13H10; Secondary 13M10.
Key words and phrases. Admissible term order, homogeneous polynomial ideal, Gröbner basis, initial ideal, k-Buchsbaum property.

By a now classical result in [**1**], for any term order $<$, $\mathrm{in}(I)$ perfect implies I perfect and if $<$ is the reverse lexicographical term order, then the converse is obtained if $x_0 < x_1 < \dots x_d$ are the smallest linear terms and form a s.o.p for I. For almost all term orders the converse implication fails (see the discussion in [**3**]). However as a generalization of the first implication, it was shown in [**7**], that if $\mathrm{in}(I)$ is k_1-Bbm, then, for any term order $<$, I is k_2-Bbm for some k_2. The main purpose of this paper is to investigate how k_1 and k_2 are related, in particular if for a fixed k_1, k_2 can grow without bound.

This can indeed happen; in general "almost anything" can occur and thus perfect ideals I are once again true to their nomenclature. In conclusion we discuss some upper bounds for k_2 and its relation to the multiplicity $e(R_{n+1}/I)$ defined by the Hilbert polynomial. In the sequel $k_i, i \in \{1, 2\}$ will denote strict Buchsbaumness, i.e. k_i is minimal.

2. Comparisons of k_1 and k_2

Our examples and constructions are mostly for ideals I with $\dim(I) = 1$. We start with an easy but useful Lemma.

Lemma 2.1. *Assume $I \subseteq R_{n+1}$ is an ideal, $\mathfrak{J} \subseteq R_{n+1}$ is a monomial ideal and $<$ a term order. Then $in(I : \mathfrak{J}) \subseteq in(I) : \mathfrak{J}$.*

Proof. Let $F \in I : \mathfrak{J}$, $m = \mathrm{in}(F) \in \mathrm{in}(I : \mathfrak{J})$, $\bar{m} \in \mathfrak{J}$, a monomial. Since $\mathrm{in}(\bar{m}) = \bar{m}$, we have $\bar{m}m = \mathrm{in}(\bar{m}F) \in \mathrm{in}(I)$, thus $m \in \mathrm{in}(I) : \bar{m}$. From this the claim follows. $\square$

We first give an example such that $k_1 - k_2$ can become arbitrarily large.

Example 2.2. Let $I(r) = (x_0x_1^r - x_2^{r+1}, x_0^r) \subseteq R_3, r \geq 2, x_1 > x_2, x_0 > x_2$. Then $\mathrm{in}(I(r)) = (x_0x_1^r, x_0^r, x_0^{r-1}x_2^{r+1}, x_0^{r-2}x_2^{2(r+1)}, \dots, x_0x_2^{(r-1)(r+1)}, x_2^{r(r+1)})$ and $\{x_1\}$ is a s.o.p for $I(r)$ and $\mathrm{in}(I(r))$. Similarly, $\mathrm{in}(I(r)) : x_1^r = \mathrm{in}(I(r)) : x_1^{r+1}$, r minimal, $\mathrm{in}(I(r)) : x_1^r = (x_0, x_2^{r(r+1)})$. $x_0(x_0^{\alpha_0}x_1^{\alpha_1}x_2^{\alpha_2}) \in \mathrm{in}(I(r))$ iff $\alpha_0 \geq r-1$ or $\alpha_1 \geq r$ or $\alpha_0 + 1 \geq r - j$ and $\alpha_2 \geq j(r+1), 1 \leq j \leq r-1$. Therefore $\mathrm{in}(I(r))$ is k_1-Bbm, $r^2 - 1 \leq k_1 \leq (r-1) + (r) + (r^2-1) - 2 = r^2 + 2r - 4$. But $I(r)$ is k_2-Bbm with $k_2 = 0$, which is immediate by using reverse lexicographical term order with x_1 the smallest linear term (see [**5**, Proposition 15.12]). For $r = 1, k_1 = k_2 = 0$.

Proposition 2.3. *For an ideal $I \subseteq R_2 = K[x_0, x_1]$ assume:*
- (i) $x_1 > x_0$ *for some term order,*
- (ii) *without loss of generality (since K is infinite), $\{x_1\}$ is a s.o.p for I and* $\mathrm{in}(I)$,
- (iii) $\mathrm{in}(I)$ *is 1-Bbm. Then I is* 0*-Bbm or 1-Bbm*

Proof. By hypothesis
$$\mathrm{in}(I) : \mathfrak{m} \subseteq \mathrm{in}(I) : x_1 \subseteq \mathrm{in}(I) : x_1^2 \subseteq \mathrm{in}(I) : \mathfrak{m},$$
thus $\mathrm{in}(I) : x_1 = in(I) : x_1^2 = \mathrm{in}(I) : \mathfrak{m}$. Let
$$F = x_1^{n-r}x_0^r + a_{r+1}x_1^{n-r-1}x_0^{r+1} + \dots + a_nx_0^n \in I : x_1, \quad 0 \leq r \leq n.$$

Then $x_1^{n-r}x_0^r \in \mathrm{in}(I : x_1) \subseteq \mathrm{in}(I) : x_1$. Thus

$$x_0^r \in \mathrm{in}(I) : x_1^{n-r+1} = \mathrm{in}(I) : x_1,$$

from which $x_1x_0^r \in \mathrm{in}(I)$. Therefore either

a) $F \equiv 0 \mod I$ or

b) $F \equiv Ax_0^n \mod I$, $A \neq 0$ ($\equiv$ denotes reduction of F by a Gröbner basis of I).

