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GATEAUX DIFFERENTIABILITY FOR FUNCTIONALS OF TYPE ORLICZ-LORENTZ

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ABSTRACT. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite nonatomic measure space and let $\Lambda_{w,\phi}$ be the Orlicz-Lorentz space. We study the Gateaux differentiability of the functional $\Psi_{w,\phi}(f) = \int_0^\infty \phi(f^*)w$. More precisely we give an exact characterization of those points in the Orlicz-Lorentz space $\Lambda_{w,\phi}$ where the Gateaux derivative exists. This paper extends known results already on Lorent spaces, $L_{w,q}$, $1 < q < \infty$. The case $q = 1$, it has been considered.

1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite nonatomic measure space, $\mathcal{M}_0 = \mathcal{M}_0(\Omega, \mathcal{A}, \mu)$ the class of μ-measurable functions on Ω that are finite μ-a.e..

As usual, for $f \in \mathcal{M}_0$ we denote by μ_f its distribution function and by f^* its decreasing rearrangement. If two functions f and g have the same distribution function we say that they are equimeasurable and we put $f \sim g$. The reader can see [**1**] for definitions and properties.

Now recall some basic notations and definitions. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, be differentiable, convex, $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$ and a weight function $w : (0, \gamma) \to \mathbb{R}_+$, for $\gamma \leq \infty$, be nonincreasing and locally integrable with respect to the Lebesgue measure m. For $f \in \mathcal{M}_0$ let

$$\Psi_{w,\phi}(f) = \int_0^\infty \phi(f^*(t))w(t)dm(t).$$

We consider the Orlicz-Lorentz space

$$\Lambda_{w,\phi} := \{f \in \mathcal{M}_0 : \Psi_{w,\phi}(\lambda f) < \infty \ \text{ for all } \ \lambda > 0\}.$$

It is clear that $w = \text{const}$, $\Lambda_{w,\phi}$ becomes an ordinary Orlicz space L_ϕ. On the other hand setting $\phi(t) = t^q$, $w(t) = t^{\frac{q}{p}-1}$ we obtain the Lorentz space $L^{p,q}$ in the case $1 \leq q \leq p < \infty$ and $\Psi_{w,\phi}(f) = \|f\|_{pq}^q$.

It is well known that the functional $\Psi_{w,\phi} : \mathcal{M}_0 \to [0, \infty]$ is an orthogonally subadditive convex modular and $\Psi_{w,\phi}(f) = \sup_{v\sim w} \int_0^\infty \phi(|f|)v$, (see [**7**]). In [**2**]

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the authors make the following assertion: If $1 < q < \infty$ and the weight w is strictly decreasing, it is known that $L_{w,q}$ has a Gateaux differentiable norm at f if and only if $\mu\{|f| = s\} = 0$ for any $s > 0$. However we observe that the Corollary 3 in [**2**] does not hold when $q = 1$. In section 4, we give an example which shows that the Corollary 3 in [**2**] is not true. So, we can not get in this case the set of points where the Gateaux derivative exists.

Our main purpose in this paper it will be to give an exact characterization of those points in $\Lambda_{w,\phi}$ where there is the Gateaux derivative of the functional $\Psi_{w,\phi}$, when ϕ is differentiable and w is a stritly decreasing function. We remark that this work generalizes the known results over differentibility in Lorentz spaces for $1 < q < \infty$.

For $f, h \in \Lambda_{w,\phi}$, we will use in this work the one-sided Gateaux derivatives $\gamma_+(f,h) = \lim_{s\to 0^+} \frac{\Psi_{w,\phi}(f+sh)-\Psi_{w,\phi}(f)}{s}$, $\gamma_-(f,h) = \lim_{s\to 0^-} \frac{\Psi_{w,\phi}(f+sh)-\Psi_{w,\phi}(f)}{s}$.

Definition 1.1. We say that a function $f \in \Lambda_{w,\phi}$ is a *smooth point* if there exists the Gateaux derivative of the functional $\Psi_{w,\phi}$ in f, i.e., if $\gamma_+(f,h) = \gamma_-(f,h)$ for all $h \in \Lambda_{w,\phi}$.We denote it by $\gamma(f,h)$.

Let $f \in \Lambda_{w,\phi}$. By redefining f, if necessary, on a set of μ-measure zero, we may assume that $|f|$ and f^* have the same non-null range, say $R(f)$. Since f^* is decreasing, if $\lambda \in R(f)$, each $I_f(\lambda) = \{t > 0 : f^*(t) = \lambda\}$ is either a singleton or an interval. The case where $I_f(\lambda)$ is an interval, can occur for at most countably many values of λ, say $W(f)$. We introduce the following sets, which will play an important role later,

$$E(f) = \Omega - \bigcup_{\lambda \in W(f)} C_f(\lambda) \quad \text{where} \quad C_f(\lambda) = \{x \in \Omega : |f(x)| = \lambda\},$$

$$\mathcal{E}^{w,\phi} = \{f \in \Lambda_{w,\phi} - \{0\} : \mu(\operatorname{supp}(f) - E(f)) = 0\}$$

and

$$\Delta^{w,\phi} = \mathcal{E}^{w,\phi} \cap \{f \in \Lambda_{w,\phi} : \mu(\Omega - \operatorname{supp}(f)) = 0 \quad \text{or} \quad m(\operatorname{supp}(f^*)) = \infty\}$$

where m is the Lebesgue measure and $\operatorname{supp}(f)$ is the support of the function f.

