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ERROR ESTIMATES FOR FINITE VOLUME SCHEME FOR PERONA-MALIK EQUATION

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Abstract. We present Perona-Malik nonlinear image selective smoothing equation (modified in the sense of Catté, Lions, Morel and Coll) which is investigated esspecially from numerical point of wiev. Error estimates in L_2 norms for fully discrete numerical finite volume scheme are derived and proved. Some numerical examples are presented.

1. Introduction

1.1. Mathematical model of the problem

We are dealing with Perona-Malik type problem discussed in [**4**] in the following form

$$\partial_t u - \nabla.(g(|\nabla G_\sigma * u|)\nabla u) = 0 \quad \text{in } Q_T \equiv I \times \Omega, \tag{1}$$

$$\partial_\nu u = 0 \quad \text{on } I \times \partial\Omega, \tag{2}$$

$$u(0,\cdot) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $\Omega \subset I\!R^d$ is a rectangular domain, $I = [0,T]$ is a scaling interval, and

(4) $g(s)$ is a Lipschitz continuous decreasing function, with Lipschitz constant L_g

(5) $g(0) = 1, 0 < g(s) \to 0$ for $s \to \infty$,

(6) $G_\sigma \in C^\infty(I\!R^d)$ is a smoothing kernel with compact support K_σ with $\int_{I\!R^d} G_\sigma(x)dx = 1$ and $G_\sigma(x) \to \delta_x$ for $\sigma \to 0$, δ_x – Dirac function at point x,

(7) initial condition u_0 is such that regularity below is fulfiled.

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We can rewrite the partial differential equation (1) in the form

$$\partial_t u - \nabla.(g(|J(u)(x)|)\nabla u) = 0 \quad \text{in } Q_T \equiv I \times \Omega, \tag{8}$$

where $J(u) : L_2(\Omega) \to (C^\infty(\Omega))^d$. In our case we use $J(u)(t,\cdot) = \nabla G_\sigma * u(t,\cdot)$ for t fixed, but we can choose any smoothing operator with these properties.
Let us define a weak solution to the problem (8),(2),(3). Equation (8) is multiplied by a test function $\varphi \in \Psi$, where Ψ is the space of smooth test functions

$$\Psi = \{\varphi \in C^{1,2}([0,T] \times \overline{\Omega}), \nabla\varphi.\vec{n} = 0 \text{ on } (0,T) \ \times \ \partial\Omega \ , \varphi(T,.) = 0\}.$$

After integrating over $[0,T]$ and Ω and after applying integration by parts and properties of a test function, we come to a definition of the weak solution.

Definition 1.1. The weak solution of the regularized Perona-Malik problem (1)–(3) is a function $u \in L_2(I, W^{1,2}(\Omega))$ satisfying the identity

$$\begin{aligned}\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}(t,x)\,dx\,dt + \int_\Omega u_0(x)\varphi(0,x)\,dx \\ - \int_0^T\int_\Omega g(|J(u(t,x))|)\nabla u(t,x)\nabla\varphi(t,x)\,dx\,dt = 0\end{aligned} \tag{9}$$

for all $\varphi \in \Psi$.

It is well known from the regularity theory of such a solution [**10**] that, because of the properties of the operator $J(u)$, the weak solution of our problem is a function $u \in L_2(I, W^{2,2}(\Omega))$ for initial condition $u_0 \in L_\infty(\Omega)$. Moreover from the embedding theory for dimension $d = 2$, or $d = 3$ follows that $u \in C(\bar{Q}_T)$.

For our error estimates we need futher regularity of the solution, more precise $u \in L_2(I, W^{2,2}(\Omega)) \cap L_\infty(I, W^{1,2}(\Omega))$ and $u_{tt} \in L_1(I, L_1(\Omega))$.