Assume b). Since $I \subseteq I : x_1$ and $F \in I : x_1$, $x_0^n \in I : x_1$, we get

$$x_0^n \in \mathrm{in}(I : x_1) \subseteq \mathrm{in}(I) : x_1 = \mathrm{in}(I) : \mathfrak{m},$$

it follows that $x_0^{n+1} \in \mathrm{in}(I)$ (otherwise $x_0^{n+1} \notin (I)$). Hence $x_0F \in I$, thus $I : x_1 \subseteq I : \mathfrak{m}$.

Next let $F = x_1^{n-r}x_0^r + a_{r-1}x_1^{n-r-1}x_0^{r+1} + \ldots + a_nx_0^n \in I : x_1^2$. As before $x_1x_0^r \in \mathrm{in}(I)$ and either a) $F \equiv 0 \mod I$ or b) $F \equiv Ax_0^n \mod I$, $A \neq 0$, and $x_0^{n+1} \in I$. In both cases $x_1F \in I$ (for b)) since $x_1x_0^r \in \mathrm{in}(I)$ and $x_o^{n+1} \in I$), hence $I : x_1^2 \subseteq I : x_1$, thus I is either 0-Bbm or 1-Bbm. □

We obtain next a family of ideals $I(n), n \geq 2$ such that:

(1) $\{z\}$ is a s.o.p for $I(n)$ and $\mathrm{in}(I(n))$.

(2) $\mathrm{in}(I(n)) : z = \mathrm{in}(I(n)) : z^2 = \mathrm{in}(I(n)) : \mathfrak{m}$, thus $\mathrm{in}(I(n))$ is 1-Bbm (even Bbm by Proposition 2.12, Chapter I in [**10**]).

(3) $I(n) : z = I(n) : z^2 \subseteq I(n) : \mathfrak{m}^n, n$ minimal, thus $I(n)$ is strictly n-Bbm.

We assume $x_1 > x_2 > \ldots > x_n$ and for notational convenience we set $z = x_0$. s-polynomials are the successor polynomials of a Gröbner algorithm. m or $\bar{m}$ will be monomials, $\partial_{x_k}(m)$ is the degree of m with respect to x_k, $\partial(m)$ its degree.

Theorem 2.4. *Let*

$$I(n)=(z(x_1 + \ldots + x_n), M_1(n), \ldots, M_h(n), \ldots, M_n(n)),$$

be an ideal of R_{n+1}, *where*

$$M_h(n) = \{m \in R_{n+1} : z \nmid m,\ x_j \nmid m,\ 1 \leq j \leq h-1,\ x_h | m,\ \partial(m) = h+1\},$$

for $1 \leq h \leq n$. *Then* $I(n)$ *satisfies the conditions* (1), (2), *and* (3).

Proof. By construction of $I(n)$, the (1) is obtained. If $m \in \mathrm{in}(I(n))$, then $z^2 \nmid m$, thus $\mathrm{in}(I(n)) : z = \mathrm{in}(I(n)) : z^2$. $\mathrm{in}(I(n)) : z = \mathrm{in}(I(n)) : \mathfrak{m}$ iff $m \in \mathrm{in}(I(n)) : z$ implies $(x_1, \ldots, x_n)m \subseteq \mathrm{in}(I(n))$. We show that the monomial sets $M_i(n)$ have enough monomials to satisfy this requirement. Since $M_1(n)$ is as claimed, we assume it to be true for $M_j(n), 1 \leq j \leq i-1$. Assume $m \in \mathrm{in}(I(n)) : z, \partial(m) = i+1$. If $x_j | m, 1 \leq j < i, j$ minimal, then, by construction, for some $\tilde{m} \in M_j(n)$, $\tilde{m} | m$, thus m is as required. It remains to be shown that $x_i | m$ otherwise. Assuming inductively that the monomials $M_j(n), 1 \leq j \leq i-1$, are obtained from nonzero polynomials $zm_j(x_j + x_{j+1} + \ldots + x_n)$, $x_h \nmid m_j, 1 \leq h < j$, it follows that also modulo reduction the i^{th} nonzero s-polynomials are of the form $zm_i(x_i + \ldots + x_n), x_h \nmid m_i$, $1 \leq h < i$, from which the claim. Therefore (2). By construction of $M_i(n)$ and the point (2), if $zx_n^d \in \mathrm{in}(I(n))$ is of smallest degree, then $d = n$. We induct on n to show that such a monomial exists. For $n = 2$ it is true. Assume it true for $n \geq 2$

and note that $(I(n+1), x_{n+1}) = (I(n), x_{n+1})$. Therefore $\text{in}(I(n+1), x_{n+1}) = \text{in}(I(n), x_{n+1}) = (\text{in}(I(n)), x_{n+1}) \supseteq \text{in}(I(n))$.

