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ESTIMATES FOR DERIVATIVES OF THE GREEN FUNCTIONS ON HOMOGENEOUS MANIFOLDS OF NEGATIVE CURVATURE

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Abstract. We consider the Green functions $\mathcal{G}$ for second-order coercive differential operators on homogeneous manifolds of negative curvature, being a semi-direct product of a nilpotent Lie group N and $A = \mathbb{R}^+$.

Estimates for derivatives of the Green functions $\mathcal{G}$ with respect to the N and A-variables are obtained. This paper completes a previous work of the author (see [**12, 13**]) where estimates for derivatives of the Green functions for the noncoercive operators has been obtained. Here we show how to use the previous methods and results from [**12**] in order to get analogous estimates for coercive operators.

1. Introduction.

Let M be a connected, simply connected homogeneous manifolds of negative curvature. Such a manifold is a solvable Lie group $S = NA$, a semi-direct product of a nilpotent Lie group N and an Abelian group $A = \mathbb{R}^+$. Moreover, for an H belonging to the Lie algebra $\mathfrak{a}$ of A, the real parts of the eigenvalues of $\mathrm{Ad}_{\exp H}|_{\mathfrak{n}}$, where $\mathfrak{n}$ is the Lie algebra of N, are all greater than 0. Conversely, every such a group equipped with a suitable left-invariant metric becomes a homogeneous Riemannian manifold with negative curvature (see [**7**]).

On S we consider a second order left-invariant operator

$$\mathcal{L} = \sum_{j=0}^{m} Y_j^2 + Y.$$

We assume that $Y_0, Y_1, \ldots, Y_m$ generate the Lie algebra $\mathfrak{s}$ of S. We can always make $Y_0, \ldots, Y_m$ linearly independent and moreover, we can choose $Y_0, Y_1, \ldots, Y_m$ so that $Y_1(e), \ldots, Y_m(e)$ belong to $\mathfrak{n}$. Let $\pi : S \to A = S/N$ be the canonical homomorphism. Then the image of $\mathcal{L}$ under π is a second order left-invariant

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operator on $\mathbb{R}^+$,

$$(a\partial_a)^2 - \gamma a\partial_a,$$

where $\gamma = \gamma_{\mathcal{L}} \in \mathbb{R}$.

Finally, $\mathcal{L}$ can be written as

$$\mathcal{L} = \mathcal{L}_\gamma = \sum_j \Phi_a(X_j)^2 + \Phi_a(X) + a^2\partial_a^2 + (1-\gamma)\partial_a, \tag{1.1}$$

where $\gamma \neq 0$, $X, X_1, \ldots, X_m$ are left-invariant vector fields on N, moreover, $X_1, \ldots, X_m$ linearly independent and generate $\mathfrak{n}$, $\Phi_a = \mathrm{Ad}_{\exp(\log a)Y_0} = e^{(\log a)\,\mathrm{ad}_{Y_0}} = e^{(\log a)D}$. $D = \mathrm{ad}_{Y_0}$ is a derivation of the Lie algebra $\mathfrak{n}$ of the Lie group N such that the real parts d_j of the eigenvalues λ_j of D are positive. By multiplying $\mathcal{L}_\gamma$ by a constant, i.e., changing Y_0, we can make d_j arbitrarily large (see [**5**]).

Let $\mathcal{G}_\gamma(xa, yb)$ be the *Green function* for $\mathcal{L}_\gamma$. $\mathcal{G}_\gamma$ is (uniquely) defined by two conditions:

i) $\mathcal{L}_\gamma\mathcal{G}_\gamma(\cdot, yb) = -\delta_{yb}$ as distributions
(functions are identified with distributions via the right Haar measure),

ii) for every $yb \in S$, $\mathcal{G}_\gamma(\cdot, yb)$ is a potential for $\mathcal{L}_\gamma$.

Let

$$\mathcal{G}_\gamma(x, a) := \mathcal{G}_\gamma(e, xa), \tag{1.2}$$

where e is the identity element of the group S. Since $\mathcal{L}_\gamma$ is left-invariant it is easily seen that

$$\mathcal{G}_\gamma(xa, yb) = \mathcal{G}_\gamma(e, yb(xa)^{-1}) = \mathcal{G}_\gamma(yb(xa)^{-1}).$$

In this paper we call $\mathcal{G}_\gamma(x, a)$ defined in (1.2) the Green function for $\mathcal{L}_\gamma$.

