Acta Math. Univ. Comenianae
Vol. LXXIV, 1(2005), pp. 15–24

GENERALIZED DIEUDONNÉ CRITERION

P. DANCHEV

ABSTRACT. An extension in the terms of σ-summable abelian p-groups of the classical Dieudonné criterion (Portugaliae Mathematica, 1952) for direct sums of p-cyclics (= cyclic p-groups) is given. Specifically, it is proved that G is a σ-summable abelian p-group if A is its balanced σ-summable abelian p-subgroup so that G/A is a σ-summable abelian p-group.

In particular, some other well-known results are also obtained.

1. INTRODUCTION.

Standardly, all groups here considered will be abelian p-primary groups written multiplicatively. All notions, notation and terminology not explicitly defined herein are the same as in the classical monographs of L. Fuchs [**4**], but transferred for a multiplicative record.

The major aim that we pursue is to extend the Dieudonné criterion about direct sums of cyclic p-groups (see [**3**] or [**4**, volume I, p. 109, Exercise 9*]) for more general classes of abelian p-torsion groups. Here this is made by concerning the class of so-called σ-summable p-groups, which sort of groups is quite large and contains as subclasses very important kinds of abelian p-groups; for instance all totally projective p-groups of cofinal with ω lengths (in particular all reduced direct sums of countable p-groups with such lengths).

We subsequently continue with two more generalizations.

2. THE GENERALIZED KOVÁCS CRITERION.

In [**10**], L. Kovács has established a necessary and sufficient condition which says that an arbitrary subgroup A of the abelian p-group G is contained in a basic subgroup of G if and only if $A = \bigcup_{n<\omega} A_n$, where $A_n \subseteq A_{n+1}$ such that $A_n \cap G^{p^n} = 1$ for every natural number n. In other words, Kovács [**10**] (see also [**4**]) showed that the countable ascending union of all p^n-high subgroups of G forms a basic subgroup for G whenever $n \in \mathbf{N}$. This is a remarkable expansion of the classical result of L. Kulikov, archived in [**4**, volume I, p. 106, Theorem 17.1], that result gives a valuable criterion for the decomposing into a direct sum of primary cyclic groups.

An important improvement of this result was obtained by K. Wallace in [**14**] in the terms of λ-basic subgroups for a cofinal with ω ordinal λ. More precisely,

---

Received March 21, 2002.
2000 *Mathematics Subject Classification.* Primary 20K10.

he had argued that if $G/G^{p^{\alpha}}$ is totally projective for each ordinal $\alpha < \lambda$, an ordinal cofinal with ω, then $\bigcup_{n<\omega} G_n$ is a λ-basic subgroup of G where, for every positive integer n, G_n is a p^{α_n}-high subgroup of G and $\lambda = \sup_{n<\omega}\{\alpha_n\}$ such that $\alpha_n < \alpha_{n+1}$; if $\lambda = \beta + \omega$ for some limit ordinal β, then $\alpha_n = \beta + m$ for some positive integer m; if $\lambda \neq \beta + \omega$ for any ordinal β, then $\alpha_n = \beta_n + \omega$ for some limit ordinal β_n.

This excellent result gives a new perspective for a global refinement of the foregoing mentioned Dieudonné's theorem, that is the theme of the next section.

3. The generalized Dieudonné criterion.

In [**3**], Jean Dieudonné finds a necessary and sufficient condition concerning the direct sums of cyclic p-groups that enlarges the well-known and documented classical criterion of L. Kulikov for the decomposition into direct sums of p-cyclics [**4**, volume I, p. 106, Theorem 17.1], namely:

Theorem (Dieudonné, 1952)**.** *Let G be an abelian p-group and let A be its subgroup such that G/A is a direct sum of cyclics. Then G is a direct sum of cyclics if and only if $A = \bigcup_{n<\omega} A_n$, $A_n \subseteq A_{n+1}$ and $A_n \cap G^{p^n} = 1$ for all $n < \omega$.*

In his paper, Dieudonné adds several interesting remarks to his theorem as well. E.g. he mentions that if A and G/A are direct sums of cyclic p-groups, the same does not hold in general for G. Besides, by an ingenious counterexample, he also demonstrates that it is not sufficient to require the former conditions even if G contains no elements of infinite heights, i.e., equivalently, even when G is separable (cf. [**4**], too).