By induction hypothesis $zx_n^n \in \text{in}(I(n))$, thus $zx_n^n \in \text{in}(I(n+i), x_{n+1})$, hence $zx_n^n \in \text{in}(I(n+1))$. From the proof of (2) we get $zx_n(x_n + x_{n+1}) \in I(n+1)$. Since $x_n x_{n+1}^n \in M_n(n+1), zx_{n+1}^{n+1} \in \text{in}(I(n+1))$, thus $M_{n+1}(n+1) = \{x_{n+1}^{n+2}\}$, which implies (3). $\square$

Remark 2.5. (0) It is possible to show that every monomial $m \in M_i(n)$ is actually obtained from a $zm \in \text{in}(I(n))$.

(1) If M_n is replaced by $M_{n+k_1} = \{x_n^{n+1+k_1}\}, k_1 \geq 1$, then for the resulting ideal $I(n, k_1), \text{in}(I(n, k_1))$ is strictly k_1-Bbm and $I(n, k_1)$ is strictly $(n+k_1)$-Bbm.

(2) For $R_{n+d} = K[z, x_1, \ldots, x_n, y_1, \ldots, y_{d-1}]$ and $I(n)$ as in Theorem 2.4, $\dim(I(n)) = d$ and (2) and (3) of Theorem 2.4 apply to $I(n)$.

For the next family of 1-dimensional ideals $I(k), k \geq 1$, we restrict ourselves to three variables, x, y, z for notational convenience. We obtain $\text{in}(I(k))$ has $k_1 = 1$, i.e. is 1-Bbm, and $I(k)$ is strictly k_2-Bbm, $k_2 = k+1$. We do not obtain the results of Remark 2.5 (1) in this case.

Theorem 2.6. *Let* $k \geq 1$,
$P_0(k) = z(x^{2k+1} + x^{(2k+1)-1}y + \ldots + xy^{2k} + y^{2k+1})$ *and* $I(k) = (P_0(k), M_k)$,
$M_k = \{x^{2k+2}, x^{2k+1}y, x^{(2k+1)-1}y^3, \ldots, x^{k+1}y^{2k+1}, xy^{2k+2}, y^{2k+3}\}$. *Assume* $x > y$. *Then:*

1. $\{z\}$ *is a s.o.p for* $I(k)$ *and* $\text{in}(I(k))$.
2. $\text{in}(I(k)) : z = \text{in}(I(k)) : z^2 = \text{in}(I(k)) : \mathfrak{m}$, *thus* $k_1 = 1$.
3. $I(k)$ *is strictly* k_2*-Bbm, and* $k_2 = k+1$.

Proof. For m and $\tilde{m}$ in M_k, we write $m < \tilde{m}$ if $\partial_y(m) < \partial_y(\tilde{m})$ (or equivalently $\partial_x(m) > \partial_x(\tilde{m})$). We proceed inductively by different steps of a Gröbner algorithm with $\rightarrow$ denoting "reduces to" and $s(F_1, F_2)$ the successor polynomial of F_1, F_2.

Step (1): $s(P_0(k), x^{2k+2}) \rightarrow P_1(k) = z(x^{(2k+1)-1}y^2 + \ldots + x^2y^{2k} + xy^{2k+1})$, $s(P_0(k), m = x^{2k+1}y) \rightarrow zy^{2k+2}$, thus $s(P_0(k), \tilde{m} > m) \rightarrow 0$ since $y^{2k+3} \in M_k$.

Step (2): $s(P_1(k), P_0(k)) \rightarrow 0$,
$s(P_1(k), m{=}x^{(2k+1)-1}y^3) \rightarrow P_2(k){=}z(x^{(2k+1)-2}y^4 + \ldots + x^2y^{2k+1})$, thus
$s(P_1(k), \tilde{m} > m) \rightarrow 0$. $s(P_1(k), \hat{m} = x^{2k+1}y) \rightarrow 0$, thus $s(P_1(k), \tilde{m} < \hat{m}) \rightarrow 0$.