1.2. Formulation of the finite volume scheme

Let τ_h be a uniform mesh of Ω with cells p of measure $m(p)$ (we assume rectangular cells here). For every cell p we consider a set of neighbours $N(p)$ consisting of all cells $q \in \tau_h$ for which the common interface of p and q, denoted by e_{pq}, is of non-zero measure $m(e_{pq})$. We denote the set of all these edges for all volumes $p \in \tau_h$ by $\mathcal{E}$ and by $e_{pq}I$ we denote the edge which connects the volumes p and q. (Clearly $e_{pq} = e_{qp} = e_{pq}I$). It is assumed that for every p, there exists a representative point $x_p \in p$, such that for every pair $p, q, q \in N(p)$, the vector $\frac{x_q - x_p}{|x_q - x_p|}$ is equal to a unit vector n_{pq} which is normal to e_{pq} and oriented from p to q. We denote $d_{pq} := |x_p - x_q|$. In a simple case of a uniform grid x_p is just the center of the pixel. Then, let x_{pq} be the point of e_{pq} intersecting the segment $\overline{x_p x_q}$. We define

$$T_{pq} := \frac{m(e_{pq})}{d_{pq}}. \tag{10}$$

Discrete approximation of a solution of partial differential equation is considered to be piecewise constant on control volumes [**5**].

Let u_p^n be the value of the numerical solution in the n-th scale step on a volume p. The finite volume semi-implicit scheme on a uniform grid is then written as follows:

Let $0 = t_0 \leq t_1 \leq \ldots \leq t_n \ldots \leq t_{N_{\max}}$, $N_{\max} \cdot k = T$ denote the scale discretization steps with $t_l = t_{l-1} + k$, where k is the discrete scale step, $l = 1, 2, \ldots, N_{\max}$. For $n = 0, \ldots, N_{\max}$ we look for u_p^{n+1}, $p \in \tau_h$, satisfying the identities

$$\left(u_p^{n+1} - u_p^n\right) m\left(p\right) = k \sum_{q \in N(p)} g_{pq}^{\sigma,n} T_{pq} \left(u_q^{n+1} - u_p^{n+1}\right), \tag{11}$$

$$u_p^0 = \frac{1}{m(p)} \int_p u_0(x) dx,$$

$$g_{pq}^{\sigma,n} := g\left(\left|J(\tilde{u}\left(t_n, x_{pq}\right))\right|\right), \tag{12}$$

where $\tilde{u}$ is a periodic extension of the discrete image computed in the n-th scale step. Its L_2 norm can be estimated with constant B by L_2 norm of function u. u_p^n is a value of the numerical solution on the volume p in the n-th scale step.

2. Stability and convergence results

We briefly mention results of Mikula and Ramarosy, see [**12**], obtained for the semi-implicit finite volume scheme concerning the stability and convergence properties. Explicit time discretization are discussed also in [**7**] and [**8**]. Stability estimates are of the following type [**12**]:

Lemma 2.1 (A priori estimates in $L_2(Q_T)$)**.** *It holds, that there exist positive constants* C_1, C_2 *such that*

(i) $\max\limits_{0 \leq l \leq N_{\max}} \sum\limits_{p \in \tau_h} \left(u_p^l\right)^2 m(p) \leq C_1,$

(ii) $\sum\limits_{l=0}^{N_{\max}} k \sum\limits_{(p,q) \in \mathcal{E}} \frac{\left(u_p^l - u_q^l\right)^2}{d_{pq}} m\left(e_{pq}\right) \leq C_2$

and the constants C_1, C_2 *do not depend on the* h, k.

Let us denote by $\overline{u}_{h,k}$ the finite volume numerical solution for some fixed space and scale mesh h and k. This solution is piecewise constant on each finite volume and in each scale step as it is usual for finite volume numerical schemes of a parabolic type. By $\bar{u}^l$ we denote the function piecewise constant on each finite volume in the l-th scale step. Then we have:

Lemma 2.2 (Convergence of $\overline{u}_{h,k}$)**.** *There exists* $u \in L^2\left(Q_T\right)$ *which is the weak solution of* (9) *such that*

$$\overline{u}_{h,k} \to u \text{ in } L^2\left(Q_T\right)$$

as $h, k \to 0$. *Furthermore, the convergence is pointwise.*

3. ERROR ESTIMATES

3.1. L_∞ stability for a discrete solution

We rewrite the original discrete equation (11) in the following way:

$$u_p^{n+1} + \frac{k}{m\,(p)} \sum_{q \in N(p)} g_{pq}^{\sigma,n} T_{pq} \left(u_p^{n+1} - u_q^{n+1}\right) = u_p^n. \tag{13}$$