The main goal of this paper is to give estimates for derivatives of the Green function (1.2) for $\mathcal{L}_\gamma$ when $\gamma \neq 0$, i.e., when $\mathcal{L}_\gamma$ is *coercive* ($\mathcal{L}$ is coercive if there is an $\varepsilon > 0$ such that $\mathcal{L} + \varepsilon I$ admits the Green function), with respect to x-variables (Theorem 3.1) and a-variable (Theorem 3.2). The case $\gamma = 0$ (i.e., noncoercive case) has been studied by the author in [**12**] and [**13**].

It is worth noting that our definition of coercivity is a little bit different than that used e.g. in [**1**]. Namely, for us, $\mathcal{L}$ is coercive if it is weakly coercive in Ancona's terminology. There is a relation between the notion of coercivity property in the sense used in the theory of partial differential eqns (i.e., that an appropriate bilinear form is coercive, [**8**]) and weak coercivity. For this the reader is referred to [**1**].

In this paper we are going to prove the following estimates. Let $\gamma > 0$. For every neighborhood $\mathcal{U}$ of the identity e of NA there is a constant $C = C(\gamma)$ such that we have

$$|\mathcal{X}^I\mathcal{G}_{-\gamma}(x, a)| \leq \begin{cases} C(|x|+a)^{-\|I\|-Q-\gamma} & \\ \times(1 + |\log(|x|+a)^{-1}|)^{\|I\|_0} & \text{for } (x, a) \in (\mathcal{Q} \cup \mathcal{U})^c, \\ C & \text{for } (x, a) \in \mathcal{Q} \setminus \mathcal{U} \end{cases} \tag{1.3}$$

and

$$(1.4)\qquad |\mathcal{X}^I\mathcal{G}_\gamma(x,a)| \le \begin{cases} C(|x|+a)^{-\|I\|-Q-\gamma}a^\gamma & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0} & \text{for } (x,a)\in(\mathcal{Q}\cup\mathcal{U})^c, \\ Ca^\gamma & \text{for } (x,a)\in\mathcal{Q}\setminus\mathcal{U}, \end{cases}$$

where $|\cdot|$ stands for a "homogeneous norm" on N, $\mathcal{Q}=\{|x|\le 1, a\le 1\}$, $\|I\|$ is a suitably defined length of the multi-index I and $\|I\|_0$ is a certain number depending on I and the nilpotent part of the derivation D. In particular, $\|I\|_0$ is equal to 0 if the action of $A=\mathbb{R}^+$ on N, given by Φ_a, is diagonal or, if $I=0$. $\mathcal{X}_1,\dots,\mathcal{X}_n$ is an appropriately chosen basis of $\mathfrak{n}$. For the precise definitions of all the notions that have appeared here see Sect. 2.

For $\partial_a^k\mathcal{G}_\gamma(x,a)$, $k\ge 0$ and $\gamma\ne 0$ we have what follows. Let $\gamma>0$.

$$(1.5)\qquad |\partial_a^k\mathcal{G}_{-\gamma}(x,a)| \le \begin{cases} Ca^{-k}(|x|+a)^{-Q-\gamma} & \text{for } (x,a)\in(\mathcal{Q}\cup\mathcal{U})^c, \\ Ca^{-k} & \text{for } (x,a)\in\mathcal{Q}\setminus\mathcal{U} \end{cases}$$

and

$$(1.6)\qquad |\partial_a^k\mathcal{G}_{\gamma}(x,a)| \le \begin{cases} Ca^{\gamma-k}(|x|+a)^{-Q-\gamma} & \text{for } (x,a)\in(\mathcal{Q}\cup\mathcal{U})^c, \\ Ca^{\gamma-k} & \text{for } (x,a)\in\mathcal{Q}\setminus\mathcal{U}, \end{cases}$$

It should be said that the estimate for the Green function itself (i.e., I=0) with $\gamma>0$, also from below, was proved by E. Damek in [**3**] and then by the author for $\gamma=0$ in [**16**] but at that time it was impossible to prove analogous estimate for derivatives with respect to x. The reason was that we did not have sufficient estimates for the derivatives of the transition probabilities of the evolution on N generated by an appropriate operator which appears as the "horizontal" component of the diffusion on $N\times\mathbb{R}^+$ generated by $a^{-2}\mathcal{L}_{-\gamma}$ (cf. [**4**]). These estimates have been obtained by the author in [**17**] and eventually led up to the estimate (1.3), (1.4), (1.5) and (1.6) for $\gamma=0$ (see [**12**] and [**13**] for mixed derivatives which required a little bit different approach). Here we are going to present how to use results from [**12**] in order to get estimates in the coercive case.