Step (i), $2 \leq i < k$: Assume for $j \leq i$ we have obtained polynomials
$P_j(k) = z(x^{(2k+1)-j}y^{2j} + x^{(2k+1)-(j+1)}y^{2j+1} + \ldots + x^jy^{2k+1})$ such that for $j < i$

(i) $s(P_h(k), P_j(k)) \rightarrow 0, \quad h < j, h \neq j$.
(ii) $s(P_j(k), x^{(2k+1)-j}y^{2j+1} = m) \rightarrow P_{j+1}(k)$
$s(P_j(k), \tilde{m} > m) \rightarrow 0, s(P_j(k), \tilde{m} < m) \rightarrow 0$.

For $i > j \geq 0, i \geq j+1$, thus $2i > j+1$ or $2i - j - 1 > 0$.

Therefore

$$\begin{aligned} s(P_j(k), P_i(k)) &= y^{2i-2j}P_j(k) - x^{i-j}P_j(k) \\ &= z(x^{2i-j-1}y^{2k+2} + \ldots + x^jy^{2k+1+2(i-j)}) \rightarrow 0 \end{aligned}$$

(Note this remains true for $i = k$.)

$$s(P_i(k), m = x^{(2k+1)-i}y^{2i+1}) = zx^i y^{2k+2} + P_{i+1}(k) \to P_{i+1}(k),$$

thus $s(P_i(k), \tilde{m} > m) {\to} 0$.

$$s(P_i(k), \hat{m} = x^{(2k+1)-i+1}y^{2i-1}) = zx^{(2k+1)-i}y^{2i+1} + P_{i+1}(k) \to 0,$$

thus $s(P_i(k), \tilde{m} < \bar{m}) \to 0$. This completes the induction.

To finish the proof we calculate first $s(P_k(k), m)$, for $m \in M_k$, where $P_k(k) = z(x^{(2k+1)-k}y^{2k} + x^k y^{2k+1})$. Since

$$s(P_k(k), m = x^{k+1}y^{2k+1}) = zx^k y^{2k+2} \to 0,$$

we get $s(P_k(k), \tilde{m} > m) \to 0$.
Similarly

$$s(P_k(k), \hat{m} = x^{2k+2}y^{2k-1}) = zx^{k+1}y^{2k+1} \to 0,$$

implies $s(P_k(k), \tilde{m} < \hat{m}) {\to} 0$.
Therefore

$$\begin{aligned} \operatorname{in}(I(k)) =\{&zx^{2k+1}, zx^{(2k+1)-1}y^2, \ldots, zx^{k+1}y^{2k}, zy^{2k+2}, x^{2k+2}, x^{2k+1}y, \\ &x^{(2k+1)-1}y^3, \ldots, x^{k+1}y^{2k+1}, xy^{2k+2}, y^{2k+3}\}. \end{aligned}$$

This implies conditions 1. and 2. Also $I(k) : z = I(k) : z^2 = (P_0(k)/z, M_k)$. By [**7**], $I(k) : z^2 \subseteq I(k) : m^{k_2}$. Since $y^k(P_0(k)/z) \to x^{k+1}y^{2k} + \ldots + y^{3k}$, but $x^{k+1}y^{2k} \notin \operatorname{in}(I(k)), k_2 > k$. $k_2 = k + 1$ is readily verified. $\square$

3. Upper bounds for k_2.

We assume as before $<$ is a term order, $I \subseteq R_{n+1} = K[x_0, \ldots, x_n]$ is a homogeneous ideal, $\dim(\operatorname{in}(I)) = \dim(I) = 1$, the field K is infinite and therefore without loss of generality $\{x_0\}$ is a s.o.p for I and $\operatorname{in}(I)$. Under these assumptions $x_i^{\delta_i} \in \operatorname{in}(I), \delta_i \geq 1, \delta_i$ minimal, $1 \leq i \leq n$. Let $\mathfrak{K} = [\sum_{i=1}^{n} (\delta_i - 1)] + 1$. Let δ_0 be minimal such that $I : x_0^{\delta_0} = I : x_0^{\delta_0+1}$ and let $\nu = (\nu_1, \ldots, \nu_l), \nu_i \leq \nu_{i+1}$ be the degree vector of $I : x_0^{\delta_0}$. Assume $I = (G)$, $G = \{G_1, \ldots, G_l\}$ is a Gröbner basis of I for the term order $<$ and let $F \overset{G_i}{\to} H$ denote "G_i reduces F to H" (reduction is on the initial term).

An elementary but useful bound for k_2 follows from:

Theorem 3.1. *Assume* $\operatorname{in}(I)$ *is* k_1-*Bbm*, $k_1 \geq 1$. *Let* $L = \mathfrak{K} + (k_1 - 1) - \nu_1$. *Then* $\mathfrak{J} = \mathfrak{m}^L(I : x_0^{\delta_0}) \subseteq I$.

Proof. Let $F \in \mathfrak{J}$, then $\partial(F) = \operatorname{degree}(F) \geq \mathfrak{K} + k_1 - 1 \geq \mathfrak{K}$. Let $\operatorname{in}(F) = x_0^{\alpha_0}m$, $x_0 \nmid m$.