In what follows, we shall state and prove a theorem-criterion which is a common expansion of Dieudonné's theorem and of some our results published in [**1**]. Moreover, the technique for an evidence that we raise is very elementary and different to that in [**3**]. In other words, by what we shall argue in the sequel, it will hold that the criteria of Dieudonné and Kulikov are, in fact, equivalent.

Adopting the facts from the previous section, namely the Kovács criterion, we observe that the Dieudonné criterion can be reformulated thus:

Theorem (Dieudonné, 1952)**.** *Suppose G is an abelian p-group and A is its subgroup so that G/A is a direct sum of cyclic groups. Then G is a direct sum of cyclic groups if and only if A can be expanded to a basic subgroup of G.*

Before stating and proving our main attainment, which motivates the present article, we need one useful

Lemma. *Assume G is an abelian p-group and N is its nice subgroup. Then* $\mathrm{length}(G/N) \leq \mathrm{length}(G)$.

Proof. Presume that $\mathrm{length}(G) = \lambda$. Therefore, as it is well-known from [**4**], $(G/N)^{p^{\lambda}} = G^{p^{\lambda}}N/N = 1$. Thus, $\mathrm{length}(G/N) \leq \lambda$, as desired. □

We begin now with two definitions to simplify the exposition.

Definition 1. The abelian p-group G is said to be σ-summable, by [**7**], if $G = \bigcup_{n<\omega} \Gamma_n$, $\Gamma_n \leq \Gamma_{n+1}$ and for every $n < \omega$ there exists an ordinal $\tau_n < \text{length}(G)$ with $\Gamma_n \cap G^{p^{\tau_n}} = 1$.

It follows directly from the definition that the σ-summable p-groups have infinite lengths (that is $\geq \omega$) cofinal with ω, every subgroup of a σ-summable p-group with equal to this of the whole group length is also σ-summable, and the unbounded direct sums of p-cyclics are σ-summable as well as all separable σ-summable groups are precisely the direct sums of cyclic groups.

Definition 2. The proper subgroup A of an arbitrary abelian p-group G is called strongly σ-summable in G if $A = \bigcup_{n<\omega} A_n$, $A_n \subseteq A_{n+1}$ and for each natural n there is an ordinal $\alpha_n < \text{length}(G)$ such that $A_n \cap G^{p^{\alpha_n}} = 1$. Certainly, if $\text{length}(A) = \text{length}(G)$ then A is likewise σ-summable.

It easily follows from the definition that a given subgroup may be strongly σ-summable in many itself containing groups, i.e., in other words, the property being strongly σ-summable subgroup is not unique. Thus the strongly σ-summability must be considered as being a personal property in each independent situation, and so it basicly depends on the embedding of one group in another. Under these reasons, if necessary and not clear from the text, we shall further specify in what group a given subgroup is strongly σ-summable. Moreover, if $A \leq G \leq K$ and A is strongly σ-summable in G, then A is strongly σ-summable in K provided G is isotype in K. If now A is strongly σ-summable in K and $\text{length}(G) = \text{length}(K)$, then A is strongly σ-summable in G.

Evidently, each proper subgroup of a σ-summable p-group is strongly σ-summable in it (see the Main Theorem below). In that aspect, although unnecessarily, we note that it follows immediately from a result of Hill [**7**] that any group can be interpreted as a strongly σ-summable subgroup which is a direct factor of some specially constructed σ-summable group with totally projective complementary factor of limit length, and which subgroup may not be strongly σ-summable in another group. Besides, each σ-summable isotype subgroup of a given group is strongly σ-summable in the same group, whereas not every strongly σ-summable subgroup, even being isotype, is itself σ-summable provided that the subgroup and the full group have distinct lengths.