The proofs of (1.3), (1.4), (1.5) and (1.6) require both analytic and probabilistic techniques. Some of them have been introduced in [**5**], [**4**] and [**16**]. This paper also heavily depend on some results from [**12**].

The structure of the paper is as follows. In Sect. 2 we state precisely notation and all necessary definitions. In particular, we recall a definition of the Bessel process which appears as the "vertical" component of the diffusion generated by $a^{-2}\mathcal{L}_{-\gamma}$ on $N\times\mathbb{R}^+$ as well as the notion of the evolution on N generated by an appropriate operator which appears as the "horizontal" component of the diffusion on $N\times\mathbb{R}^+$ mentioned in the Introduction above (cf. [**4, 12**]).

Finally, in Section 3 we state precisely the estimates (1.3), (1.4) (see Theorem 3.1) and (1.5), (1.6) (see Theorem 3.2) and we give their proofs.

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2. Preliminaries

2.1. NA groups

The best reference for this Sect. is [**5**] and [**6**]. Let N be a connected and simply connected nilpotent Lie group. Let D be a derivation of the Lie algebra $\mathfrak{n}$ of N. For every $a \in \mathbb{R}^+$ we define an automorphism Φ_a of $\mathfrak{n}$ by the formula

$$\Phi_a = e^{(\log a)D}.$$

Witing $x = \exp X$ we put

$$\Phi_a(x) := \exp \Phi_a(X).$$

Let $\mathfrak{n}^{\mathbb{C}}$ be the complexification of $\mathfrak{n}$. Define

$$\mathfrak{n}^{\mathbb{C}}_{\lambda} = \{X \in \mathfrak{n}^{\mathbb{C}} : \exists k > 0 \text{ such that } (D - \lambda I)^k = 0\}.$$

Then

$$\mathfrak{n} = \bigoplus_{\mathrm{Im}\lambda \geq 0} V_\lambda, \tag{2.1}$$

where

$$V_\lambda = \begin{cases} V_{\bar{\lambda}} = (\mathfrak{n}^{\mathbb{C}} \oplus \mathfrak{n}^{\mathbb{C}}_{\bar{\lambda}}) \cap \mathfrak{n} & \text{if } \mathrm{Im}\lambda \neq 0, \\ \mathfrak{n}^{\mathbb{C}}_{\lambda} \cap \mathfrak{n} & \text{if } \mathrm{Im}\lambda = 0. \end{cases}$$

We assume that the real parts d_j of the eigenvalues λ_j of the matrix D are strictly greater than 0. We define the number

$$Q = \sum_j \mathrm{Re}\ \lambda_j = \sum_j d_j \tag{2.2}$$

and we refer to this as a *"homogeneous dimension"* of N. In this paper $D = \mathrm{ad}_{Y_0}$ (see Introduction). Under the assumption on positivity of d_j, (2.1) is a gradation of $\mathfrak{n}$.

We consider a group S which is a *semi-direct* product of N and the multiplicative group $A = \mathbb{R}^+ = \{\exp tY_0 : t \in \mathbb{R}\}$:

$$S = NA = \{xa : x \in N, a \in A\}$$

with multiplication given by the formula

$$(xa)(yb) = (x\Phi_a(y)\ ab).$$

In N we define a *"homogeneous norm"*, $|\cdot|$ (cf. [**5, 4**]) as follows. Let $(\cdot,\cdot)$ be a fixed inner product in $\mathfrak{n}$. We define a new inner product

$$\langle X, Y\rangle = \int_0^1 \Big(\Phi_a(X), \Phi_a(Y)\Big)\frac{da}{a} \tag{2.3}$$

and the corresponding norm

$$\|X\| = \langle X, X\rangle^{1/2}.$$

We put

$$|X| = \left(\inf\{a > 0 : \|\Phi_a(X)\| \geq 1\}\right)^{-1}.$$

One can easily show that for every $Y \neq 0$ there exists precisely one $a > 0$ such that $Y = \Phi_a(X)$ with $|X| = 1$. Then we have $|Y| = a$.

