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WEIGHTED ENDPOINT ESTIMATES FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS

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Abstract. In this paper, we prove weighted endpoint estimates for multilinear Littlewood-Paley operators.

1. Introduction and Results

Let ψ be a fixed function on R^n which satisfies the following properties:

(1) $\int \psi(x)dx = 0$,
(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y)-\psi(x)| \le C|y|(1+|x|)^{-(n+2)}$ when $2|y|<|x|$;

Let m be a positive integer and A be a function on R^n. The multilinear Littlewood-Paley operator is defined by

$$g_\mu^A(f)(x) = \left[\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}}\right]^{1/2}, \quad \mu>1,$$

where

$$F_t^A(f)(x,y) = \int_{R^n} \frac{R_{m+1}(A;x,z)}{|x-z|^m}\psi_t(y-z)f(z)dz,$$

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t>0$. We denote by $F_t(f)(y) = f*\psi_t(y)$. We also define

$$g_\mu(f)(x) = \left(\int\int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley operator (see [**17**]).

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Let H be the Hilbert space $H = \left\{ h : ||h|| = \left(\int \int_{R_+^{n+1}} |h(t)|^2 dydt/t^{n+1} \right)^{1/2} < \infty \right\}$. Then for each fixed $x \in R^n$, $F_t^A(f)(x,y)$ may be viewed as a mapping from $(0,+\infty)$ to H, and it is clear that

$$\begin{aligned} g_\mu^A(f)(x) &= \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^A(f)(x,y) \right\|, \\ g_\mu(f)(x) &= \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t(f)(y) \right\|. \end{aligned}$$

We also consider the variant of g_μ^A, which is defined by

$$\tilde{g}_\mu^A(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{n\mu} |\tilde{F}_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}, \quad \mu > 1,$$

where

$$\tilde{F}_t^A(f)(x,y) = \int_{R^n} \frac{Q_{m+1}(A;x,z)}{|x-z|^m} \psi_t(y-z) f(z) dz$$

and

$$Q_{m+1}(A;x,z) = R_m(A;x,z) - \sum_{|\alpha|=m} D^\alpha A(x)(x-z)^\alpha.$$

Note that when $m = 0$, g_μ^A is just the commutator of Littlewood-Paley operator (see [**1**], [**14**], [**15**]). It is well known that multilinear operators, as an extension of commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see [**4**] – [**8**], [**12**], [**13**]). In [**11**], [**16**], the endpoint boundedness properties of commutators generated by the Calderon-Zygmund operator and BMO functions are obtained. The main purpose of this paper is to study the weighted endpoint boundedness of the multilinear Littlewood-Paley operators. Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, Q will denote a cube of R^n with side parallel to the axes. For a cube Q and any locally integral function f on R^n, we denote that $f(Q) = \int_Q f(x)dx$, $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q| dy$. Moreover, for a weight functions $w \in A_1$ (see [**10**]), f is said to belong BMO(w) if $f^\# \in L^\infty(w)$ and define $||f||_{\mathrm{BMO}(w)} = ||f^\#||_{L^\infty(w)}$, if $w = 1$, we denote that BMO(R^n) = BMO(w). Also, we give the concepts of atomic and weighted H^1 space. A function a is called a $H^1(w)$ atom if there exists a cube Q such that a is supported on Q, $||a||_{L^\infty(w)} \le w(Q)^{-1}$ and $\int a(x)dx = 0$. It is well known that, for $w \in A_1$, the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization (see [**2**]).

We shall prove the following theorems in Section 3.

Theorem 1. *Let $D^\alpha A \in$ BMO(R^n) for $|\alpha| = m$ and $w \in A_1$. Then g_μ^A is bounded from $L^\infty(w)$ to* BMO(w).