Apparently, as in the preceding section, because each subgroup A_n can be expanded to a p^{α_n}-high, whence isotype, subgroup of G, we derive at once that every strongly σ-summable p-subgroup may be embedded in an isotype strongly σ-summable p-subgroup.

These technical claims shall be used below. It is not hard to observe that the equalities can be taken over the group socle only (for instance see [**7**]).

Now, we can attack our central assertion which encompasses similar results in the simpler context of direct sums of p-torsion cyclics and which says:

Main Theorem. *Let G be an abelian p-group of limit length and let A be its nice subgroup with the property that G/A is a σ-summable factor-group. Then G is σ-summable if and only if A is strongly σ-summable in G.*

Proof. Necessity. Utilizing [**7**], we may write $G = \bigcup_{n<\omega} \Gamma_n$, $\Gamma_n \subseteq \Gamma_{n+1}$ and there is an ordinal $\tau_n <$ length(G) with $\Gamma_n \cap G^{p^{\tau_n}} = 1$. Furthermore,

$$A = \bigcup_{n<\omega} (\Gamma_n \cap A),$$

hence $\Gamma_n \cap A \cap G^{p^{\tau_n}} \subseteq \Gamma_n \cap G^{p^{\tau_n}} = 1$. So, A is strongly σ-summable in G.

Sufficiency. Following [**7**], we write down

$$G/A = \bigcup_{n<\omega} (G_n/A),$$

where $G_n \subseteq G_{n+1}$ and $(G_n/A) \cap (G/A)^{p^{\alpha_n}} = 1$ for some existing ordinal number $\alpha_n <$ length(G/A). By virtue of the Lemma, length$(G) \geq$ length(G/A) whence $\alpha_n <$ length(G). Moreover, according to a lemma of Hill [**4**, volume II, p. 91, Lemma 79.2], we compute

$$(G_n/A) \cap (G^{p^{\alpha_n}} A/A) = [G_n \cap (G^{p^{\alpha_n}} A)]/A = 1,$$

so complying with the modular law $G_n \cap G^{p^{\alpha_n}} \subseteq A \subseteq G_n$, for all $n \in \mathbf{N}$.

Besides, it is obvious that $G = \bigcup_{n<\omega} G_n$. On the other hand, by hypothesis, $A = \bigcup_{m<\omega} A_m$, $A_m \subseteq A_{m+1}$ and there exists an ordinal $\beta_m <$ length(G) so that $A_m \cap G^{p^{\beta_m}} = 1$. Next, it is no harm in presuming that $\beta_k \geq \alpha_k$, for almost all naturals k.

After this, for each so k, we construct a new system of generating subgroups C_k of G in the following manner:

First of all, consider the subsets $M_k \subseteq G_k \setminus A$ so that

$$\bigcup_{k<\omega} M_k = \bigcup_{k<\omega} (G_k \setminus A) \text{ and so that } \langle M_k \rangle \cap A \subseteq A_k.$$

In other words, the set M_k consists of such (eventually not all) elements of G_k which do not belong to A but such that arbitrary finite products of their degrees lie in $(G_k \setminus A) \cup A_k$, that is $\langle M_k \rangle \subseteq M_k \cup A_k \subseteq (G_k \setminus A) \cup A_k$.

Certainly, the sets M_n are not unique. In fact, it is of a real possibility to exist another set M'_n such that the following properties hold:

$$M'_n \cap M_n = \emptyset, \quad \langle M'_n \rangle \cap A \subseteq A_n \quad \text{but} \quad (\langle M'_n \rangle \langle M_n \rangle) \cap A \subseteq A_{n+1}$$

properly, namely there is $a \in [(\langle M'_n \rangle \langle M_n \rangle) \cap A] \setminus A_n$. Since $G_k \subseteq G_{k+1}$ and $A_k \subseteq A_{k+1}$, we can choose $M_k \subseteq M_{k+1}$ whence $M_k \subseteq \langle M_k \rangle \subseteq \langle M_{k+1} \rangle$.