(i) $\alpha_0 = 0$. Since $x_i^{\delta_i} \in \operatorname{in}(I)$, there exists $G_j \in G$ such that $F \overset{G_j}{\to} F'$, $\operatorname{in}(F) > \operatorname{in}(F')$, $\partial(F_1) = \partial(F) \geq \mathfrak{K} + k_1 - 1 \geq \mathfrak{K}$.

(ii) $\alpha_0 > 0$. If $\alpha_0 < k_1$, then $\partial(m) \geq \mathfrak{K}$, therefore as in (i) $F \overset{G_i}{\to} F'$, $\operatorname{in}(F) > \operatorname{in}(F')$, $\partial(F) = \partial(F') \geq \mathfrak{K} + k_1 - 1 \geq \mathfrak{K}$.

If $\alpha_0 \geq k_1$, then, since

$$\begin{aligned} m \in \mathrm{in}(I : x_0^{\delta_0}) : x_0^{\alpha_0} \subseteq \mathrm{in}(I) : x_0^{\delta_0+\alpha_0} &= \mathrm{in}(I) : x_0^{k_1} = \mathrm{in}(I) : x_0^{k_1+1} \\ &= \mathrm{in}(I) : \mathfrak{m}^{k_1} = \mathrm{in}(I) : \mathfrak{m}^{k_1+1} \end{aligned}$$

(since $\mathrm{in}(I)$ is k_1-Bbm), $x_0^{k_1} m \in \mathrm{in}(I)$, thus $F \xrightarrow{G_i} F'$, $\mathrm{in}(F) > \mathrm{in}(F')$, $\partial(F) = \partial(F_1) \geq \mathfrak{K} + k_1 - 1 \geq \mathfrak{K}$. From this $F \in I$. □

Corollary 3.2. *Under the hypothesis of Theorem 3.1, $k_2 \leq L$. In particular if $k_1 = 1$, then $k_2 \leq L = \mathfrak{K} - \nu_1$.*

Proof. This follows immediately from Theorem 3.1. □

Definition 3.3. $e(R_{n+1}/I)$ will denote the multiplicity as defined by the Hilbert polynomial.

An important result due to Macaulay is $e(R_{n+1}/I) = e(R_{n+1}/in(I))$. By [**8**] if $\mathrm{in}(I)$ is 1-Bbm and $\dim(in(I)) = 1$, then $k_2 \leq e(R_{n+1}/I) = e(R_{n+1}/in(I))$. (The proof uses the fact that $[H^0_{\mathfrak{m}}(R_{n+1}/I)]_n = [H^0_{\mathfrak{m}}(R_{n+1}/\mathrm{in}(I)]_n = 0$ for $n \leq 0$, n denoting the n^{th} graded piece of the 0^{th} local cohomology module $H^0_{\mathfrak{m}}(\ldots)$, and $k_2 \leq a(H^0_{\mathfrak{m}}(R_{n+1}/I)) \leq a(H^0_{\mathfrak{m}}(R_{n+1}/\mathrm{in}(I))) \leq e(R_{n+1}/(\mathrm{in}(I))$ by Lemma 3.1 in [**8**], $a(\ldots)$ denoting the last nonzero graded piece.) We will improve on this bound in the sequel. Presently we relate the multiplicity to the bound L of Corollary 3.2.

Lemma 3.4. *Assume $\mathfrak{Q}_0 \neq (x_1, \ldots, x_n) \subseteq R_{n+1}$ is a $(x_1, \ldots, x_n)$-primary monomial ideal with $\{x_0\}$ a s.o.p.*

Let $\mathfrak{Q}_0 = (x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, M)$, $\alpha_i \geq 1$, $1 \leq i \leq n$, and $m \in M$ implies $m = x_0^{\beta_0} x_1^{\beta_1} \ldots x_n^{\beta_n}$, $\beta_i < \alpha_i$, $1 \leq i \leq n$. Then, if $l(\ldots)$ denotes length, we have:

(i) $l((x_1, \ldots, x_n)/\mathfrak{Q}_0) \geq \sum_{i=1}^{n} (\alpha_i - 1)$,

(ii) $l(x_1, \ldots, x_n/\mathfrak{Q}_0) = \sum_{i=1}^{u} (\alpha_i - 1)$ *iff* $x_i x_j \in \mathfrak{Q}_0, i \neq j, 1 \leq i, j \leq n$.

Proof. (i) Lowering the exponent in $x_i^{\alpha_i}$ by one, results in a proper inclusion, thus (i).