Theorem 2. *Let $D^\alpha A \in \mathrm{BMO}(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then $\tilde{g}_\mu^A$ is bounded from $H^1(w)$ to $L^1(w)$.*

Theorem 3. *Let $D^\alpha A \in \mathrm{BMO}(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then g_μ^A is bounded from $H^1(w)$ to weak $L^1(w)$.*

Theorem 4. *Let $D^\alpha A \in \mathrm{BMO}(R^n)$ for $|\alpha| = m$ and $w \in A_1$.*

(i) *If for any $H^1(w)$-atom a supported on certain cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\int_{(4Q)^c}\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-u)^\alpha}{|x-u|^m}\int_Q \psi_t(y-z)D^\alpha A(z)a(z)dz\right\| w(x)dx \le C,$$

then g_μ^A is bounded from $H^1(w)$ to $L^1(w)$;

(ii) *If for any cube Q and $u \in 3Q \setminus 2Q$, there is*

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\sum_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A(x)-(D^\alpha A)_Q)\right.\\
&\qquad\left.\cdot\int_{(4Q)^c}\frac{(u-z)^\alpha}{|u-z|^m}\psi_t(y-z)f(z)dz\right\| w(x)dx\\
\le\ & C||f||_{L^\infty(w)},
\end{aligned}$$

then $\tilde{g}_\mu^A$ is bounded from $L^\infty(w)$ to $\mathrm{BMO}(w)$.

Remark. In general, g_μ^A is not bounded from $H^1(w)$ to $L^1(w)$.

2. Some Lemmas

We begin with two preliminary lemmas.

Lemma 1. *(see [**7**].) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then*

$$|R_m(A:x,y)| \le C|x-y|^m\sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}(x,y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2. *Let $w \in A_1$, $1 < p < \infty$ $1 < r \le \infty$, $1/q = 1/p + 1/r$ and $D^\alpha A \in \mathrm{BMO}(R^n)$ for $|\alpha| = m$. Then g_μ^A is bounded from $L^p(w)$ to $L^q(w)$, that is*

$$||g_\mu^A(f)||_{L^q(w)} \le C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^p(w)}.$$

Proof. By Minkowski inequality and the condition of ψ, we have

$$
\begin{aligned}
& g_\mu^A(f)(x) \\
\leq & \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_{R_+^{n+1}} |\psi_t(y-z)|^2 \left(\frac{t}{t+|x-y|} \right)^{n\mu} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\
\leq & C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left(\int_0^\infty \int_{R^n} \frac{t^{-2n}}{(1+|y-z|/t)^{2n+2}} \right. \\
& \qquad \left. \cdot \left(\frac{t}{t+|x-y|} \right)^{n\mu} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\
\leq & C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^m} \left[\int_0^\infty \left(t^{-n} \int_{R^n} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \right. \right. \\
& \qquad \left. \left. \cdot \frac{dy}{(t+|y-z|)^{2n+2}} \right) tdt \right]^{1/2} dz,
\end{aligned}
$$

noting that

$$
\begin{aligned}
& t^{-n} \int_{R^n} \left(\frac{t}{t+|x-y|} \right)^{n\mu} \frac{dy}{(t+|y-z|)^{2n+2}} \\
\leq & CM \left(\frac{1}{(t+|\cdot - z|)^{2n+2}} \right)(x) \leq C \frac{1}{(t+|x-z|)^{2n+2}}
\end{aligned}
$$

and

$$
\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2}} = C|x-z|^{-2n},
$$

we obtain

$$
\begin{aligned}
g_\mu^A(f)(x) & \leq C \int_{R^n} \frac{|f(z)|}{|x-z|^m} |R_{m+1}(A;x,z)| \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+2}} \right)^{1/2} dz \\
& = C \int_{R^n} \frac{|f(z)||R_{m+1}(A;x,z)|}{|x-z|^{m+n}} dz,
\end{aligned}
$$

thus, the lemma follows from [**8**] [**9**]. □

3. Proof of Theorems

Proof of Theorem 1. It is only to prove that there exists a constant C_Q such that