However, the given construction needs more detailed explanations for its correctness. Foremost, we observe that such a minimal index $t \geq 1$ from which we may start does exist, hence we deduce that $M_k \neq \emptyset$ for almost all naturals k. Indeed, if $x_1 \in G_1 \setminus A$, $y_1 \in G_1 \setminus A$ and if $x_1^{\pm\varepsilon} y_1^{\pm\delta} \in A$ for some $0 \leq \varepsilon \leq$ order(x_1) and $0 \leq \delta \leq$ order(y_1), it follows at once that $x_1^{\pm\varepsilon} y_1^{\pm\delta} \in A_t$ for some $t \geq 1$. But we have $x_1 \in G_t \setminus A$ and $y_1 \in G_t \setminus A$, so we may fix the index t. Thereby, it is easy to see that both x_1 and y_1 belong to some M_t, that is, $\{x_1, y_1\} \subseteq M_t$. Of course, we can begin with a finite or an empty set M_t. Besides, it is well to reveal how the groups $\langle M_k \rangle$ are naturally formed. In order to do this, we emphasize that the selection of the generating elements relies on ordinary mathematical induction about the subscript k. And so, we assume that we have already inductively constructed via the same idea as above the set $M_k = \{c_i \in G_k \setminus A | i \in I\}$ with the property that all possible finite products between degrees of the c_i's belong to $(G_k \setminus A) \cup A_k$ for some $k \geq t$ and, after re-indexing if it is needed, that the ascending chain $M_t \subseteq \ldots \subseteq M_k$ is just defined.

Using this, it remains to explain how we now continue the process by interpolating the set M_k to the set M_d for $d \geq k+1$. To this goal, if $g_d \in G_d \setminus A$ is another element (notice that $g_d \in G_k$ is possible), we will distinguish two basis situations. Firstly, if for some arbitrary $0 < \varepsilon < \text{order}(g_d)$ it holds $g_d^{\varepsilon} c_{i_1}^{\varepsilon_1} \ldots c_{i_s}^{\varepsilon_s} \notin A$ over all possible permutations of the different indices $i_1, \ldots, i_s$, everything is good. Secondly, in the remaining case when $\langle g_d, c_{j_1}, \ldots, c_{j_l} \rangle \cap A \subseteq A_d$, that is $g_d^{\varepsilon} c_{j_1}^{\delta_1} \ldots c_{j_l}^{\delta_l} \in A_d$ for any $0 < \varepsilon < \text{order}(g_d)$, for some fixed $d \geq k$ (we may fix such an index d since the nontrivial degrees g_d^{ε} are of finite number) and some combination $(j_1, \ldots, j_l)$, we infer in virtue of the induction hypothesis that

$$g_d^{\varepsilon} c_{i_1}^{\varepsilon_1} \ldots c_{i_s}^{\varepsilon_s} = g_d^{\varepsilon} c_{j_1}^{\delta_1} \ldots c_{j_l}^{\delta_l} . c_{i_1}^{\varepsilon_1} \ldots c_{i_s}^{\varepsilon_s} . c_{j_1}^{-\delta_1} \ldots c_{j_l}^{-\delta_l} \in (G_d \setminus A) \cup A_d$$

for an arbitrary permutation $(i_1, \ldots, i_s)$. Hence, $\{g_d, c_i | i \in I\} \subseteq M_d$ etcetera repeating the same procedure to the obtaining

$$M_d = \{u_j \in G_d \setminus A | j \in J\}, \quad |J| \geq |I| \quad \text{and} \quad \langle M_d \rangle \cap A \subseteq A_d.$$

This completes the induction step. So, $M_d = M_k \cup S_d$ such that $S_d \cap M_k = \emptyset$ and such that $S_d \subseteq G_d \setminus A$ possesses the above described intersection property $\langle S_d, c_{j_1}, \ldots, c_{j_l} \rangle \cap A \subseteq A_d$; thus $g_d \in S_d$.