(ii) $\Leftarrow$. For $\sum_{i=1}^{n} (\alpha_i - 1) = 1$, $\mathfrak{Q}_0 \subset (x_1, \ldots, x_n)$ is a saturated chain. Let $\sum_{i=1}^{n} (\alpha_i - 1) = h + 1$, $h \geq 1$. Without loss of generality assume $\delta_1 \geq 2$. Consider $\mathfrak{Q}_0 \subset (\mathfrak{Q}_0, x_1^{\delta_1 - 1}) = \mathfrak{Q}_1$. $x_i x_j \in \mathfrak{Q}_0, i \neq j$, $1 \leq i, j \leq n$, implies if $m \notin \mathfrak{Q}_0$ and $m \neq x_1^{\delta_1 - 1}$, then

$$m = x_j^{\delta_j - \beta_j}, \quad 1 \leq \beta_j, \quad 2 \leq j \leq n, \quad \text{or} \quad m = x_1^{\delta_1 - \beta_1}, \quad 2 \leq \beta_1,$$

thus $\mathfrak{Q}_0 \subset \mathfrak{Q}_1$ is saturated, from which the implication by induction.
$\Rightarrow$. Suppose without loss of generality that $x_1 x_2 \notin \mathfrak{Q}_0$, thus $\alpha_1 \geq 2$ and $\alpha_2 \geq 2$. But then

$$(\mathfrak{Q}_0, x_1^2) \subset (\mathfrak{Q}_0, x_1^2, x_1 x_2) \subset (\mathfrak{Q}_0, x_1),$$

which constradicts the hypothesis. □

Assume $\{x_0\}$ is a s.o.p for $\mathrm{in}(I)$ and $\mathrm{in}(I)$ is strictly k_1-Bbm, i.e.

$$\mathrm{in}(I) : x_0^{k_1} = \mathrm{in}(I) : x_0^{k_1+1} = \mathrm{in}(I) : \mathfrak{m}^{k_1} = \mathrm{in}(I) : \mathfrak{m}^{k_1+1}$$

and k_1 is minimal. Let $\mathrm{in}(I) = (x_1^{\delta_1}, \ldots, x_n^{\delta_n}, M)$, $1 \le \delta_i$, $1 \le i \le n$, and for $m \in M$, $m = x_0^{\beta_0} x_1^{\beta_1} \ldots x_n^{\beta_n}$, $\beta_0 \le k_1$, $\beta_i < \delta_i$, $1 \le i \le n$. Then $\mathrm{in}(I) = (\mathfrak{Q}_0 = (x_1^{\delta_1}, \ldots, x_n^{\delta_n}, M|_{x_0=1})) \cap \mathfrak{Q}_1$, $\mathfrak{Q}_1 = R_{n+1}$ or a trivial component.

Definition 3.5. Let

$$\begin{aligned} D(k_1) = \{m : m \; &= \; x_0^{\beta_0} x_j^{\delta_j - \varepsilon_j}, \;\; 1 \le \beta_0, \varepsilon_j \le k_1, \;\; \varepsilon_j < \delta_j, \;\; \varepsilon_j \;\text{ maximal}, \\ & \quad 1 \le j \le n, \; m \in M\}. \end{aligned}$$

Define $\sigma(k_1) = \sum\limits_{j=1}^{n} \varepsilon_j$. Put $\varepsilon_j = 0$, if ε_j does not occur in $D(k_1)$.

Theorem 3.6. $e(R_{n+1}/I) \ge \mathfrak{K} - \sigma(k_1)$.

Proof. Let $\mathrm{in}(I) = \mathfrak{Q}_0 \cap \mathfrak{Q}_1$ be a primary decomposition with $\mathfrak{Q}_0$ $(x_1, \ldots, x_n)$-primary (thus unique), $\mathfrak{Q}_1$ either the trivial component or R_{n+1}. By [**9**] (see also the monomial construction there) and Lemma 3.4

$$\begin{aligned} e \; &= \; e(R_{n+1}/\mathrm{in}(I)) = 1 + l((x_1, \ldots, x_n)/\mathfrak{Q}_0) \\ &\ge \; 1 + \sum_{j=1}^{n} [(\delta_i - \varepsilon_i) - 1] = 1 + \sum_{j=1}^{n} (\delta_i - 1) - \sigma(k_1) = \mathfrak{K} - \sigma(k_1). \end{aligned}$$

□

Corollary 3.7. *For $k_1 = 1$, n fixed, $e - k_2$ increases beyond bound with increasing ν_1.*

Proof. For $k_1 = 1$, L of Corollary 3.2 is $\mathfrak{K} - \nu_1$. By Theorem 3.6

$$e = e(R_{n+1}/I) \ge (\mathfrak{K} - \nu_1) + (\nu_1 - \sigma(1)) \ge k_2 + \nu_1 - \sigma(1).$$