$$
\frac{1}{w(Q)} \int_Q |g_\mu^A(f)(x) - C_Q| w(x) dx \leq C ||f||_{L^\infty(w)}
$$

holds for any cube Q. Fix a cube $Q = Q(x_0, l)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_m(A;x,y) = R_m(\tilde{A};x,y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write $F_t^A(f) = F_t^A(f_1) + F_t^A(f_2)$ for $f_1 = f\chi_{\tilde{Q}}$

and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$, then

$$\begin{aligned}
&\frac{1}{w(Q)}\int_Q |g_\mu^A(f)(x) - g_\mu^A(f_2)(x_0)|w(x)dx \\
= \quad & \frac{1}{w(Q)}\int_Q \Bigg| ||(\frac{t}{t+|x-y|})^{n\mu/2}F_t^A(f)(x,y)|| \\
& \qquad - ||(\frac{t}{t+|x_0-y|})^{n\mu/2}F_t^A(f_2)(x_0,y)||\Bigg| w(x)dx \\
\leq \quad & \frac{1}{w(Q)}\int_Q g_\mu^A(f_1)(x)w(x)dx \\
& + \frac{1}{w(Q)}\int_Q \Bigg|\Bigg|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f_2)(x,y) \\
& \qquad - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t^A(f_2)(x_0,y)\Bigg|\Bigg| w(x)dx \\
:= \quad & I(x) + II(x).
\end{aligned}$$

Now, let us estimate I and II. First, by the L^∞ boundedness of g_μ^A (Lemma 2), we gain

$$I \leq ||g_\mu^A(f_1)||_{L^\infty(w)} \leq C||f||_{L^\infty(w)}.$$

To estimate II, we write

$$\begin{aligned}
& \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} F_t^A(f_2)(x,y) - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t^A(f_2)(x_0,y) \\
= \quad & \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \int \left[\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m}\right]\psi_t(y-z)R_m(\tilde{A};x,z)f_2(z)dz \\
& + \left(\frac{t}{t+|x-y|}\right)^{n\mu/2} \int \frac{\psi_t(y-z)f_2(z)}{|x_0-z|^m}[R_m(\tilde{A};x,z) - R_m(\tilde{A};x_0,z)]dz \\
& + \int \left[\left(\frac{t}{t+|x-y|}\right)^{n\mu/2} - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\right] \frac{\psi_t(y-z)R_m(\tilde{A};x_0,z)f_2(z)}{|x_0-z|^m}dz \\
& - \sum_{|\alpha|=m}\frac{1}{\alpha!}\int \Bigg[(\frac{t}{t+|x-y|})^{n\mu/2}\frac{(x-z)^\alpha}{|x-z|^m} \\
& \qquad - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2}\frac{(x_0-z)^\alpha}{|x_0-z|^m}\Bigg]\psi_t(y-z)D^\alpha\tilde{A}(z)f_2(z)dz \\
:= \quad & II_1^t(x) + II_2^t(x) + II_3^t(x) + II_4^t(x),
\end{aligned}$$

Note that $|x-z| \sim |x_0-z|$ for $x \in \tilde{Q}$ and $z \in R^n \setminus \tilde{Q}$, similar to the proof of Lemma 2 and by Lemma 1, we have

$$\begin{aligned}
&\frac{1}{w(Q)} \int_Q ||II_1^t(x)||w(x)dx \\
&\quad \leq \frac{C}{w(Q)} \int_Q \left(\int_{R^n\setminus\tilde{Q}} \frac{|x-x_0||f(z)|}{|x-z|^{n+m+1}} |R_m(\tilde{A};x,z)|dz \right) w(x)dx \\
&\quad \leq \frac{C}{w(Q)} \int_Q \left(\sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0||f(z)|}{|x-z|^{n+m+1}} |R_m(\tilde{A};x,z)|dz \right) w(x)dx \\
&\quad \leq C \sum_{k=0}^{\infty} \frac{kl(2^k l)^m}{(2^k l)^{n+m+1}} \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} (\int_{2^{k+1}\tilde{Q}} |f(z)|dz) \\
&\quad \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} ||f||_{L^\infty(w)} \sum_{k=0}^{\infty} k2^{-k} \\
&\quad \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} ||f||_{L^\infty(w)};
\end{aligned}$$