Finally, we define the homogeneous norm on N. For $x = \exp X$ we put

$$|x| = |X|.$$

Notice that if the action of $A = \mathbb{R}^+$ on N (given by Φ_a) is diagonal the norm we have just defined is the usual homogeneous norm on N and the number Q in (2.2) is simply the homogeneous dimension of N (see [**6**]).

Having all that in mind we define appropriate derivatives (see also [**5**]). We fix an inner product (2.3) in $\mathfrak{n}$ so that V_{λ_j}, $j = 1, \dots, k$ are mutually orthogonal and an orthonormal basis $\mathcal{X}_1, \dots, \mathcal{X}_n$ of $\mathfrak{n}$. The enveloping algebra $\mathfrak{U}(\mathfrak{n})$ of $\mathfrak{n}$ is identified with the polynomials in $\mathcal{X}_1, \dots, \mathcal{X}_n$. In $\mathfrak{U}(\mathfrak{n})$ we define $\langle \mathcal{X}_1 \otimes \dots \otimes \mathcal{X}_r, \mathcal{Y}_1 \otimes \dots \otimes \mathcal{Y}_r \rangle = \prod_{j=1}^r \langle \mathcal{X}_j, \mathcal{Y}_j \rangle$. Let V_j^r be the symmetric tensor product of r copies of V_{λ_j}. For $I = (i_1, \dots, i_k) \in (\mathbb{N} \cup \{0\})^k$ let

$$\mathcal{X}^I = \mathcal{X}_1^{(i_1)} \dots \mathcal{X}_k^{(i_k)}, \text{ where } \mathcal{X}_j^{(i_j)} \in V_j^{i_j}.$$

Then for $\mathcal{X} \in V_{\lambda_j}$

$$\|\Phi_a(\mathcal{X})\| \leq c \exp(d_j \log a + D_j \log(1 + |\log a|)),$$

where $d_j = \mathrm{Re}\lambda_j$ and $D_j = \dim V_{\lambda_j} - 1$, and so

$$\|\Phi_a(\mathcal{X}^I)\| \leq \exp\left(\sum_{j=1}^k i_j(d_j \log a + D_j \log(1 + |\log a|))\right) \prod_{j=1}^k \|\mathcal{X}_j^{(i_j)}\|. \tag{2.4}$$

2.2. Bessel process

Let b_t denotes the Bessel process with a parameter $\alpha \geq 0$ (cf. [**10**]), i.e., a continuous Markov process with the state space $[0, +\infty)$ generated by $\partial_a^2 + \frac{2\alpha+1}{a}\partial_a$. The transition function with respect to the measure $y^{2\alpha+1}dy$ is given, e.g. in [**2, 10**], by:

$$p_t(x,y) = \begin{cases} \frac{1}{2t} \exp\left(\frac{-x^2-y^2}{4t}\right) I_\alpha\left(\frac{xy}{2t}\right) \frac{1}{(xy)^\alpha} & \text{for } x, y > 0, \\ \frac{1}{2^\alpha (2t)^{\alpha+1} \Gamma(\alpha+1)} \exp\left(\frac{-y^2}{4t}\right) & \text{for } x = 0,\ y > 0, \end{cases}$$

where

$$I_\alpha(x) = \sum_{k=0}^\infty \frac{(x/2)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}$$

is *the Bessel function* (see [**9**]). Therefore for $x \geq 0$ and a measurable set $B \subset (0, \infty)$:

$$\mathbf{P}_x(b_t \in B) = \int_B p_t(x,y) y^{2\alpha+1} dy.$$

If b_t is the Bessel process with a parameter α starting from x, i.e. $b_0 = x$, then we will write that $b_t \in \mathrm{BESS}_x(\alpha)$ or simply $b_t \in \mathrm{BESS}(\alpha)$ if the starting point is not important or is clear from the context.

Properties of the Bessel process are very well known and their proofs are rather standard. They can be found e.g. in [**10, 4, 15, 14**]. However, in our paper we will not explicitly make use of any particular property of the Bessel process. What we only need is the possibility to generalize some lemmas from Section 5 in [**12**] (see Proposition 3.3 in Section 3 below).