Further, define the wanted subgroups like this: $C_k = \langle A_k, M_k \rangle = A_k \langle M_k \rangle$. Plainly, $C_k \subseteq G_k$. Moreover, $G = A \cup [(G \setminus A) \cup \{1\}]$ and with the aid of the standard foregoing procedure of distribution of elements, we yield

$$G \setminus A = (\bigcup_{k<\omega} G_k) \setminus A = \bigcup_{k<\omega} (G_k \setminus A) = \bigcup_{k<\omega} M_k.$$

To finalize the distributive method, we observe that if $g \in G$ then $g \in (G_k \setminus A) \cup A$ for some $k \geq 1$ and henceforth the above demonstrated scheme of selecting works. We therefore obtain

$$G = \bigcup_{k<\omega} C_k, \quad C_k \subseteq C_{k+1}.$$

Clearly, $C_k \cap A = A_k$. In fact, via the exploiting of the modular law,

$$C_k \cap A = [A_k \langle M_k \rangle] \cap A = A_k [\langle M_k \rangle \cap A] = A_k.$$

Furthermore, what is mandatory to prove is that $C_k \cap G^{p^{\beta_k}} = 1$. In order to do this, by what we have above established,

$$C_k = (C_k \cap A) \cup [(C_k \setminus A) \cup \{1\}] = A_k \cup [(C_k \setminus A) \cup \{1\}]$$

and consequently

$$C_k \cap G^{p^{\beta_k}} = (A_k \cap G^{p^{\beta_k}}) \cup ((C_k \setminus A) \cap G^{p^{\beta_k}}) \cup \{1\} \subseteq ((G_k \setminus A) \cap G^{p^{\alpha_k}}) \cup \{1\} = 1,$$

as expected, and we are done. This concludes the proof in all generality. □

The author feels that the following example is helpful. It uncovers the above made selection of the special generators and also demonstrably shows that the conditions on the generating members of the groups C_k cannot be decreased. In precise words, the restrictions cannot be ignored and must be fulfilled for all possible products of the generators, but not only for the independent degrees of the different single generator elements.

Example 1. Let us take $g \in G_k \setminus A$ such that $g^\varepsilon \in (G_k \setminus A) \cup A_k$ for every $0 \leq \varepsilon \leq \text{order}(g)$. Suppose there exists an element $a \in A \setminus A_k$ such that $\text{order}(a) = \text{order}(g)$, that is a and g are of the same order, and $a^\varepsilon \in A \setminus A_k \Leftrightarrow g^\varepsilon \in G_k \setminus A$. Then $(g^{-1}a)^\varepsilon \in G_k \setminus A$ if $g^\varepsilon \in G_k \setminus A$, but $(g^{-1}a)^\varepsilon \in A_k$ otherwise when $g^\varepsilon \in A_k$. Hence, $g \in C_k$ and $g^{-1}a \in C_k$, so $a = gg^{-1}a \in C_k \cap (A \setminus A_k)$. But it is a contradiction with our constructing idea, that substantiates the above given claim.

Remark. When $A = G^{p^\alpha} \neq 1$ for any arbitrary but a fixed ordinal α, the condition G/G^{p^α} to be σ-summable can be removed (see, for instance, [**1**]); thus G is σ-summable $\iff$ G^{p^α} is σ-summable. This is so because of the argument that G^{p^α} being strongly σ-summable in G implies that it is σ-summable. Indeed, we can write $G^{p^\alpha} = \bigcup_{n<\omega} G_n$, where $G_n \subseteq G_{n+1}$ and $G_n \cap G^{p^{\delta_n}} = 1$ for some $\delta_n < \text{length}(G)$. Since $G_n \neq 1$ we have $\delta_n > \alpha$, otherwise $\delta_n \leq \alpha$ insures $G_n = G_n \cap G^{p^\alpha} \subseteq G_n \cap G^{p^{\delta_n}} = 1$ that is wrong. Next, since both the lengths of G^{p^α} and G are cofinal with ω, one may choose $\epsilon_n < \text{length}(G^{p^\alpha})$ such that $\delta_n \leq \alpha + \epsilon_n < \text{length}(G)$. Consequently, we calculate

$$G_n \cap (G^{p^\alpha})^{p^{\epsilon_n}} = G_n \cap G^{p^{\alpha+\epsilon_n}} \subseteq G_n \cap G^{p^{\delta_n}} = 1,$$

and the claim really sustained.