For $II_2^t(x)$, by the formula (see [**7**]):

$$R_m(\tilde{A};x,z) - R_m(\tilde{A};x_0,z) = R_m(\tilde{A};x,x_0) + \sum_{0<|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A};x_0,z)(x-x_0)^\beta$$

and Lemma 1, we get

$$\begin{aligned}
|R_m(\tilde{A};x,z) \quad &- \quad R_m(\tilde{A};x_0,z)| \\
&\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} (|x-x_0|^m + \sum_{0<|\beta|<m} |x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}),
\end{aligned}$$

thus, for $x \in Q$,

$$\begin{aligned}
&||II_2^t(x)|| \\
&\quad \leq C \int_{R^n} \frac{|f_2(z)|}{|x-z|^{m+n}} |R_m(\tilde{A};x,z) - R_m(\tilde{A};x_0,z)|dz \\
&\quad \leq C \sum_{|\alpha|=m} |D^\alpha A||_{\mathrm{BMO}} \int_{R^n} \frac{|x-x_0|^m + \sum_{0<|\beta|<m} |x_0-z|^{m-|\beta|}|x-x_0|^{|\beta|}}{|x_0-z|^{m+n}} |f_2(z)|dz \\
&\quad \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \sum_{k=0}^{\infty} \frac{kl^m}{(2^k l)^{m+n}} \int_{2^{k+1}\tilde{Q}} |f(z)|dz \\
&\quad \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} ||f||_{L^\infty(w)} \sum_{k=1}^{\infty} k2^{-km} \\
&\quad \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} ||f||_{L^\infty(w)};
\end{aligned}$$

For $II_3^t(x)$, by the inequality: $a^{1/2}-b^{1/2}\le(a-b)^{1/2}$ for $a\ge b>0$, we obtain, similar to the estimate of Lemma 2 and II_1,

$$\begin{aligned}
&||II_3^t(x)||\\
&\le C\int_{R^n}\left(\int_{R_+^{n+1}}\left[\frac{t^{n\mu/2}|x-x_0|^{1/2}|\psi_t(y-z)||f_2(z)|}{(t+|x-y|)^{(n\mu+1)/2}|x_0-z|^m}|R_m(\tilde{A};x_0,z)|\right]^2\frac{dydt}{t^{n+1}}\right)^{1/2}dz\\
&\le C\int_{R^n}\frac{|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\\
&\qquad\cdot\left(\int\int_{R_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{n\mu+1}\frac{t^{-n}dydt}{(t+|y-z|)^{2n+2}}\right)^{1/2}dz\\
&\le C\int_{R^n}\frac{|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^m}\left(\int_0^\infty\frac{dt}{(t+|x-z|)^{2n+2}}\right)^{1/2}dz\\
&\le C\int_{R^n}\frac{|f_2(z)||x-x_0|^{1/2}|R_m(\tilde{A};x_0,z)|}{|x_0-z|^{m+n+1/2}}dz\\
&\le C\sum_{k=0}^\infty\frac{kl^{1/2}(2^kl)^m}{(2^kl)^{n+m+1/2}}\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}(\int_{2^{k+1}\tilde{Q}}|f(z)|dz)\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)}\sum_{k=0}^\infty k2^{-k/2}\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)};
\end{aligned}$$