Let us now agree on some terminology. A function $\sigma : (\Omega, \mu) \to (\mathcal{S}, \nu)$ is called *a measure preserving transformation (m.p.t)* if for each ν-measurable set $I \subset \mathcal{S}$, $\sigma^{-1}(I)$ is μ-measurable and $\mu(\sigma^{-1}(I)) = \nu(I)$. It is very important to emphasize that any m.p.t. induce equimeasurability, that is, if $g \in \mathcal{M}_0(\mathcal{S}, \nu)$ then $|g| \circ \sigma$ is a μ-measurable function on Ω and $|g| \circ \sigma \sim |g|$.

If $f \in \Lambda_{w,\phi}$, then $\lim_{t\to\infty} f^*(t) = 0$. In consequence, by Ryff Theorem (see [**1**]) there is a m.p.t. $\sigma : \operatorname{supp}(f) \to \operatorname{supp}(f^*)$ such that $|f| = f^* \circ \sigma$ μ-a.e. on $\operatorname{supp}(f)$. We denote such a σ by σ_f and we observe that σ_f satisfies $\mu_f(|f|) \leq \sigma_f$ on $\operatorname{supp}(f)$.

2. A characterization of the smooth points in $\Lambda_{w,\phi}$ when $\phi'_+(0) = 0$

In this section, we obtain a characterization of smooth points in the Orlicz-Lorentz space $\Lambda_{w,\phi}$, when $\phi'_+(0) = 0$. More precisely, we prove that the set of smooth points in this $\Lambda_{w,\phi}$, is $\mathcal{E}^{w,\phi}$.

For the proof of the main theorem, we need some auxiliary lemmas.

Lemma 2.1. *Let $f, h \in \Lambda_{w,\phi}$, $A := \operatorname{supp}(h) - \operatorname{supp}(f)$ and s a nonzero real number be. If $\mu(A) > 0$, then there is a m.p.t σ_{f+sh} such that $|f + sh| = (f + sh)^* \circ \sigma_{f+sh}$ μ-a.e on $\operatorname{supp}(f + sh)$ and $\sigma_{h\chi_A} \le \sigma_{f+sh}$ μ-a.e on A, where $\sigma_{h\chi_A}$ is given by Ryff in* [**1**].

Proof. From [**1**] as $a_\lambda = \mu(C_{h\chi_A}(\lambda)) < \infty$,

$$\sigma_{h\chi_A}(x) = \begin{cases} \inf\{t : x \in E_{\lambda,t}\} + \mu_{h\chi_A}(\lambda) & \text{if } \lambda \in W(h\chi_A), x \in C_{h\chi_A}(\lambda) \\ \mu_{h\chi_A}(\lambda) & \text{otherwise} \end{cases}$$

where $\{E_{\lambda,t} : 0 \le t \le a_\lambda\}$ is an increasing family of μ-measurable subsets (i.f.m.s) of $C_{h\chi_A}(\lambda)$ such that $\mu(E_{\lambda,t}) = t$ for $0 \le t \le a_\lambda$.

For each $\lambda \in R(f + sh)$ we define a function $\alpha_\lambda : C_{f+sh}(\lambda) \to I_{f+sh}(\lambda)$ in the following way. If $\lambda \notin W(f+sh)$, then $I_{f+sh}(\lambda)$ is a singleton and we define $\alpha_\lambda(x) = \mu_{f+sh}(\lambda)$. Now suppose that $\lambda \in W(f + sh)$. Since $|s|W(h\chi_A) \subset W(f + sh)$ then $\lambda \notin |s|W(h\chi_A)$ or there is a $\beta \in W(h\chi_A)$ such that $\lambda = |s|\beta$. In the first case, let α_λ be the m.p.t given in [**1**], proposition II.7.4. For the second case, we have $C_{h\chi_A}(\beta) \subset C_{f+sh}(\lambda)$. We call $D := C_{f+sh}(\lambda) - C_{h\chi_A}(\beta)$ and $k_\lambda = \mu(D)$. If $k_\lambda = 0$ then $a_\beta = \mu(C_{f+sh}(\lambda))$. Here we consider $\{E_{\beta,t} : 0 \le t \le a_\beta\}$ an i.f.m.s of $C_{f+sh}(\lambda)$ with $\mu(E_{\beta,t}) = t$, $0 \le t \le a_\beta$ and the mapping $\alpha_\lambda(x) = \inf\{t : x \in E_{\beta,t}\} + \mu_{f+sh}(\lambda)$ is a m.p.t Finally, if $k_\lambda > 0$ let $\{R_{\lambda,t} : 0 \le t \le k_\lambda\}$ be an i.m.f.s of D such that $\mu(R_{\lambda,t}) = t$, $0 \le t \le k_\lambda$. Then