2.3. Evolutions

Let $X, X_1, \ldots, X_m$ be as in (1.1). Let $\sigma : [0,\infty) \longrightarrow [0,\infty)$ be a continuous function such that $\sigma(t) > 0$ for every $t > 0$. We consider the family of evolutions operators $L_{\sigma(t)} - \partial_t$, where

$$L_{\sigma(t)} = \sigma(t)^{-2}\left(\sum_j \Phi_{\sigma(t)}(X_j)^2 + \Phi_{\sigma(t)}(X)\right). \tag{2.5}$$

Since we may assume that $X_1, \ldots, X_m$ are linearly independent we select $X_{m+1}, \ldots, X_n$ so that $X_1, \ldots, X_n$ form a basis of $\mathfrak{n}$. For a multi-index $I = (i_1, \ldots, i_n)$, $i_j \in \mathbb{Z}^+$ and the basis $X_1, \ldots, X_n$ of the Lie algebra $\mathfrak{n}$ of N we write: $X^I = X_1^{i_1} \ldots X_n^{i_n}$ and $|I| = i_1 + \ldots + i_n$. For $k = 0, 1, \ldots, \infty$ we define:

$$C^k = \{f : X^I f \in C(N), \text{ for } |I| < k+1\}$$

and

$$C^k_\infty = \{f \in C^k : \lim_{x\to\infty} X^I f(x) \text{ exists for } |I| < k+1\}.$$

For $k < \infty$ the space C^k_∞ is a Banach space with the norm

$$\|f\|_{C^k_\infty} = \sum_{|I|\le k} \|X^I f\|_{C(N)}.$$

Let $\{U^\sigma(s,t) : 0 \le s \le t\}$ be the unique family of bounded operators on $C_\infty = C^0_\infty$ which satisfy

i) $U^\sigma(s,s) = I$,
ii) $U^\sigma(s,r)U^\sigma(r,t) = U^\sigma(s,t)$, $s < r < t$,
iii) $\partial_s U^\sigma(s,t)f = -L_{\sigma(s)}U^\sigma(s,t)f$ for every $f \in C_\infty$,
iv) $\partial_t U^\sigma(s,t)f = U^\sigma(s,t)L_{\sigma(t)}f$ for every $f \in C_\infty$,
v) $U^\sigma(s,t) : C^2_\infty \longrightarrow C^2_\infty$.

$U^\sigma(s,t)$ is a convolution operator. Namely, $U^\sigma(s,t)f = f * p^\sigma(t,s)$, where $p^\sigma(t,s)$ is a smooth density of a probability measure. By ii) we have $p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s)$ for $t > r > s$. Existence of the family $U^\sigma(s,t)$ follows from [**11**].

3. The main results and their proofs

In this section we obtain pointwise estimates for derivatives of the Green function (1.2) in the coercive case (i.e. $\gamma \neq 0$).

For a positive $\delta < 1/2$ define

$$\begin{aligned} T_\delta =& \{(x,a) \in N \times \mathbb{R}^+ : 1-\delta < a < 1+\delta, |x| < \delta\}, \\ \mathcal{Q} =& \{(x,a) \in N \times \mathbb{R}^+ : |x| \le 1, a \le 1\}. \end{aligned}$$

Theorem 3.1. *For a multi-index $I = (i_1, \dots, i_k)$, $\gamma > 0$ and all operators $\mathcal{X}^I = \mathcal{X}_1^{(i_1)} \dots \mathcal{X}_k^{(i_k)}$, where $\mathcal{X}_j^{(i_j)} \in V_j^{i_j}$, with $\|\mathcal{X}^I\| \le 1$, there are constants C such that*

$$|\mathcal{X}^I \mathcal{G}_{-\gamma}(x,a)| \le \begin{cases} C(|x|+a)^{-\|I\|-Q-\gamma} & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0} & \text{for } (x,a) \in (\mathcal{Q} \cup T_\delta)^c, \\ C & \text{for } (x,a) \in \mathcal{Q} \setminus T_\delta \end{cases}$$

and

$$|\mathcal{X}^I \mathcal{G}_{\gamma}(x,a)| \le \begin{cases} C(|x|+a)^{-\|I\|-Q-\gamma} a^\gamma & \\ \times(1+|\log(|x|+a)^{-1}|)^{\|I\|_0} & \text{for } (x,a) \in (\mathcal{Q} \cup T_\delta)^c, \\ Ca^\gamma & \text{for } (x,a) \in \mathcal{Q} \setminus T_\delta \end{cases}$$

where $\|I\| = \sum_{j=1}^k i_j d_j$, $d_j = Re\lambda_j$, and $\|I\|_0 = \sum_{j=1}^k i_j D_j$, $D_j = dim V_{\lambda_j} - 1$.