For $II_4^t(x)$, similar to the estimates of $II_1^t(x)$ and $II_3^t(x)$, we have

$$\begin{aligned}
||II_4^t(x)|| &\le C\int_{R^n\setminus\tilde{Q}}\left(\frac{|x-x_0|}{|x-z|^{n+1}}+\frac{|x-x_0|^{1/2}}{|x-z|^{n+1/2}}\right)\sum_{|\alpha|=m}|D^\alpha\tilde{A}(z)||f(z)|dz\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)}\sum_{k=0}^\infty k(2^{-k}+2^{-k/2})\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{\mathrm{BMO}}||f||_{L^\infty(w)}.
\end{aligned}$$

Combining these estimates, we complete the proof of Theorem 1. □

Proof of Theorem 2. It suffices to show that there exists a constant $C>0$ such that for every $H^1(w)$-atom a (that is that a satisfies: $\operatorname{supp} a\subset Q=Q(x_0,r)$, $||a||_{L^\infty(w)}\le w(Q)^{-1}$ and $\int a(y)dy=0$ (see [**8**])), we have

$$||\tilde{g}_\mu^A(a)||_{L^1(w)}\le C.$$

We write

$$\int_{R^n} \tilde{g}_\mu^A(a)(x)w(x)dx = \left[\int_{|x-x_0|\le 2r} + \int_{|x-x_0|>2r}\right] \tilde{g}_\mu^A(a)(x)w(x)dx := J + JJ.$$

For J, by the following equality

$$Q_{m+1}(A;x,y) = R_{m+1}(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),$$

we have, similar to the proof of Lemma 2,

$$\tilde{g}_\mu^A(a)(x) \le g_\mu^A(a)(x) + C \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |a(y)|dy,$$

thus, $\tilde{g}_\mu^A$ is L^∞-bounded by Lemma 2 and [**3**]. We see that

$$J \le C||\tilde{g}_\mu^A(a)||_{L^\infty(w)} w(2Q) \le C||a||_{L^\infty(w)} w(Q) \le C.$$

To obtain the estimate of JJ, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2B} x^\alpha$. Then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$. We write, by the vanishing moment of a and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha D^\alpha A(x)$, for $x \in (2Q)^c$,

$$\begin{aligned}
&\tilde{F}_t^A(a)(x,y) \\
&= \int \frac{\psi_t(y-z) R_m(\tilde{A};x,z)}{|x-z|^m} a(z)dz - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \frac{\psi_t(y-z) D^\alpha \tilde{A}(z)(x-z)^\alpha}{|x-z|^m} a(z)dz \\
&= \int \left[\frac{\psi_t(y-z) R_m(\tilde{A};x,z)}{|x-z|^m} - \frac{\psi_t(y-x_0) R_m(\tilde{A};x,x_0)}{|x-x_0|^m}\right] a(z)dz \\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \left[\frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-x_0)(x-x_0)^\alpha}{|x-x_0|^m}\right] D^\alpha \tilde{A}(x) a(z)dz,
\end{aligned}$$

thus, similar to the proof of II in Theorem 1, we obtain

$$\begin{aligned}
&||\tilde{F}_t^A(a)(x,y)|| \\
&\le C \frac{|Q|^{1+1/n}}{w(Q)} \Big(\sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} |x-x_0|^{-n-1} + |x-x_0|^{-n-1} |D^\alpha \tilde{A}(x)|\Big),
\end{aligned}$$

note that if $w \in A_1$, then $\frac{w(Q_2)}{|Q_2|} \frac{|Q_1|}{w(Q_1)} \le C$ for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$. Thus, by Holder' inequality and the reverse of Holder' inequality for $w \in A_1$, taking $p > 1$ and $1/p + 1/p' = 1$, we obtain

$$\begin{aligned} JJ &\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} \sum_{k=1}^{\infty} 2^{-k} \left(\frac{|Q|}{w(Q)} \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right) \\ &\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} 2^{-k} \frac{|Q|}{w(Q)} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^\alpha \tilde{A}(x)|^p dx \right)^{1/p} \\ &\qquad \cdot \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} w(x)^{p'} dx \right)^{1/p'} \\ &\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}} \sum_{k=1}^{\infty} k 2^{-k} \left(\frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \frac{|Q|}{w(Q)} \right) \leq C, \end{aligned}$$

which together with the estimate for J yields the desired result. This finishes the proof of Theorem 2. □