$$U_{\lambda,t} = \begin{cases} E_{\beta,t} & \text{if } 0 \le t \le a_\beta \\ R_{t-a_\beta} \cup C_{h\chi_A}(\beta) & \text{if } a_\beta < t \le a_\beta + k_\lambda \end{cases}$$

is an i.f.m.s of $C_{f+sh}(\lambda)$ such that $\mu(U_{\lambda,t}) = t$, $0 \le t \le a_\beta + k_\lambda$. So, $\alpha_\lambda(x) = \inf\{t : x \in U_{\beta,t}\} + \mu_{f+sh}(\lambda)$ is a m.p.t.

Now, we define $\sigma_{f+sh} : \operatorname{supp}(f + sh) \to \operatorname{supp}(f + sh)^*$ by

$$\sigma_{f+sh}(x) = \alpha_\lambda(x), \qquad (\lambda \in R(f + sh),\ x \in C_{f+sh}(\lambda)).$$

Clearly, σ_{f+sh} is a m.p.t and $|f + sh| = (f + sh)^* \circ \sigma_{f+sh}$ μ-a.e on $\operatorname{supp}(f + sh)$. On the other hand, for $x \in A$ and $\lambda = |h(x)|$, we have $x \in C_{h\chi_A}(\lambda) \subset C_{f+sh}(|s|\lambda)$. Since $\mu_{sh\chi_A} \le \mu_{f+sh}$, if $\lambda \in W(h\chi_A)$, we get $\sigma_{h\chi_A}(x) \le \inf\{t : x \in E_{\lambda,t}\} + \mu_{f+sh}(|s|\lambda) = \sigma_{f+sh}(x)$. If $\lambda \notin W(h\chi_A)$, $\sigma_{h\chi_A}(x) = \mu_{h\chi_A}(\lambda) \le \mu_{f+sh}(|s|\lambda) \le \sigma_{f+sh}(x)$. The proof is complete. □

Henceforth, we consider in this paper the m.p.t σ_{f+sh} given in lemma 2.1

Lemma 2.2. *Let $f, h \in \Lambda_{w,\phi}$. If f^* is continuous at t_0 then*

$$\lim_{s\to 0}(f + sh)^*(t_0) = f^*(t_0).$$

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be an arbitrary sequence such that $\lim_{n\to\infty} s_n = 0$. As $\lim_{n\to\infty} |f + s_n h| = |f|$, it is well known that $f^*(t) \le \underline{\lim}_n (f + s_n h)^*(t)$ for all $t \ge 0$.

Using a property of the decreasing rearrangement we obtain for $m \in \mathbb{N}$ and $t \ge 0$, $(f+s_nh)^*(t) = (f+s_nh)^*\left(\frac{m}{m+1}t + \frac{1}{m+1}t\right) \le f^*\left(\frac{m}{m+1}t\right) + |s_n| h^*\left(\frac{1}{m+1}t\right)$. Therefore, we have $\overline{\lim}_n (f+s_nh)^*(t) \le f^*\left(\frac{m}{m+1}t\right)$. As f^* is continuous in t_0, taking limit for $m \to \infty$, we have

$$\overline{\lim_n}(f + s_n h)^*(t_0) \le f^*(t_0).$$

The proof is complete. □

Lemma 2.3. *Let A, $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be subsets of Ω such that $\mu(B_n - A_n) = 0$ for all $n \in \mathbb{N}$ and $A \subset \underline{\lim}_n B_n =: B_0$. If $A_0 := \underline{\lim}_n A_n$ then $\mu(A) = \mu(A \cap A_0)$.*

Proof. We note that

$$A = (A \cap A_0) \cup (A \cap \cup_{n=1}^{\infty} (B_n - A_n)). \tag{2.1}$$

In fact, if $x \in A$, there exists $m \in \mathbb{N}$ such that $x \in \bigcap_{n=m}^{\infty} B_n$. Suppose that $x \notin A_0$ then there exists $k \in \mathbb{N}$, $k \ge m$ such that $x \notin A_k$. Thus $x \in B_k - A_k$. The reciprocal inclusion is obvious, so (2.1) is true.
Since $\mu(B_n - A_n) = 0$ for all $n \in \mathbb{N}$, (2.1) implies that $\mu(A) = \mu(A \cap A_0)$. □

Lemma 2.4. *Let $f \in \Lambda_{w,\phi}$ and $(s_n)_{n\in\mathbb{N}}$ be such that $\mu(E(f)) > 0$ and $\lim_{n\to\infty} s_n = 0$. Then for all $h \in \Lambda_{w,\phi}$,*

$$\lim_{n\to\infty} \sigma_{f+s_nh}(x) = \sigma_f(x)$$

μ-a.e on $E(f) \cap \operatorname{supp}(f)$.