Theorem 3.2. *For every nonnegative integer k and $\gamma > 0$ there is a constant C such that*

$$|\partial_a^k \mathcal{G}_{-\gamma}(x,a)| \le \begin{cases} Ca^{-k}(|x|+a)^{-Q-\gamma} & \text{for } (x,a) \in (\mathcal{Q} \cup T_\delta)^c, \\ Ca^{-k} & \text{for } (x,a) \in \mathcal{Q} \setminus T_\delta \end{cases}$$

and

$$|\partial_a^k \mathcal{G}_{\gamma}(x,a)| \le \begin{cases} Ca^{\gamma-k}(|x|+a)^{-Q-\gamma} & \text{for } (x,a) \in (\mathcal{Q} \cup T_\delta)^c, \\ Ca^{\gamma-k} & \text{for } (x,a) \in \mathcal{Q} \setminus T_\delta. \end{cases}$$

Let $\alpha \ge 0$ and $\gamma > 0$. Along with the operator $\mathcal{L}_{-\gamma}$ defined in (1.1) we consider the corresponding operator L_α,

$$L_\alpha = a^{-2} \sum_j \Phi_a(X_j)^2 + a^{-2}\Phi_a(X) + \partial_a^2 + \frac{2\alpha+1}{a}\partial_a = a^{-2}\mathcal{L}_{-\gamma}, \tag{3.1}$$

where $\alpha = \gamma/2$. The Green function G_α for L_α is given by

$$G_\alpha(x,a;y,b) = \int_0^\infty p_t(x,a;y,b)dt,$$

where $T_t f(x,a) = \int f(y,b) p_t(x,a;y,b) dy b^{2\alpha+1} db$ is the heat semigroup on $L^2(N \times \mathbb{R}^+, dy b^{2\alpha+1} db)$ with the infinitesimal generator L_α.

On $N \times \mathbb{R}^+$ we define *dilations*:

$$D_t(x,a) = (\Phi_t(x), ta), \qquad t > 0.$$

It is not difficult to check that although the operator L_α is not left-invariant it has some homogeneity with respect to the family of dilations introduced above:

$$L_\alpha(f \circ D_t) = t^2 L_\alpha f \circ D_t.$$

This implies that

$$G_\alpha(x,a;y,b) = t^{-Q-2\alpha} G_\alpha(D_{t^{-1}}(x,a); D_{t^{-1}}(y,b)). \tag{3.2}$$

It turns out (see (1.17) in [**4**]) that

$$\mathcal{G}_{-\gamma}(x,a) = G_{\gamma/2}(e,1;x,a) = G^*_{\gamma/2}(x,a;e,1), \tag{3.3}$$

where G^*_α is the Green function for the operator

$$L^*_\alpha = a^{-2}\sum \Phi_a(X_j)^2 - a^{-2}\Phi_a(X) + \partial_a^2 + \frac{2\alpha+1}{a}\partial_a$$

conjugate to L_α with respect to the measure $a^{2\alpha+1}dadx$. Moreover,

$$G^*_\alpha(x,a;e,1) = \lim_{\eta\to 0}\int_0^\infty \mathbf{E}_1 p^\sigma(t,0)(x) m_\alpha(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t)dt, \tag{3.4}$$

where

$$m_\alpha(I) = \int_I a^{2\alpha+1}da \tag{3.5}$$

and the expectation is taken with respect to the distribution of the Bessel process with the parameter α starting from 1, i.e., $\mathrm{BESS}_1(\alpha)$ on the space $C((0,\infty),(0,\infty))$, $p^\sigma(t,0)$ is the transition function of the evolution generated by the operator (2.5) and $I_{a,\eta} = [a-\eta, a+\eta]$.

Since $\mathcal{L}_{-\gamma}(\cdot) = a^{-\gamma}\mathcal{L}_\gamma(a^\gamma\cdot)$ it follows that

$$\mathcal{G}_\gamma(xa,yb) = a^\gamma \mathcal{G}_{-\gamma}(xa,yb)b^{-\gamma} \tag{3.6}$$

and therefore, by (3.3) and (3.6),

$$\mathcal{G}_\gamma(x,a) = G^*_{\gamma/2}(x,a;e,1)a^\gamma. \tag{3.7}$$

Before we go the proofs we note the following important proposition which gives estimates on the set $\mathcal{Q}\setminus T_\delta$ of some functional of the evolution p^σ.