Since $\sigma(1) \le n$, $e - k_2 \ge \nu_1 - n$, we get the claim. □

Example 3.8. Let $I(m, m, p) = (x_1^{m-1}(x_1^p + x_0^p), x_0(x_1^p + x_0^p), x_2, \ldots, x_{n-1})$, $p \ge 1$, $n \ge 2$, $m \ge 2$. Assume $x_1 > x_0$. It follows readily that $\mathrm{in}(I(m, n, p)) = (x_1^{p+m-1}, x_0 x_1^p, x_2, \ldots, x_{n-1})$, therefore $\{x_0\}$ is a s.o.p for $I(m, n, p)$ and $\mathrm{in}(I(m, n, p))$. $\mathrm{in}(I(m, n, p)) : x_0 = (x_1^p, x_2, \ldots, x_{n-1}) = \mathrm{in}(I(m, n, p)) : x_0^2 \subseteq \mathrm{in}(I(m, n.p)) : \mathfrak{m}^{m-1}, (m-1)$ minimal. $I(m, n, p) : x_0 = (x_1^p + x_0^p, x_2, \ldots, x_{n-1}) = I(m, n, p) : x_0^2 \subseteq I(m, n, p) : \mathfrak{m}^{m-1}, (m-1)$ minimal. Thus $k_1 = k_2 = m - 1$. Also always $e = e(R_n/I(m, n, p)) = p$. Therefore, since m and p are independent parameters, in general there is no relationship between e and k_1, k_2. We calculate next L of Corollary 3.2. We consider two cases:

(i) $n > 2$. Then $L = \mathfrak{K} + k_1 - \nu_1 - 1 = (p + m - 1) + (m - 1) - 1 - 1 = (p - 1) + 2m - 3 \ge k_2 = m - 1$. For $m = 2$ and $k_1 = k_2 = 1, L = p = e$.

(ii) $n = 2$. Then $L = (p + m - 1) + (m - 1) - p - 1 = 2m + 3 \ge m - 1$.

For $m = 2$, thus $k_1 = k_2 = 1, L = 1 \le p = e$, thus the difference between L and e becomes arbitrarily large with increasing p.

Example 3.9. The ideals $I(k)$ of Theorem 2.6 are as in Corollary 3.7.

For $I(n)$ in Theorem 2.4, n is not fixed. We therefore investigate for $k_1 = 1$ another relation between k_2 and $e(R_{n+1}/in(I)) = e$ (from now on). For this we separate monomials m into:

(i) $m \in \text{in}(I)$.
(ii) $m \notin \text{in}(I)$, but $m \in \text{in}(I) : x_0$ ($\{x_0\}$ a s.o.p for $\text{in}(I)$ and I).
(iii) $m \notin \text{in}(I) : x_0$.

Note that a monomial m such that $m \in \text{in}(I) : x_0$ and $x_0|m$ implies $m \in \text{in}(I)$.

Definition 3.10. A monomial m as in (ii) is called an *obstruction*.

Lemma 3.11. *If* $m = x_1^{\alpha_1} \cdot \ldots \cdot x_i^{\alpha_i} \cdot \ldots \cdot x_n^{\alpha_n}, \alpha_i \geq 1$, *is an obstruction, then*

$$x_1^{\alpha_1} \cdot \ldots \cdot x_i^{\alpha_i - 1} \cdot \ldots \cdot x_n^{\alpha_n} \notin \text{in}(I) : x_0.$$

Proof. $x_1^{\alpha_1} \cdot \ldots \cdot x_i^{\alpha_i-1} \cdot \ldots \cdot x_n^{\alpha_n} \in \text{in}(I) : x_0$ implies $x_0 x_1^{\alpha_1} \cdot \ldots \cdot x_i^{\alpha_i-1} \cdot \ldots \cdot x_n^{\alpha_n} \in \text{in}(I)$, thus $x_1^{\alpha_1} \cdot \ldots \cdot x_i^{\alpha_i} \cdot \ldots \cdot x_n^{\alpha_n} \in \text{in}(I)$, a contradiction.

In what follows, $\text{in}(I) = \mathfrak{Q}_0 \cap \mathfrak{Q}_1, \mathfrak{Q}_0$ $(x_1, \ldots, x_n)$-primary, $\mathfrak{Q}_1$ a trivial component. Note : (i) $\text{in}(I) : x_0 = \mathfrak{Q}_0$. (ii) If $\mathfrak{Q}_1 = R_{n+1}$, then $\text{in}(I)$ is perfect, which, since $k_1 = 1$, is not the case. $\square$

Lemma 3.12. (i) $1 \notin \text{in}(I) : x_0$.
(ii) *m an obstruction and $x_i|m, x_j|m, i \neq j$ implies $m/x_i \neq m/x_j$ are not in* $\text{in}(I) : x_0$.
(iii) $x_i^{\alpha_i}, \alpha_i \geq 2$, *such that $x_i^{\alpha_i - 1}$ is the only monomial of degree $\alpha_i - 1$ not in* $\text{in}(I) : x_0$, *implies $x_i^{\alpha_i}$ is the only monomial of degree α_i not in* $\text{in}(I)$ *and* $k_2 \leq \alpha_i + 1 - \nu_1$.