We can especially restate the Main Theorem in a weaker form into two parts as follows:

Main Corollary. *Let G be an abelian p-group and let A be its balanced (= nice and isotype) subgroup so that G/A is σ-summable. Then A being σ-summable yields that G is σ-summable. The converse holds when A and G have equal lengths.*

Proof. Since A is balanced in G, it easily follows that $\text{length}(G)$ is limit because both A and G/A have such lengths. Since A is both σ-summable and isotype in G, it is straightforward that A is strongly σ-summable in G as above commented. The Main Theorem now applies to show that G is σ-summable, as claimed. □

Note. In accordance with [**7**], as it was just remarked earlier, we can restrict our computations only on the socle of the whole group. Thereby, the definitions of σ-summable p-groups and strongly σ-summable p-subgroups as well as the formulation of the two types central assertions may be stated only in the terms of the operator $[p]$.

Further, we concentrate on a counterexample demonstrating that over an arbitrary subgroup A of G the conditions on A and G/A to be σ-summable are not sufficient to provide us with enough information to decide whether G is σ-summable as well.

Example 2. We shall show below that if both A and G/A are σ-summable groups, then G need not be σ-summable when A is not nice and isotype in G. This example contrasts with the corresponding one of Dieudonné [**3**] formulated for direct sums of cyclic p-groups. This can be explained via the difference of the lengths in the two situations, and more specially that our lengths are strictly more than ω.

In fact, suppose $G = \bigoplus_{i\in I} G_i$, where G_i are reduced countable abelian p-groups such that for all $i \in I$: length$(G_i) = \omega^2 + n$ for some fixed natural number n, whence length$(G) = \omega^2 + n$. Let us assume also that $A = \bigoplus_{i\in I} A_i$, where A_i are reduced countable abelian p-groups so that for any $i \in I$: length$(A_i) = \omega \cdot 2$, hence length$(A) = \omega \cdot 2$, and assume that length$(G_i/A_i) = \omega^2$. Consequently, by the constructions,

$$G/A \cong \bigoplus_{i\in I}(G_i/A_i)$$

possesses length equal to ω^2 and thus G/A along with A are σ-summable p-groups, while G is not σ-summable because its length $\omega^2 + n$ is not cofinal with ω. The application of a result due to Hill-Megibben [**8**] or [**12**, Theorem 2.1] is a guarantor that A is not balanced in G, since otherwise A must be a direct factor of G that is false. Another reason ensuring that A is not isotype in G is that for the isotype subgroup A it holds that

$$G^{p^{\omega^2}} \cong G^{p^{\omega^2}}/A^{p^{\omega^2}} \cong (G^{p^{\omega^2}}A)/A \subseteq (G/A)^{p^{\omega^2}} = 1$$

whence $G^{p^{\omega^2}} = A^{p^{\omega^2}} = 1$, thus length$(G) \leq \omega^2$, contrary to our supposition.

We emphasize that $A \neq G^{p^\alpha}$ for any ordinal α because otherwise we have $G^{p^{\alpha+\omega\cdot 2}} = 1$ with length$(G) = \alpha + \omega \cdot 2$ cofinal with ω, which is impossible.

We note that if the lengths of the subgroups G_i are taken to be limit, G is a direct sum of σ-summable groups. We also indicate that A is nice in G if and only if each A_i is nice in G_i, respectively.

It is simply observed that analogous examples may be extracted when length$(G) = \Omega$, the first uncountable limit ordinal that is not cofinal with ω, or even when length(G) is cofinal with ω. They parallel with the Dieudonné's one [**3**].

4. Applications and consequences.

First and foremost, we are in a position to proceed by proving the following.