Proof. Let $A := E(f) \cap \{x \in \operatorname{supp}(f) : |f(x)| = f^* \circ \sigma_f(x)\}$, $B_n := \operatorname{supp}(f + s_n h)$ and $A_n := \{x \in \operatorname{supp}(f+s_nh) : |f(x) + s_n h(x)| = (f+s_nh)^* \circ \sigma_{f+s_nh}(x)\}$. Clearly $A \subset \underline{\lim}_n B_n$. Since $|f + s_n h| = (f+s_nh)^* \circ \sigma_{f+s_nh}$ μ-a.e on $\operatorname{supp}(f + s_n h)$, we have $\mu(B_n - A_n) = 0$ for all $n \in \mathbb{N}$. As $\mu(E(f) \cap \operatorname{supp}(f)) = \mu(A)$, from lemma 2.3 we get

$$\mu(E(f) \cap \operatorname{supp}(f)) = \mu(A \cap \underline{\lim_n} A_n).$$

We will prove that $\lim_{n\to\infty} \sigma_{f+s_nh}(x) = \sigma_f(x)$ for all $x \in A \cap \underline{\lim}_n A_n$. Assume that it is false, then there exist $x \in A \cap \underline{\lim}_n A_n$, $\epsilon > 0$ and a subsequence of $(s_n)_{n\in\mathbb{N}}$ which we again denote by $(s_n)_{n\in\mathbb{N}}$ such that $|\sigma_{f+s_nh}(x) - \sigma_f(x)| > \epsilon$. Consider the following sets

$$\mathcal{N} = \{n \in \mathbb{N} : \sigma_{f+s_nh}(x) > \sigma_f(x) + \epsilon\} \text{ and } \mathcal{M} = \{n \in \mathbb{N} : \sigma_{f+s_nh}(x) < \sigma_f(x) - \epsilon\}.$$

Clearly some of these sets must be infinite. Suppose that card($\mathcal{N}$)$= \infty$. Since the points of noncontinuity of f^* are at most countably, there exists $0 < \delta < \epsilon$, such that f^* is continuous in $\sigma_f(x) + \delta$.

As $x \in A$, from lemma 2.2 we have

$$|f(x)| = f^*(\sigma_f(x)) > f^*(\sigma_f(x)+\frac{\delta}{2}) \geq f^*(\sigma_f(x)+\delta) = \lim_{n\to\infty}(f+s_nh)^*(\sigma_f(x)+\delta).$$

Since $x \in \underline{\lim}_n A_n$, for n large enough in $\mathcal{N}$ we obtain

$$|f(x)+s_nh(x)| = (f+s_nh)^* \circ \sigma_{f+s_nh}(x).$$

Therefore

$$\lim_{n\to\infty}(f+s_nh)^*(\sigma_f(x)+\delta) \geq \lim_{n\to\infty}(f+s_nh)^*(\sigma_{f+s_nh}(x)) = |f(x)|.$$

It is a contradiction, so the card($\mathcal{N}$) must be finite.
We now suppose that card($\mathcal{M}$)$= \infty$. Then

$$|f(x)| = f^*(\sigma_f(x)) < f^*(\sigma_f(x)-\frac{\epsilon}{2}) \leq f^*(\sigma_f(x)-\epsilon) \leq \underline{\lim}_n(f+s_nh)^*(\sigma_f(x)-\epsilon).$$

For n large enough in $\mathcal{M}$, $(f+s_nh)^*(\sigma_f(x)-\epsilon) \leq |f(x)+s_nh(x)|$, and we obtain

$$\underline{\lim}_n(f+s_nh)^*(\sigma_f(x)-\epsilon) \leq |f(x)|,$$

which is other contradiction, so the card($\mathcal{M}$) must be finite. The proof is complete. □

Lemma 2.5. *Let $f, h \in \Lambda_{w,\phi}$ be with $\mu(E(f)) > 0$. If $C := E(f) \cap \operatorname{supp}(f)$ then*

$$\lim_{s\to 0} \int_{\operatorname{supp}(f+sh)\cap C} w(\sigma_{f+sh})\frac{\phi(|f+sh|)-\phi(|f|)}{s}d\mu = \int_C w(\sigma_f)\phi'(|f|)\operatorname{sg}(f)hd\mu$$

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be an arbitrary sequence such that $\lim_{n\to\infty} s_n = 0$ and let $(w_n)_{n\in\mathbb{N}}$ be a sequence of μ-measurable functions on Ω defined by

$$w_n(x) = \begin{cases} w(\sigma_{f+s_nh}(x)) & \text{if } x \in \operatorname{supp}(f+s_nh)\cap C \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$\int_{\operatorname{supp}(f+s_nh)\cap C} w(\sigma_{f+s_nh})\frac{\phi(|f+s_nh|)-\phi(|f|)}{s_n}d\mu = \int_C w_n\frac{\phi(|f+s_nh|)-\phi(|f|)}{s_n}d\mu.$$

