Acta Math. Univ. Comenianae
Vol. LXXIII, 1(2004), pp. 69–73


# DISTRIBUTIVE PAIRS IN BIATOMIC LATTICES

B. N. WAPHARE AND V. V. JOSHI

Abstract. We prove that a biatomic lattice $L$ is distributive if and only if every pair of atoms of $L$ is distributive. This result has been used to obtain characterizations of distributive pairs in terms of semi-distributive pairs, del-relation and perspectivity.

In an atomistic lattice (every non-zero element is the join of atoms contained in it) $L$, for a pair of non-zero elements $a, b \in L$ we write $(a, b)P$, if for every atom $p \leq a \vee b$ there exist atoms $q$, $r$ such that $p \leq q \vee r$, $q \leq a$ and $r \leq b$. $L$ is called *biatomic* if $(a, b)P$ holds for all non-zero elements $a, b \in L$.

In [**2**], Bennett studied the class of biatomic lattices and provided many important examples. In fact, the same class with the nomenclature "additive lattices" is also studied by Bennett [**1**]. Biatomic lattices are also defined in terms of $P$-relation.

Properties and characterizations of $P$-relation can be found in Maeda [**7**] ( see also Maeda [**8**]) for lattices and in Thakare, Wasadikar and Maeda [**11**] for join-semilattices.

The following concepts can be found in Maeda and Maeda [**6**] and Maeda [**9**].

For a lattice $L$ and $a,\ b \in L$ we write:

$(a, b)D$ (*distributive pair*) if, $(a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x)$ for every $x$;

$(a, b)SD$ (*semi-distributive pair*) if, $\{(a \vee b) \wedge x\} \vee b = (a \wedge x) \vee b$ for every $x$;

$(a, b)M$ (*modular pair*) if, $c \vee (a \wedge b) = (c \vee a) \wedge b$ for every $c \leq b$;

$a \nabla b$ (*del-relation*) if, $(a \vee x) \wedge b = b \wedge x$ for every $x$;

$a \tilde{\nabla} b$ if, $(a \vee x) \wedge (b \vee x) = x$ for every $x$.

Dually, we have the concepts of *dually distributive pair* $(a, b)D^*$, *dually semi-distributive pair* $(a, b)SD^*$ and *dually modular pair* $(a, b)M^*$ etc.

A lattice is said to be *distributive* if $(a, b)D$ holds for all $a, b$.

It is easy to prove that $(a, b)D \Rightarrow (a, b)SD$ but not conversely; also a lattice is distributive if $(a, b)SD$ holds for all $a, b \in L$; see Maeda [**7**].

Maeda [**9**] essentially proved that for elements $a, b$ in a biatomic lattice $L$, $(a, b)M^*$ holds if $(p, q)M^*$ holds for atoms $p \leq a$ and $q \leq b$. This motivates us to

Received June 23, 2003.
2000 *Mathematics Subject Classification.* Primary 06D10, 06D99; Secondary 06C05.
*Key words and phrases.* Biatomic lattice, distributive pair, modular pair, del-relation, exchange property.

prove analogues results for different concepts in lattices. In fact, in this paper, we prove the following result in biatomic lattices.

**Theorem 1.** *In a biatomic lattice $L$, the following statements are true for $a, b \in L$.*

($\alpha$) *If $(p, q)D$ holds for all atoms $p \leq a$ and $q \leq b$ then $(a, b)D$ holds.*

($\beta$) *$p\nabla q$ holds for all atoms $p \leq a$ and $q \leq b$ if and only if $a\nabla b$ holds.*

We use this result to obtain characterizations of distributive pairs in terms of semi-distributive pairs, del-relation and perspectivity.

For undefined notations and terminology the reader is referred to Maeda and Maeda [**6**].

To prove Theorem 1 we need:

**Lemma 2.** *(Maeda [**9**]). Let $a, b$ be elements of an atomistic lattice $L$. The following conditions are equivalent.*

1. *$(a, b)D$.*
2. *$(a, b)SD$.*
3. *For an atom $p \in L$, $p \leq a \vee b$ implies $p \leq a$ or $p \leq b$.*

*Proof of Theorem 1.* ($\alpha$): Suppose $(p, q)D$ holds for all atoms $p, q$ with $p \leq a$ and $q \leq b$. Let $p$ be an atom and $p \leq a \vee b$. In view of Lemma 2, it is sufficient to prove that $p \leq a$ or $p \leq b$. Suppose $p \nleq b$. Since $L$ is biatomic, there exist atoms $q, r$ such that $p \leq q \vee r$ with $q \leq a$ and $r \leq b$. Clearly, $p \neq r$. Using $p \leq q \vee r$, $p \neq r$ and $(q, r)D$ we have,

$p = (q \vee r) \wedge p = (q \wedge p) \vee (r \wedge p) = q \wedge p.$

Thus $p = q \leq a$ as required.