Now let u_p^{n+1} be the maximal value for the fixed scale step $n+1$ and $p \in \tau_h$. Then the second term on the left hand side of (13) is nonnegative and:

$$u_p^{n+1} \leq u_p^n.$$

Recursively we have

$$||\overline{u}^{n+1}||_{L_\infty} \leq |u_p^{n+1}| \leq |u_p^0| \leq ||\overline{u}^0||_{L_\infty} \leq C. \tag{14}$$

3.2. Error estimates

Let now $t_n < t \leq t_{n+1}$. Multiplying PDE (8) by v_p^{n+1} and then integrating over volume p and using integration by parts, we have:

$$\int\limits_p \partial_t u(t,x)\, v_p^{n+1}\, dx - \int\limits_{\partial p} g(|J(u)|)\nabla u \cdot \mathbf{n_p} v_p^{n+1}\, dx = 0, \tag{15}$$

where ∂p is the boundary of the volume p and $\mathbf{n_p}$ is the outward unit normal vector to the boundary of volume p and further analogously $\mathbf{n_{pq}}$ will be the outward unit normal vector to the edge e_{pq}. We can write

$$\partial p = \bigcup_{q \in N(p)} e_{pq}.$$

For the discrete scheme we have

$$\frac{\left(u_p^{n+1} - u_p^n\right) v_p^{n+1} m\,(p)}{k} - \sum_{q \in N(p)} g_{pq}^{\sigma,n} T_{pq} \left(u_q^{n+1} - u_p^{n+1}\right) v_p^{n+1} = 0. \tag{16}$$

Now we denote

$$e_p^n = u(t_n, x_p) - u_p^n,$$

where x_p is a representative point of a volume p, $p \in \tau_h$.
Then posing $v_p^n = e_p^n$ and subtracting (16) from (15) we obtain:

$$\int\limits_p \frac{\left(e_p^{n+1} - e_p^n\right)}{k} e_p^{n+1} + \sum_{q \in N(p)} \int\limits_{e_{pq}} \left(g_{pq}^{\sigma,n} \frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} - g(|J(u)|)\nabla u \cdot \mathbf{n_{pq}} \right) e_p^{n+1}$$

$$= \int\limits_p \left(\frac{u(t_{n+1}, x_p) - u(t_n, x_p)}{k} - \partial_t u \right) e_p^{n+1}.$$

Now after summation over all volumes $p \in \tau_h$ and integration over $I_n = \langle t_n, t_{n+1})$ for all $n = 0, 1, \ldots, m-1$ and rearranging the equation we obtain:
(17)

$$
\begin{aligned}
&\int\limits_{\Omega} |\overline{e}^m|^2 + \sum_{n=0}^{m-1} \int\limits_{\Omega} |\overline{e}^{n+1} - \overline{e}^n|^2 \\
&\qquad + 2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{p \in \tau_h} \sum_{q \in N(p)} \int\limits_{e_{pq}} \left(g_{pq}^{\sigma,n} \frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} - g(|J(u)|) \nabla u \cdot \mathbf{n_{pq}} \right) e_p^{n+1} \\
&\qquad = \int\limits_{\Omega} |\overline{e}^0|^2 + 2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{p \in \tau_h} \int\limits_{p} \left(\frac{u(t_{n+1}, x_p) - u(t_n, x_p)}{k} - \partial_t u \right) e_p^{n+1}.
\end{aligned}
$$