Corollary. *Let G be an abelian p-group of countable limit length λ such that G/A is a reduced direct sum of countable groups of limit length for some its nice subgroup A. Then G is a reduced direct sum of countable groups if and only if A may be expanded to a λ-basic subgroup of G.*

Proof. Necessity. Since G is σ-summable, we write $G = \bigcup_{n<\omega} G_n$, $G_n \subseteq G_{n+1}$ and there exists an ordinal $\alpha_n <$ length(G) so that $G_n \cap G^{p^{\alpha_n}} = 1$. Therefore,

$$A = \bigcup_{n<\omega} A_n, \text{ where } A_n = G_n \cap A \text{ and } A_n \cap G^{p^{\alpha_n}} \subseteq G_n \cap G^{p^{\alpha_n}} = 1.$$

Thus A is strongly σ-summable in G. But G is a C_λ-group and owing to [**14**], it contains a λ-basic subgroup. That is why, by making use of a mild modification of the result due to Wallace, formulated in Section 2, the proof of this point is over.

Sufficiency. Invoking [**14**], G is itself a C_λ-group, i.e. G/G^{p^α} is a direct sum of countable groups for all $\alpha < \mathrm{length}(G)$. But, it is easy to observe that G/A is σ-summable. Moreover, conforming with the Wallace's theorem from the previous paragraph 2, A must be strongly σ-summable in G. By what we have already shown in the Main Theorem, the group G is σ-summable, hence a result due to Linton-Megibben and Hill-Megibben (see, for example, [**11**] and [**12**]) can be applied to get the claim. □

We process now the confirmation of some classical facts in the current direction.

a) A theorem of Dieudonné.

Here we shall examine the separable case, i.e. the groups with length ω. We start with a verification of the result of Dieudonné which is on the focus of our interest; a strengthening of the Dieudonné criterion in another way was established in [**9**] as well.

Indeed, let us presume that $G^{p^\omega} = 1$. It is well-known by using of the classical Kulikov's criterion [**4**, volume I, p. 106, Theorem 17.1] that all separable σ-summable groups are precisely the direct sums of cyclics.

For any subgroup A of G, we take G/A to be a direct sum of cyclic groups, and A to be a subgroup of the basic subgroup of G. It is a routine matter to observe that A must be nice in G, because $(G/A)^{p^\omega} = 1 = (G^{p^\omega}A)/A$. That is why our criterion is applicable to end the claim that G is a direct sum of cyclic groups.

An other corollary is in the case when $A = G[p]$. Since

$$G^{p^\omega} = 1 \quad \text{and} \quad G/G[p] \cong G^p,$$

it is trivial that $G[p]$ is nice in G. By presuming G^p is a direct sum of cyclics, and

$$G[p] = \bigcup_{n<\omega} A_n, \quad A_n \subseteq A_{n+1} \ \text{ and } \ A_n \cap G^{p^n}[p] = 1.$$

Furthermore, applying our Main Theorem or its above established consequence due to Dieudonné, we extract that G is a direct sum of cyclics.

Commentary. The fact that G^p being a direct sum of cyclics implies the same property for G, is a result from [**4**, volume I, p. 111, Proposition 18.3]. In this aspect, the fact that the conditions $G[p] = \bigcup_{n<\omega} A_n$, $A_n \subseteq A_{n+1}$ and $A_n \cap G^{p^n}[p] = 1$ yield G is a direct sum of cyclics, is the classical criterion due to Kaplansky-Honda (see, for instance, [**5**] and the second volume of [**4**, p. 123, Theorem 84.1]).

b) A theorem of Nunke.

Here we shall consider the special case in the above found Corollary when $A = G^{p^\alpha}$ for any ordinal number α. It is simple to be seen that G^{p^α} is nice in G (also cf. [4]). Now, we come to the significant Nunke's theorem (see [**13**] or [**4**, volume II, p. 112, Exercise 3]).