Proof. (i) is true since $\{x_0\}$ is a s.o.p for $\text{in}(I)$. (ii) follows from Lemma 3.11. (iii) Let $\tilde{m} \neq x_i^{\alpha_i}$ be of degree α_i. $x_i|\tilde{m}$ implies $\tilde{m}/x_i \neq x_i^{\alpha_i-1}$, thus $\tilde{m}/x_i \in \text{in}(I) : x_0$, hence $x_i\tilde{m}/x_i = \tilde{m} \in \text{in}(I)$. $x_i \nmid \tilde{m}$, then for some $x_j \neq x_i$ $\tilde{m}/x_j \neq x_i^{\alpha_i-1}$, thus, as before, $x_j\tilde{m}/x_j = \tilde{m} \in \text{in}(I)$. Consider

$$m \in \text{in}(\mathfrak{m}^{\alpha_i+1-\nu_1}(I : x_0^{\delta_0} = I : x_0^{\delta_0+1})) \cap K[x_1, \ldots, x_n] \subseteq \text{in}(I : x_0^{\delta_0}) \subseteq \text{in}(I) : x_0^{\delta_0}$$
$$= \text{in}(I) : x_0, \partial(m) = \alpha_i + 1,$$

thus of minimal degree. By the above and since $k_1 = 1$, we get $m \in \text{in}(I)$; thus $\text{in}(\mathfrak{m}^{\alpha_i+1-\nu_1}(I : x_0^{\delta_0})) \subseteq \text{in}(I)$ since if $m \in \text{in}(I) : x_0$ and $x_0|m$, then $m \in \text{in}(I)$. Therefore $k_2 \leq \alpha_i + 1 - \nu_1$. $\square$

Theorem 3.13. *For* $k_1 = 1$, $k_2 \leq e/2$ *if* $2 \leq \nu_1$ *and,* $k_2 \leq (e + 2)/2$ *if* $\nu_1 = 1$.

Proof. For $k_2 = 0$, the bounds obviously are correct. Let $k_2 = 1$. If $\nu_1 = 1$, the bound is correct. If $2 \leq \nu_1$ and $\text{in}(I) : x_0 = \mathfrak{Q}_0 \neq (x_1, \ldots, x_n)$, the bound is correct. If $\mathfrak{Q}_0 = (x_1, \ldots, x_n) = \text{in}(I) : x_0$ and $\nu_1 \geq 2$, then all quadratic monomials, except x_0^2, are in $\text{in}(I)$. Therefore, $I : x_0 \subseteq I$, by reduction with a Gröbner basis in (I), thus $k_2 = 0$ which contradics $k_2 = 1$.

Assume $k_2 \geq 2$. We consider the obstructions of lowest degree in

$$\mathrm{in}(\mathfrak{m}^\rho(I : x_0^{\delta_0})) \subseteq in(I : x_0^{\delta_0}) \subseteq \mathrm{in}(I) : x_0^{\delta_0} = \mathrm{in}(I) : x_0, 0 \leq \rho \leq k_2 - 1.$$

Starting with $\rho = 0$, we obtain obstructions m_0 of degree d_0, giving rise to monomials $\tilde{m}_0 \notin \mathrm{in}(I) : x_0$ of degree $d_0 - 1$. Since $\mathfrak{m}(m_0) \subseteq \mathrm{in}(I)$, we obtain a sequence of monomials $\tilde{m} \notin \mathrm{in}(I) : x_0$ of degrees $d_0 - 1 < d_1 - 1 < \cdots < d_{k_2-1} - 1$. Possibilities for a single such monomial, by Lemma 3.12 are:

(i) $d_0 = \nu_1 = 1, m_0 = 1$,

(ii) $x_i^{\alpha_i - 1} = x_1^{d_{k_2-1}-1}$.

If $\nu_1 \geq 2$, we can add the monomial 1 to the possibility (ii), thus $2k_2 \leq e$ (the count starts at 0). If $\nu_1 = 1$, we obtain $2(k_2 - 1) \leq e$, which finishes the proof. □

Example 3.14. For I and $\mathrm{in}(I)$ as in Theorem 3.13, if $2 < e$, then $k_2 < e$. We give two examples with $e = k_2 = 1$ and $e = k_2 = 2$.

1. If $m = 2$, $p = 1$, $n > 2$ in Example 3.8, then $k_1 = k_2 = e = 1 = \nu_1$.
2. Let $n = 2$ for $I(n)$ of Theorem 2.4. Then $I(2) = (z(x_1 + x_2),\ M_1 = \{x_1^2, x_1x_2\},\ M_2 = \{x_3^3\})$, $\mathrm{in}(I(2)) = (zx_1, zx_1^2, x_1^2, x_1x_2, x_2^3)$. Therefore $\nu_1 = k_1 = 1$ and $k_2 = e = 2$.

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