Proposition 3.3. *For every $1>\delta>1/2$, for every $0<\chi_0\le 1$, $0<r_0\le 1$ and for every multi-index I such that $|I|>0$ there exists a constant C such that for every $(x,a)\in\mathcal{Q}\setminus T_\delta$,*

$$\sup_{0<\eta<\delta/2}\Big|\int_0^\infty \mathbf{E}_\chi X^I p^\sigma(t,0)(x) m_\alpha(I_{a,\eta})^{-1} 1_{I_{a,\eta}}(\sigma_t)dt\Big| \le C.$$

Sketch of the proof. It is enough to notice that Lemmas 5.1–5.5 in [**12**] remain valid if we replace $\mathrm{BESS}_1(0)$ by $\mathrm{BESS}_1(\alpha)$, $\alpha>0$ and $m=m_0$ by m_α defined in (3.5). □

After this preparatory facts we are ready to give

Proof of Theorem 3.1. For $r\ge 0$, define

$$V_r = \{(x,a)\in N\times\mathbb{R}^+ : |(x,a)| = r\},$$

where $|(x,a)| = |x| + a$.

Let $0<\delta<1/2$ and a multi-index I be fixed.

Case 1. We consider the set

$$S_1 = \mathcal{Q}\setminus T_\delta.$$

By Proposition 3.3 it follows that there exists a positive constant C such that

$$|\mathcal{X}^I G^*_{\gamma/2}(x,a;e,1)| \le C \tag{3.8}$$

for every $(x,a) \in \tilde{S}_1 := S_1 \cap \{(x,a) \in N \times \mathbb{R}^+ : a \le 1-\delta\}$. But $S_1 \setminus \mathrm{Int}\tilde{S}_1$ is a compact set and $G^*_{\gamma/2}$ is a continuous function so we get (3.8) on S_1. Therefore on S_1 we have that

$$|\mathcal{X}^I \mathcal{G}_{-\gamma}(x,a)| = |\mathcal{X}^I G^*_{\gamma/2}(x,a;e,1)| \le C$$

and

$$|\mathcal{X}^I \mathcal{G}_{\gamma}(x,a)| = |\mathcal{X}^I G^*_{\gamma/2}(x,a;e,1)a^\gamma| \le Ca^\gamma.$$

Case 2. We consider the set

$$S_2 = \{(x,a) \in N \times \mathbb{R}^+ : |x| \ge 1, |x| \ge a\}.$$

(Of course, $S_2 \cap T_\delta = \emptyset$.) Every element $(x,a) \in N \times \mathbb{R}^+$ can be written as

$$(x,a) = D_t(y,b), \text{ where } (y,b) \in V_1 \text{ and } t = |(x,a)| = |x| + a.$$

By homogeneity of G_α (see (3.2)) and (2.4) we get

$$\begin{aligned}
|\mathcal{X}^I G^*_{\gamma/2}(x,a;e,1)| =& |\mathcal{X}^I G^*_{\gamma/2}(D_t(y,b), D_t(e,t^{-1}))| \\
=& |\Phi_{t^{-1}}(\mathcal{X}^I)(G^*_{\gamma/2} \circ D_t)(y,b;e,t^{-1})| \\
\le& t^{-\|I\|}(1+|\log t^{-1}|)^{\|I\|_0} \\
& \times \sup_{\|\mathcal{Y}\| \le 1} |\mathcal{Y}^I (G^*_{\gamma/2} \circ D_t)(y,b;e,t^{-1})| \\
\le& (|x|+a)^{-\|I\|-Q-\gamma}(1+|\log|(x,a)|^{-1}|)^{\|I\|_0} \\
& \times \sup_{\|\mathcal{Y}\| \le 1} |\mathcal{Y}^I G^*_{\gamma/2}(y,b;e,|(x,a)|^{-1})|.
\end{aligned} \tag{3.9}$$

Virtually the same argument as in the proof of Theorem 6.1 in [**12**] together with Proposition 3.3 give us

$$|\mathcal{X}^I G^*_{\gamma/2}(x,a;e,1)| \le C(|x|+a)^{-\|I\|-Q-\gamma}(1+|\log|(x,a)|^{-1}|)^{\|I\|_0}.$$