Let $H := \{t \in \operatorname{supp}(f^*) : w \text{ is continuous at } t\}$ and $D := A \cap \underline{\lim}_n A_n \cap \sigma_f^{-1}(H)$ where A and A_n are the sets defined in lemma 2.4 Then $\mu(D) = \mu(C)$. In fact, by the proof of the lemma 2.4 we have $\mu(A \cap \underline{\lim}_n A_n) = \mu(C)$. On the other hand, as $\operatorname{supp}(f) - \sigma_f^{-1}(H) \subset \sigma_f^{-1}(\operatorname{supp}(f^*) - H)$ and $\mu(\sigma_f^{-1}(\operatorname{supp}(f^*)-H)) = m(\operatorname{supp}(f^*)-H) = 0$, we get $\mu(D) = \mu(A \cap \underline{\lim}_n A_n)$.

So, if $x \in D$ we get

$$\lim_{n\to\infty} w_n\phi(2(|f|+|h|)) = w(\sigma_f)\phi(2(|f|+|h|)) \quad \textit{on } D$$

and

$$\lim_{n\to\infty} w_n\frac{\phi(|f+s_nh|)-\phi(|f|)}{s_n} = w(\sigma_f)\phi'(|f|)\operatorname{sg}(f)h \quad \textit{on } D.$$

Moreover, for all $n \in \mathbb{N}$ such that $0 < |s_n| \leq 1$, $\left| w_n \frac{\phi(|f+s_n h|)-\phi(|f|)}{s_n} \right| \leq w_n \phi(2(|f|+|h|))$. On the other hand,

$$\int_{\mathrm{supp}(f)} w(\sigma_f)\phi(2(|f|+|h|))d\mu \leq \int_0^\infty w\phi(2(|f|+|h|)^*)dt < \infty. \tag{2.2}$$

Then by the generalized Lebesgue convergence theorem we have

$$\lim_{n\to\infty} \int_C w_n \frac{\phi(|f+s_n h|)-\phi(|f|)}{s_n} d\mu = \int_C w(\sigma_f)\phi'(|f|)\,\mathrm{sg}(f) h d\mu.$$

□

Remark. We observe that the lemma 2.5 holds when $\phi'_+(0) > 0$.

Theorem 2.6. *If $f \in \mathcal{E}_{w,\phi}$ and $h \in \Lambda_{w,\phi}$ then*

$$\gamma(f,h) = \int_{\mathrm{supp}\,(f)} w(\sigma_f)\phi'(|f|)\,\mathrm{sg}(f) h d\mu.$$

Proof. Since

$$w\phi(f^*) \sim w(\sigma_f)\phi(|f|) \quad \text{and} \quad w\phi((f+sh)^*) \sim w(\sigma_{f+sh})\phi(|f+sh|)$$

we have

$$\Psi_{w,\phi}(f) = \int_{\mathrm{supp}\,(f)} w(\sigma_f)\phi(|f|)d\mu, \ \Psi_{w,\phi}(f+sh) = \int_{\mathrm{supp}\,(f+sh)} w(\sigma_{f+sh})\phi(|f+sh|)d\mu.$$

As $(w(\sigma_f))^* = w$ in $[0, \mu(\mathrm{supp}\,(f))$, by the Hardy-Littlewood inequality (see [**1**])

$$\int_{\mathrm{supp}\,(f)} w(\sigma_f)\phi(|f+sh|)d\mu \leq \int_0^\infty w(t)\phi((f+sh)^*(t))dt.$$

In consequence, for all $s > 0$ we get

$$\frac{1}{s}\left(\int_{\mathrm{supp}\,(f+sh)} w(\sigma_{f+sh})\phi(|f+sh|)d\mu - \int_{\mathrm{supp}\,(f)} w(\sigma_f)\phi(|f+sh|)d\mu \right) \geq 0.$$

So

$$\frac{\Psi_{w,\phi}(f+sh) - \Psi_{w,\phi}(f)}{s} \geq \int_{\mathrm{supp}\,(f)} w(\sigma_f)\frac{\phi(|f+sh|)-\phi(|f|)}{s} d\mu. \tag{2.3}$$

Analogously with $f+sh$ to instead of f, we get for $s > 0$ the inequality

$$\frac{\Psi_{w,\phi}(f+sh) - \Psi_{w,\phi}(f)}{s} \leq \int_{\mathrm{supp}\,(f+sh)} w(\sigma_{f+sh})\frac{\phi(|f+sh|)-\phi(|f|)}{s} d\mu. \tag{2.4}$$

Similarly for all $s < 0$

$$\int\limits_{\mathrm{supp}\,(f+sh)} w(\sigma_{f+sh})\frac{\phi(|f+sh|)-\phi(|f|)}{s}d\mu \le \frac{\Psi_{w,\phi}(f+sh)-\Psi_{w,\phi}(f)}{s}$$

$$(2.5) \qquad \le \int\limits_{\mathrm{supp}\,(f)} w(\sigma_f)\frac{\phi(|f+sh|)-\phi(|f|)}{s}d\mu.$$