($\beta$): Suppose $a\nabla b$ holds and $p$, $q$ are atoms such that $p \leq a$ and $q \leq b$. For any $x \in L$ we have

$(p\vee x)\wedge q = [(a\vee x)\wedge(p\vee x)]\wedge(b\wedge q) = (a\vee x)\wedge b\wedge(p\vee x)\wedge q) \overset{a\nabla b}{=} x\wedge(p\vee x)\wedge b\wedge q = x \wedge b \wedge q = x \wedge q.$

Thus $p\nabla q$ holds.

Conversely, suppose that $p\nabla q$ holds for all atoms $p \leq a$ and $q \leq b$. To prove $a\nabla b$, it is sufficient to show that $(a\vee x)\wedge b \leq x\wedge b$. Suppose $(a\vee x)\wedge b \nleq x\wedge b$. Since $L$ is atomistic, there exists an atom $r$ such that $r \leq (a \vee x) \wedge b$ and $r \nleq x \wedge b$. Since $L$ is biatomic and $r \leq a \vee x$, there exist atoms $p, q$ such that $r \leq p \vee q$, with $p \leq a$ and $q \leq x$. Clearly $r \neq q$. By $p\nabla r$ and $r \leq p\wedge q$, we have $r = (p\vee q)\wedge r = q\wedge r = 0$, a contradiction. □

We supply an example to show that the assertions of Theorem 1 are not true in a general atomistic lattice.

**Example.** Let $X$ be an infinite set with $A, B$ complementary infinite subsets of $X$. Consider the set $L = \{C \cup D \mid C \subseteq A,\ C = B \text{ or } C = X,\ D \text{ finite}\}$ ordered by set inclusion. In Janowitz and Cote [**5**], it is proved that, $L$ is an atomistic lattice in which every finite element (an element is called finite if it is either 0 or a join of finitely many atoms) $s$ is a standard element (*i.e.* $(s, x)D$ holds for all

$x \in L$; *see* Grätzer [**4**]). Therefore $(p, q)D$ holds for all atoms $p$, $q$ of $L$. But the lattice is not distributive as the pair $(C, B)$ is not distributive where $C$ is an infinite proper subset of $A$.

Also, it is shown in Janowitz and Cote [**5**] that $B\nabla A$ does not hold. Now, we observe that $p\nabla q$ holds for all distinct atoms $p$, $q$ in $L$. For this, note that in $L$, for an atom $p$, $(p, x)D$ holds for all $x \in L$ and therefore $(x, p)M^*$ holds. Now, we prove $p\nabla q$. By $(x, p)M^*$, $(p \vee q) \wedge (x \vee p) = (((p \vee q) \wedge x) \vee p)$. Also, by $(p, q)D$, $(p \vee q) \wedge x = (p \wedge x) \vee (q \wedge x)$. Therefore $(((p \vee q) \wedge x) \vee p) = p \vee (q \wedge x)$. Thus $(p \vee q) \wedge (x \vee p) = p \vee (q \wedge x)$. Taking meet with $q$ and using $(p, q)M$ we have the desired result.

Using Theorem 1($\alpha$) we obtain:

**Theorem 3.** *A biatomic lattice $L$ is distributive if and only if $(p, q)D$ holds for all atoms $p, q \in L$.*

We provide a relationship between distributive pairs and the concept of perspectivity.

Let $a$ and $b$ be elements of a lattice $L$ with 0. We say that $a$, $b$ are *perspective* and write $a \sim b$, when $a \vee x = b \vee x$ and $a \wedge x = b \wedge x = 0$ for some $x \in L$.

**Lemma 4.** *Let $a$ and $b$ be elements of a modular atomistic lattice $L$. The following three statements are equivalent.*

1. *$a\nabla b$.*
2. *There do not exist non-zero elements $a_1$ and $b_1$ such that $a_1 \sim b_1$, $a_1 \leq a$ and $b_1 \leq b$.*
3. *There do not exist atoms $p$ and $q$ such that $p \sim q$, $p \leq a$ and $q \leq b$.*

*Proof.* Using Lemma 11.1 of Maeda and Maeda [**6**] and the fact that del-relation is symmetric in modular lattices, the result can be proved on the similar lines of Theorem 10.5 of Maeda and Maeda [**6**]. □

**Remark 5.** Note that the above result can be found in Maeda and Maeda [**6**] for an atomistic $SSC^*$(dually section semi-complemented) lattice. Stern [**10**] essentially proved that a modular atomistic lattice of finite length is dually atomistic (therefore $SSC^*$). However, this assertion is not true if we drop the assumption of finiteness. In this context we provide the following example.