Thus by (3.3) and (3.7) we get

$$|\mathcal{X}^I \mathcal{G}_{-\gamma}(x,a)| \le C(|x|+a)^{-\|I\|-Q-\gamma}(1+|\log|(x,a)|^{-1}|)^{\|I\|_0} \tag{3.10}$$

and

$$|\mathcal{X}^I \mathcal{G}_{\gamma}(x,a)| \le Ca^\gamma(|x|+a)^{-\|I\|-Q-\gamma}(1+|\log|(x,a)|^{-1}|)^{\|I\|_0} \tag{3.11}$$

Case 3. Finally we consider the set

$$S_3 = \{(x,a) \notin T_\delta : a \ge |x|, a \ge 1\}.$$

Because $V_1 \cap T_\delta \ne \emptyset$ we write every element $(x,a) \in N \times \mathbb{R}^+$ as a dilation of some element from $V_{1/2}$:

$$(x,a) = D_t(y,b), \text{ where } (y,b) \in V_{1/2} \text{ and } t = 2|(x,a)| = 2|x| + 2a.$$

By homogeneity, we can write analogously to (3.9),

$$|\mathcal{X}^I G^*_{\gamma/2}(x,a;e,1)| \le 2^{-\|I\|-Q-\gamma}(|x|+a)^{-\|I\|-Q-\gamma} \times (1+|\log|(x,a)|^{-1}|)^{\|I\|_0} \sup_{\|\mathcal{Y}\|\le 1} |\mathcal{Y}^I G^*_{\gamma/2}(y,b;e,\tilde{\beta})|. \tag{3.12}$$

where $\tilde{\beta} = 2^{-1}(|x|+a)^{-1}$. Now using Proposition 3.3 we proceed exactly in the same way as in the proof of Theorem 6.1 in [**12**] and we get that there exists a constant C such that $\sup_{\|\mathcal{Y}\|\le 1} |\mathcal{Y}^I G^*(y,b;e,\tilde{\beta})|$ in (3.12) is less than or equal to C. Hence we get (3.10) and (3.11) on S_3 and the proof is done. □

Proof of Theorem 3.2. Estimates for $\partial_a^k \mathcal{G}_{\pm\gamma}$ follows easily from estimates for $\mathcal{G}_{\pm\gamma}$ (given in the previous theorem) and the Harnack inequality, exactly in the same way as it was shown in [**12**] for $\mathcal{G}_0$. However, for the sake of completeness we repeat the argument here since it is very short.

We may assume that $k>0$ since for $k=0$ the result follows from the previous Theorem.

It can be easily proved by induction that for every integer $k\ge 1$ we have

$$(a\partial_a)^k = a^k\partial_a^k + \sum_{j=1}^{k-1} c_j a^j \partial_a^j,$$

where $c_j \in \mathbb{Z}$. Therefore

$$a^k\partial_a^k = (a\partial_a)^k - \sum_{j=1}^{k-1} c_j a^j \partial_a^j.$$

Applying the above formula recursively to the terms $a^j\partial_a^j$ we get that

$$a^k\partial_a^k = (a\partial_a)^k + \sum_{j=1}^{k-1} \alpha_j (a\partial_a)^j, \tag{3.13}$$

where $\alpha_j \in \mathbb{Z}$. Note that the operator $a\partial_a$ is left-invariant. Thus, by (3.13) and the Harnack inequality (cf. [**18**]) we have for every $(x,a)\notin T_\delta$,

$$\begin{aligned} |a^k\partial_a^k \mathcal{G}_{\pm_\gamma}(x,a)| \le& |(a\partial_a)^k \mathcal{G}_{\pm\gamma}(x,a)| + \sum_{j=1}^{k-1} |\alpha_j||(a\partial_a)^j \mathcal{G}_{\pm\gamma}(x,a)| \\ \le& C_0\mathcal{G}_{\pm\gamma}(x,a) + \sum_{j=1}^{k-1} |\alpha_j| C_j \mathcal{G}_{\pm\gamma}(x,a) \le C\mathcal{G}_{\pm\gamma}(x,a). \end{aligned}$$

This inequality together with the estimate for $\mathcal{G}_{\pm\gamma}$ complete the proof. □

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