For all $s,\ 0<|s|\le 1$, $\left|w(\sigma_f)\frac{\phi(|f+sh|)-\phi(|f|)}{s}\right| \le w(\sigma_f)\phi(2(|f|+|h|))$. Thus, from (2.4) and the Lebesgue convergence theorem we get

$$\lim_{s\to 0}\int_{\mathrm{supp}\,(f)} w(\sigma_f)\frac{\phi(|f+sh|)-\phi(|f|)}{s}d\mu = \int_{\mathrm{supp}\,(f)} w(\sigma_f)\phi'(|f|)\,\mathrm{sg}(f)h d\mu.$$

We now consider $A := \mathrm{supp}\,(h) - \mathrm{supp}\,(f)$ and the following functions

$$H(s) = \int\limits_{\mathrm{supp}\,(f+sh)} w(\sigma_{f+sh})\frac{\phi(|f+sh|)-\phi(|f|)}{s}d\mu,$$

$$K(s) = \int_A w(\sigma_{f+sh})\frac{\phi(|sh|)}{s}d\mu,$$

and

$$L(s) = \int_{\mathrm{supp}\,(f+sh)\cap E(f)\cap\mathrm{supp}\,(f)} w(\sigma_{f+sh})\frac{\phi(|f+sh|)-\phi(|f|)}{s}d\mu.$$

Since $f\in\mathcal{E}_{w,\phi}$ we have $H(s) = L(s)+K(s)$. In addition, by lemma 2.5 we get

$$\lim_{s\to 0} L(s) = \int_{\mathrm{supp}\,(f)} w(\sigma_f)\phi'(|f|)\,\mathrm{sg}(f)h d\mu.$$

If $\mu(A)=0$, then $K(s)=0$. In otherwise, from lemma 2.1, we get

$$\left|w(\sigma_{f+sh})\frac{\phi(|sh|)}{s}\right| \le w(\sigma_{h\chi_A})\frac{\phi(|sh|)}{|s|}\ \mu\text{-a.e on } A.$$

Moreover, $\lim_{s\to 0} w(\sigma_{h\chi_A})\frac{\phi(|sh|)}{|s|} = 0$, $w(\sigma_{h\chi_A})\frac{\phi(|sh|)}{|s|} \le w(\sigma_{h\chi_A})\phi(|h|)$ if $|s|\le 1$ and

$$\int_A w(\sigma_{h\chi_A})\phi(|h|)d\mu \le \Psi_{w,\phi}(h) < \infty.$$

Thus by generalized Lebesgue convergence theorem, $\lim_{s\to 0} K(s) = 0$, which implies that

$$\lim_{s\to 0} H(s) = \int_{\mathrm{supp}\,(f)} w(\sigma_f)\phi'(|f|)\,\mathrm{sg}(f)h d\mu.$$

Finally from (2.3), (2.4) and (2.5) we obtain

$$\gamma(f,h) = \int_{\mathrm{supp}\,(f)} w(\sigma_f)\phi'(|f|)\,\mathrm{sg}(f)h d\mu.$$

□

Corollary 2.7. *For all $f \in \mathcal{E}^{w,\phi}$,*

$$\Gamma(f) = w(\mu_f(|f|))\phi'(|f|)\,\mathrm{sg}(f)$$

where Γ denotes the Gateaux gradient operator.

Proof. Since $\sigma_f(x) = \mu_f(|f(x)|)$ on $\mathrm{supp}(f)$, the corollary is an immediate consequence of theorem 2.6 □

Next, we establish the main result of this section.

Theorem 2.8. *Suppose that w is a strictly decreasing function. $f \in \Lambda_{w,\phi}$ is a smooth point if only if $f \in \mathcal{E}^{w,\phi}$.*

Proof. The sufficient condition follows from theorem 2.6.

Suppose now that $f \notin \mathcal{E}^{w,\phi}$. If $f = 0$ then for any $h \in \Lambda_{w,\phi} - \{0\}$, $\gamma_+(f,h) \neq \gamma_-(f,h)$, so f is not a smooth point. Assume $f \neq 0$. Then there exists $r > 0$ such that $\mu(C_f(r)) = a > 0$. As μ is nonatomic, there is a set $A \subset C_f(r)$ such that $\mu(A) = \frac{a}{4}$. We define $h \in \Lambda_{w,\phi}$ by $h(x) = \chi_A(x)\,\mathrm{sg}(f(x))$. For $s > 0$, we call $f_s = f + sh$ and $g_s = f - sh$. From the definition of h, for sufficiently small positive s, we have $|f_s| = |f| + s|h|$ and $|g_s| = |f| - s|h|$. Next, if $s < r$ and $\mu_f(r) < \mu_f\,(r+s) + \frac{a}{4}$, we obtain

$$\mu_{f_s}(\lambda) = \begin{cases} \mu_f(\lambda) + \frac{a}{4} & \text{if } r \leq \lambda < r+s \\ \mu_f(\lambda) & \text{otherwise} \end{cases}$$

and

$$f_s^*(t) = \begin{cases} r+s & \text{if } \mu_f\,(r+s) \leq t < \mu_f\,(r+s) + \frac{a}{4} \\ f^*\left(t - \frac{a}{4}\right) & \text{if } \mu_f\,(r+s) + \frac{a}{4} \leq t < \mu_f(r) + \frac{a}{4} \\ f^*(t) & \text{otherwise} \end{cases}$$