**Example.** Let $X$ be an infinite set. Put $L = \{ F \mid F$ is a finite subset of $X\} \cup \{\phi\}$. Then $L$ forms a lattice under the set inclusion. Moreover, it is easy to observe that $L$ is an atomistic modular lattice which is not $SSC^*$.

The following result is proved in Bennett [**2**].

**Lemma 6.** *In an atomistic lattice $L$ the following statements are equivalent.*

1. *$L$ is modular.*
2. *$L$ is biatomic with the exchange property (If $p$ and $q$ are atoms, $p \not\leq a$ and $p \leq a \vee q \Rightarrow q \leq a \vee p$.).*

Observe that Lemma 6 can also be deduced immediately from Lemma 4 of Maeda [**8**]; (see also Maeda [**7**]).

We also need the following lemma which is essentially proved by Crawley and Dilworth [**3**, p. 145].

**Lemma 7.** *Let $L$ be a modular lattice with 0 and $a, b \in L$ with $a \wedge b = 0$. Then $(a, b)D$ if and only if $a \nabla b$.*

Now, we prove our main result.

**Theorem 8.** *Let $L$ be a biatomic lattice with the exchange property. Let $a, b \in L$ and $a \wedge b = 0$. Then the following statements are equivalent.*

(1) $(a, b)D$.

(2) $(a, b)SD$.

(3) *$p \leq a \vee b$ imply $p \leq a$ or $p \leq b$ for an atom $p \in L$.*

(4) $a \tilde{\nabla} b$.

(5) $a \nabla b$.

(6) *$(p, q)D$ for all atoms $p \leq a$ and $q \leq b$.*

(7) *$(p, q)SD$ for all atoms $p \leq a$ and $q \leq b$.*

(8) *$p \nabla q$ for all atoms $p \leq a$ and $q \leq b$.*

(9) *$p \tilde{\nabla} q$ for all atoms $p \leq a$ and $q \leq b$.*

(10) *There do not exist atoms $p$ and $q$ such that $p \sim q$, $p \leq a$ and $q \leq b$.*

(11) *There do not exist non-zero elements $a_1$ and $b_1$ such that $a_1 \sim b_1$, $a_1 \leq a$ and $b_1 \leq b$.*

*Proof.* Equivalence of the first three statements follows from Lemma 2. The statements (1) and (5) are equivalent by Lemma 6 and Lemma 7.

(4) ⇒ (5) is obvious.

(5) ⇒ (4) : Suppose $a \nabla b$ holds. By $(b, x)M^*$ (whih holds due to Lemma 6) and $a \nabla b$ we get $(a \vee x) \wedge (b \vee x) = [(a \vee x) \wedge b] \vee x = (x \wedge b) \vee x = x$. Thus the statements (1) to (5) are equivalent. On the similar lines equivalence of the statements (6) to (9) can be proved. By Theorem 1($\beta$), the statements (5) and (8) are equivalent. Equivalence of the statements (5), (10) and (11) follows from Lemma 4. □

**Acknowledgements**. The authors are thankful to Professor N. K. Thakare and the learned referee for their valuable suggestions.

## References


**1.** Bennett M. K., *On generating affine geometries*, Alg. Univ., **4** (1974), 207–219.

**2.** ———, *Biatomic lattices*, Alg. Univ., **24** (1987), 60–73.

**3.** Crawley P. and Dilworth R. P., *Algebraic Theory of Lattices*, Prentice Hall, Inc., Englewood Cliffs, N.J., 1973.

**4.** Grätzer G., *General Lattice Theory*, Birkhäuser Verlag, Second Edition (1998).

**5.** Janowitz M. F. and Cote N. H., *Finite-distributive atomistic lattices*, Portug. Math., **35** (1976), 81–91.

**6.** Maeda F. and Maeda S., *Theory of Symmetric Lattices*, Springer-Verlag, Berlin Heidelberg, New York. (1970).
**7.** Maeda S., *On finite modular atomistic lattices*, Alg. Univ. **12** (1981), 76–80.
**8.** ———, *On modularity in atomistic lattices*, in Contribution to lattice theory, Colloq. Math. Soc. Janos Bolyai, **33** (1983), 627–636.
**9.** ———, *On distributive pairs in lattices*, Acta Math. Acad. Sci. Hung. **45** (1985), 133–140.
**10.** Stern M., *On dually atomistic lattices*, Discrete Math. **93** (1991), 97–100.
**11.** Thakare N. K., Wasadikar M. P. and Maeda S., *On modular pairs in semilattices*, Alg. Univ. **19** (1984), 255–265.

B. N. Waphare, Department of Mathematics, University of Pune, Pune 411 007, INDIA., *e-mail*: bnwaph@math.unipune.ernet.in

V. V. Joshi, Department of Mathematics, Government College of Engineering, Pune 411 005, INDIA., *e-mail*: vinayakjoshi111@yahoo.com