Proof of Theorem 3. By the equality

$$R_{m+1}(A; x, y) = Q_{m+1}(A; x, y) + \sum_{|\alpha|=m} \frac{1}{\alpha!} (x - y)^\alpha (D^\alpha A(x) - D^\alpha A(y))$$

and similar to the proof of Lemma 2, we get

$$g_\mu^A(f)(x) \leq \tilde{g}_\mu^A(f)(x) + C \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x - y|^n} |f(y)| dy,$$

by Theorem 1 and 2 with [**3**], we obtain

$$\begin{aligned} &w(\{x \in R^n : g_\mu^A(f)(x) > \lambda\}) \\ &\quad \leq w(\{x \in R^n : \tilde{g}_\mu^A(f)(x) > \lambda/2\}) \\ &\qquad + w(\{x \in R^n : \sum_{|\alpha|=m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x - y|^n} |f(y)| dy > C\lambda\}) \\ &\quad \leq C||f||_{H^1(w)}/\lambda. \end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. **(i)** It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom a with supp$a \subset Q = Q(x_0, d)$, there is

$$||g_\mu^A(a)||_{L^1(w)} \leq C.$$

Let $\tilde{A}(x)=A(x)-\sum\limits_{|\alpha|=m}\frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}}x^\alpha$. We write, by the vanishing moment of a and for $u\in 3Q\setminus 2Q$,

$$\begin{aligned}
&F_t^A(a)(x,y)\\
&= \chi_{4Q}(x)F_t^A(a)(x,y)\\
&\quad+\chi_{(4Q)^c}(x)\int_{R^n}\left[\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-z|^m}-\frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m}\right]a(z)dz\\
&\quad-\chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left[\frac{(x-z)^\alpha}{|x-z|^m}-\frac{(x-u)^\alpha}{|x-u|^m}\right]\psi_t(y-z)D^\alpha A(z)a(z)dz\\
&\quad-\chi_{(4Q)^c}(x)\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(y-z)D^\alpha A(z)a(z)dz,
\end{aligned}$$

then

$$\begin{aligned}
&g_\mu^A(a)(x)\\
&= \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^A(a)(x,y)\right\|\\
&\le \chi_{4Q}(x)\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}F_t^A(a)(x,y)\right\|\\
&\quad+\chi_{(4Q)^c}(x)\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\int_{R^n}\left[\frac{R_m(\tilde{A};x,z)\psi_t(y-z)}{|x-z|^m}\right.\right.\\
&\qquad\qquad\left.\left.-\frac{R_m(\tilde{A};x,u)\psi_t(y-u)}{|x-u|^m}\right]a(z)dz\right\|\\
&\quad+\chi_{(4Q)^c}(x)\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\left[\frac{(x-z)^\alpha}{|x-z|^m}-\frac{(x-u)^\alpha}{|x-u|^m}\right]\right.\\
&\qquad\qquad\left.\cdot\,\psi_t(y-z)D^\alpha A(z)a(z)dz\right\|\\
&\quad+\chi_{(4Q)^c}(x)\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{(x-u)^\alpha}{|x-u|^m}\psi_t(y-z)a(z)dz\right\|\\
&= L_1(x)+L_2(x)+L_3(x,u)+L_4(x,u).
\end{aligned}$$

By the $L^\infty(w)$-boundedness of g_μ^A, we get

$$\begin{aligned}
\int_{R^n}L_1(x)w(x)dx &= \int_{4Q}g_\mu^A(a)(x)w(x)dx\le ||g_\mu^A(a)||_{L^\infty(w)}w(4Q)\\
&\le C||a||_{L^\infty(w)}w(Q)\le C;
\end{aligned}$$