Thus,

$$\begin{aligned} \frac{\Psi_{w,\phi}(f_s) - \Psi_{w,\phi}(f)}{s} &= \int_{\mu_f(r+s)}^{\mu_f(r)} w(t)\frac{\phi(r+s) - \phi(f^*(t))}{s}dt \\ &+ \int_{\mu_f(r)}^{\mu_f(r+s)+\frac{a}{4}} w(t)\frac{\phi(r+s) - \phi(r)}{s}dt \\ &+ \int_{\mu_f(r+s)+\frac{a}{4}}^{\mu_f(r)+\frac{a}{4}} w(t)\frac{\phi(f^*(t-\frac{a}{4})) - \phi(r)}{s}dt. \end{aligned}$$

It is easy to prove that the first and third terms in the equality above tend to zero as $s \to 0$. A straightforward computation leads to

$$\gamma_+(f,h) = \phi'(r)\int_{\mu_f(r)}^{\mu_f(r)+\frac{a}{4}} w(t)dt.$$

Since $\phi(t) > 0$ for $t > 0$, $\phi'(r) > 0$. On the other hand, if $s < r$ and $\mu_f\,(r-s) - \frac{a}{4} < \mu_f(r) + a$, we obtain, in a similar way,

$$\begin{aligned}\frac{\Psi_{w,\phi}(g_s) - \Psi_{w,\phi}(f)}{-s} &= \int_{\mu_f(r)+\frac{3}{4}a}^{\mu_f(r-s)-\frac{a}{4}} w(t)\frac{\phi(r) - \phi(f^*(t+\frac{a}{4}))}{s}dt \\ &+ \int_{\mu_f(r-s)-\frac{a}{4}}^{\mu_f(r)+a} w(t)\frac{\phi(r)-\phi(r-s)}{s}dt \\ &+ \int_{\mu_f(r)+a}^{\mu_f(r-s)} w(t)\frac{\phi(f^*(t)) - \phi(r-s)}{s}dt.\end{aligned}$$

Therefore

$$\gamma_-(f,h) = \phi'(r)\int_{\mu_f(r)+\frac{3}{4}a}^{\mu_f(r)+a} w(t)dt.$$

Since w is strictly decreasing, $\gamma_-(f,h) < \gamma_+(f,h)$. We conclude that f is not a smooth point. □

Remark. We observe that the necessary condition of theorem 2.8 holds when $\phi'_+(0) > 0$.

3. A characterization of the smooth points in $\Lambda_{w,\phi}$ when $\phi'_+(0) > 0$

In this section we give a characterization of smooth points in Orlicz-Lorentz space $\Lambda_{w,\phi}$ when $\phi'_+(0) > 0$. We will use a similar technical to the developed in the section two. We assume here that $\lim\limits_{t\to\infty} w(t) = 0$.

We begin with two auxiliary lemmas.

Lemma 3.1. *Let $f, h \in \Lambda_{w,\phi}$. If μ_f is continuous at λ then*

$$\lim_{s\to 0}\mu_{f+sh}(\lambda) = \mu_f(\lambda).$$

Proof. Let s be such that $0 < |s| < 1$. Using properties of the distribution function we have

$$\begin{aligned}\mu_{f+sh}(\lambda) &= \mu_{f+sh}\left((1-\sqrt{|s|})\lambda + \sqrt{|s|}\lambda\right) \\ &\le \mu_f\left((1-\sqrt{|s|})\lambda\right) + \mu_h\left(\frac{\sqrt{|s|}}{|s|}\lambda\right),\end{aligned} \tag{3.1}$$

$\lim_{s\to 0}\mu_h\left(\frac{\sqrt{|s|}}{|s|}\lambda\right) = 0$ and $\mu_f(\lambda) \le \underline{\lim}_s\,\mu_{f+sh}(\lambda)$. In addition, by hypotheses, we have $\lim_{s\to 0}\mu_f((1-\sqrt{|s|})\lambda) = \mu_f(\lambda)$. From (3.1), $\overline{\lim}_s\,\mu_{f+sh}(\lambda) \le \mu_f(\lambda)$. The proof follows immediately. □