Theorem (Nunke, 1967)**.** *Suppose G is an abelian reduced p-group of countable limit length so that both G/G^{p^α} and G^{p^α} are direct sums of countable groups for any ordinal α. Then G is a direct sum of countable groups.*

Proof. By virtue of a transfinite induction on the length of G, we detect that G/G^{p^β} is a direct sum of countable groups for all $\beta < \text{length}(G)$, since so are

$$G/G^{p^\alpha} \cong G/G^{p^\beta}/(G/G^{p^\beta})^{p^\alpha} = G/G^{p^\beta}/G^{p^\alpha}/G^{p^\beta}$$

and

$$(G/G^{p^\beta})^{p^\alpha} = G^{p^\alpha}/G^{p^\beta}$$

whenever $\beta > \alpha$. Employing [**1**, Proposition of p. 6] or our central Theorem, we deduce that G is a σ-summable p-group. Hence, results due to Linton-Megibben [**11**] and Hill [**6**] along with [**13**] will imply that, G is a direct sum of countable groups, as required. □

c) A theorem of Megibben.

A version of the generalized Kulikov criterion, formulated as Theorem B in [**12**], states as follows:

Theorem (Megibben, 1969)**.** *Suppose G is a reduced abelian p-group with countable limit length λ. Then G is a direct sum of countable groups provided G is summable and, for each $\alpha < \lambda$, G contains a p^α-high subgroup which is a direct sum of countable groups.*

This significant result may be enlarged if the word "σ-summable“ supersedes the word "summable“ from the text etc. This can be done, as an example, by showing that G possesses a λ-basic subgroup and so referring to [**14**], G is a C_λ-group. Therefore, we apply [**12, 11, 6**] to complete the proof. However, this was an intensive work of other appropriate research study (see cf. [**2**]; Generalized Megibben Criterion).

Acknowledgments. The author owes warm thanks to the specialist referees for the valuable expert reports considerably improved the original manuscript and to Prof. Dr. Brendan Goldsmith from Dublin Institute of Technology for his care on the present paper.

References

1. Danchev P. V., *Theorems of the type of Nunke for abelian groups,* Compt. rend. Acad. bulg. Sci. **53**(10) (2000), 5–8.

2. ———, *Commutative group algebras of direct sums of σ-summable abelian p-groups*, Math. J. Okayama Univ. **45**(2) (2003), 1–15.

3. Dieudonné J., *Sur les p-groupes abéliens infinis*, Portugal. Math. **11**(1) (1952), 1–5.

4. Fuchs L., *Infinite Abelian Groups I and II*, Mir, Moscow 1974 and 1977 (In Russian).

5. Hill P. D., *Sufficient conditions for a group to be a direct sum of cyclic groups*, Rocky Mount. J. Math. **1**(2) (1971), 345–351.

6. ———, *Criteria for total projectivity*, Canad. J. Math. **33**(4) (1981), 817–825.

7. ———, *A note on σ-summable groups*, Proc. Amer. Math. Soc. **126**(11) (1998), 3133–3135.

8. Hill P. D. and Megibben C. K., *Extending automorphisms and lifting decompositions in abelian groups*, Math. Ann. **175** (1968), 159–168.

9. Kertész A., *On a theorem of Kulikov and Dieudonné*, Acta Sci. Math. Szeged **15** (1953), 61–69.

10. Kovács L. G., *On subgroups of the basic subgroup*, Publ. Math. Debrecen **5** (1958), 261–264.

11. Linton R. and Megibben C., *Extensions of totally projective groups*, Proc. Amer. Math. Soc. **64**(1) (1977), 35–38.

12. Megibben C. K., *The generalized Kulikov criterion*, Canad. J. Math. **21** (1969), 1192–1205.

13. Nunke R. J., *Homology and direct sums of countable abelian groups*, Math. Z. **101**(3) (1967), 182–212.

14. Wallace K. D., C_λ*-groups and* λ*-basic subgroups*, Pacific J. Math. **43**(3) (1972), 799–809.

P. Danchev, 13, General Kutuzov street, block 7, floor 2, flat 4, 4003 Plovdiv, Bulgaria,
e-mail: `pvdanchev@yahoo.com`