Similar to the proof of Theorem 1, we obtain

$$\int_{R^n} L_2(x)w(x)dx \le C$$

and

$$\int_{R^n} L_3(x,u)w(x)dx \le C.$$

Thus, using the condition of $L_4(x,u)$, we obtain

$$\int_{R^n} g_\mu^A(a)(x)w(x)dx \le C.$$

(ii) For any cube $Q = Q(x_0, d)$, let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$. We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $u \in 3Q \setminus 2Q$,

$$\begin{aligned}
&\tilde{F}_t^A(f)(x,y)\\
&= \tilde{F}_t^A(f_1)(x,y) + \int_{R^n} \frac{R_m(\tilde{A};x,z)}{|x-z|^m}\psi_t(y-z)f_2(z)dz\\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \left[\frac{(x-z)^\alpha}{|x-z|^m} - \frac{(u-z)^\alpha}{|u-z|^m}\right]\psi_t(y-z)f_2(z)dz\\
&\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \frac{(u-z)^\alpha}{|u-z|^m}\psi_t(y-z)f_2(z)dz,
\end{aligned}$$

then

$$\begin{aligned}
&\left|\tilde{g}_\mu^A(f)(x) - g_\mu\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right|\\
&= \left|\left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\tilde{F}_t^A(f)(x,y)\right\|\right.\\
&\qquad \left. - \left\|\left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right\|\right|\\
&\le \left\|\left(\frac{t}{t+|x-y|}\right)^{n\mu/2}\tilde{F}_t^A(f)(x,y)\right.\\
&\qquad \left. - \left(\frac{t}{t+|x_0-y|}\right)^{n\mu/2} F_t\left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m}f_2\right)(x_0)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\le \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} \tilde{F}_t^A(f_1)(x,y) \right\| \\
&\quad + \left\| \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\mu/2} \frac{R_m(\tilde{A};x,z)}{|x-z|^m} \psi_t(y-z) \right. \right. \\
&\qquad \left. \left. - \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} \frac{R_m(\tilde{A};x_0,z)}{|x_0-z|^m} \psi_t(x_0-z) \right] f_2(z)dz \right\| \\
&\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} \right. \\
&\qquad \left. \cdot \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \left[\frac{(y-z)^\alpha}{|y-z|^m} - \frac{(u-z)^\alpha}{|u-z|^m} \right] \psi_t(y-z) f_2(z)dz \right\| \\
&\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} \right. \\
&\qquad \left. \cdot \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{R^n} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(y-z) f_2(z)dz \right\| \\
&= M_1(x) + M_2(x) + M_3(x,u) + M_4(x,u).
\end{aligned}
$$

By the $L^\infty(w)$-boundedness of $\tilde{g}_\mu^A$, we get

$$\frac{1}{w(Q)} \int_Q M_1(x) w(x)dx \le ||\tilde{g}_\mu^A(f_1)||_{L^\infty(w)} \le C||f||_{L^\infty(w)};$$

Similar to the proof of Theorem 1, we obtain

$$\frac{1}{w(Q)} \int_Q M_2(x) w(x)dx \le C||f||_{L^\infty(w)}$$

and

$$\frac{1}{w(Q)} \int_Q M_3(x,u) w(x)dx \le C||f||_{L^\infty(w)}.$$

Thus, using the condition of $M_4(x,u)$, we obtain

$$\frac{1}{w(Q)} \int_Q \left| \tilde{g}_\mu^A(f)(x) - g_\mu \left(\frac{R_m(\tilde{A};x_0,\cdot)}{|x_0-\cdot|^m} f_2 \right)(x_0) \right| w(x)dx \le C||f||_{L^\infty(w)}.$$

This completes the proof of Theorem 4. □

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