Lemma 3.2. *Let $f \in \mathcal{E}^{w,\phi}$ and $h \in \Lambda_{w,\phi}$ be. If $m(\mathrm{supp}\,(f^*)) = \infty$ then*

$$\lim_{s\to 0}\sigma_{f+sh}(x) = \infty \quad \text{for all } x \in \mathrm{supp}\,(h) - \mathrm{supp}\,(f).$$

Proof. If $\operatorname{supp}(h) - \operatorname{supp}(f) = \varnothing$, it is trivial. In either case, let $x \in \operatorname{supp}(h) - \operatorname{supp}(f)$. Given $M > 0$, since $m(\operatorname{supp}(f^*)) = \infty$ we can choice s_1 such that $\mu_f(|s_1 h(x)|) > M$. As $f \in \mathcal{E}^{w,\phi}$, μ_f is continuous. Then by lemma 3.1 $\lim_{s\to 0} \mu_{f+sh}(|s_1 h(x)|) = \mu_f(|s_1 h(x)|)$. Thus $\mu_{f+sh}(|s_1 h(x)|) > M$ for sufficiently small s. It follows that $M < \mu_{f+sh}(|sh(x)|)$ for sufficiently small s. Finally, for the last remark in the introduction we have $\mu_{f+sh}(|sh(x)|) \le \sigma_{f+sh}(x)$. The proof is complete. □

Theorem 3.3. *If $f \in \Delta^{w,\phi}$ and $h \in \Lambda_{w,\phi}$ then*

$$\gamma(f,h) = \int_{\operatorname{supp}(f)} w(\sigma_f)\phi'(|f|)\operatorname{sg}(f) h \, d\mu.$$

Proof. We proceed analogously to the proof of theorem 2.6. According to the remark of lemma 2.5, we only need to prove that $\lim_{s\to 0} K(s) = 0$. Let $A := \operatorname{supp}(h) - \operatorname{supp}(f)$. If $\mu(\Omega - \operatorname{supp}(f)) = 0$ then $\mu(A) = 0$. Therefore $K(s) \equiv 0$. Suppose that $\mu(A) > 0$ and $m(\operatorname{supp}(f^*)) = \infty$. From lemma 2.1, $\sigma_{h\chi_A}(x) \le \sigma_{f+sh}(x)$ for all $x \in A$. In addition, $\int_A w(\sigma_{h\chi_A})\phi(|h|)d\mu \le \Psi_{w,\phi}(h) < \infty$. Therefore by lemma 3.2 and the Lebesgue convergence theorem, $\lim_{s\to 0} K(s) = 0$. □

Corollary 3.4. *For all $f \in \Delta^{w,\phi}$,*

$$\Gamma(f) = w(\mu_f(|f|))\phi'(|f|)\operatorname{sg}(f)$$

where Γ denotes the Gateaux gradient operator.

Theorem 3.5. *Suppose that w is a strictly decreasing function. $f \in \Lambda_{w,\phi}$ is a smooth point if only if $f \in \Delta^{w,\phi}$.*

Proof. The sufficient condition is immediate consequence of theorem 3.3.

Assume that $f \notin \Delta^{w,\phi}$. If $f \notin \mathcal{E}^{w,\phi}$ then f is not a smooth point because the remark of theorem 2.8. We now suppose that $f \in \mathcal{E}^{w,\phi}$, so $\mu(\Omega - \operatorname{supp}(f)) > 0$ and $m(\operatorname{supp}(f^*)) < \infty$. Take a set $A \subset \Omega - \operatorname{supp}(f)$ with $0 < \mu(A) < \infty$ and let $h(x) = \chi_A(x)$. A straightforward computation leads to

$$\lim_{s\to 0^\pm} \frac{\Psi_{w,\phi}(f+sh) - \Psi_{w,\phi}(f)}{s} = \pm\phi'_+(0)\int_{\mu_f(0)}^{\mu_f(0)+\mu(A)} w(t)dt \tag{3.2}$$

i.e., $\gamma_+(f,h) \neq \gamma_-(f,h)$. □

Remark. It is well known that $\mathcal{E}^{w,\phi}$ and $\Delta^{w,\phi}$ are dense sets in the Lorentz space because the points of Gateaux-differentiability of the norm in a separable space form always a dense set.

4. An example

In [**2**], the authors have established the following corollary for $1 \le q < \infty$:

Corollary. Let f and h be any elements in $L_{w,q}$, Then

$$\lim_{s\downarrow 0} \frac{\|f+sh\|^q - \|f\|^q}{s} = \lim_{s\uparrow 0} \frac{\|f+sh\|^q - \|f\|^q}{s}$$

if and only if for all $\lambda > 0$, $\mu(|f| = \lambda) > 0$ implies $h \cdot \mathrm{sg}(f)$ is constant on $|f|^{-1}(\lambda)$.

However, next we give an example which shows that the Corollary does not hold for $q = 1$.

Example. Let $f \in L_{w,1}(0, \infty)$ be defined by $f(x) = (1-x)\chi_{[0,1]}(x)$. According to corollary 3, f should be a smooth point, however $f \notin \Delta^{p,1}$.

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