The third term on the left hand side of the last equation can be expressed as it is usual in the finite volume theory, see [**5**]:

$$
\begin{aligned}
\text{"Third"} &= 2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{p \in \tau_h} \sum_{q \in N(p)} \int\limits_{e_{pq}} \left(g_{pq}^{\sigma,n} \frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} - g(|J(u)|) \nabla u \cdot \mathbf{n_{pq}} \right) e_p^{n+1} \\
&= 2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq} I} \left(g_{pq}^{\sigma,n} \frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} - g(|J(u)|) \nabla u \cdot \mathbf{n_{pq}} \right) (e_p^{n+1} - e_q^{n+1}).
\end{aligned}
$$

After rearranging we get:

$$
\begin{aligned}
&\text{"Third``} = 2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq} I} g(|J(u)|) \frac{\left(e_p^{n+1} - e_q^{n+1}\right)^2}{d_{pq}} \\
&+2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq} I} \left(g_{pq}^{\sigma,n} - g(|J(u)|)\right) \frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} (e_p^{n+1} - e_q^{n+1}) \\
&+2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq} I} g(|J(u)|) \left(\frac{u(t_{n+1}, x_q) - u(t_{n+1}, x_p)}{d_{pq}} - \nabla u \cdot \mathbf{n_{pq}} \right) (e_p^{n+1} - e_q^{n+1}).
\end{aligned}
$$

Involving these terms to the (17) equation we obtain:

$$
\int\limits_{\Omega} |\overline{e}^m|^2 + \sum_{n=0}^{m-1} \int\limits_{\Omega} |\overline{e}^{n+1} - \overline{e}^n|^2 + 2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq} I} g(|J(u)|) \frac{\left(e_p^{n+1} - e_q^{n+1}\right)^2}{d_{pq}}
$$

$$
\begin{aligned}
&= \int_\Omega |\overline{e}^0|^2 + 2\sum_{n=0}^{m-1}\int_{I_n}\sum_{p\in\tau_h}\int_p \left(\frac{u(t_{n+1},x_p)-u(t_n,x_p)}{k} - \partial_t u\right) e_p^{n+1} \\
&+ 2\sum_{n=0}^{m-1}\int_{I_n}\sum_{\mathcal{E}}\int_{e_{pq}I} \left(g_{pq}^{\sigma,n} - g(|J(u)|)\right) \frac{u_q^{n+1}-u_p^{n+1}}{d_{pq}}(e_p^{n+1}-e_q^{n+1}) \\
&+ 2\sum_{n=0}^{m-1}\int_{I_n}\sum_{\mathcal{E}}\int_{e_{pq}I} g(|J(u)| \left(\frac{u(t_{n+1},x_q)-u(t_{n+1},x_p)}{d_{pq}} - \nabla u\cdot\mathbf{n_{pq}}\right)(e_p^{n+1}-e_q^{n+1}),
\end{aligned}
$$

or briefly

$$
\begin{aligned}
&\int_\Omega |\overline{e}^m|^2 + \sum_{n=0}^{m-1}\int_\Omega |\overline{e}^{n+1}-\overline{e}^n|^2 + 2\sum_{n=0}^{m-1}\int_{I_n}\sum_{\mathcal{E}}\int_{e_{pq}I} g(|J(u)|)\frac{\left(e_p^{n+1}-e_q^{n+1}\right)^2}{d_{pq}} \\
&= \int_\Omega |\overline{e}^0|^2 + I + II + III.
\end{aligned}
$$

Remark. To obtain an appropriate error estimate we must take into account the regularity of the solution which plays an important role in error analysis. Error estimates could be done also better, but further regularity for time derivative is needed. If we suppose $u_0 \in L_\infty(\Omega)$ only, no further regularities are available.

Now we must estimate each of the last three terms on the right hand side.

$$
\begin{aligned}
I =& 2\sum_{n=0}^{m-1}\int_{I_n}\sum_{p\in\tau_h}\int_p \left(\frac{u(t_{n+1},x_p)-u(t_n,x_p)}{k} - \partial_t u\right) e_p^{n+1} \\
=& 2\sum_{n=0}^{m-1}\sum_{p\in\tau_h}\int_p \left(u(t_{n+1},x_p) - u(t_{n+1},x) + u(t_n,x) - u(t_n,x_p)\right) e_p^{n+1} \\
=& 2\sum_{p\in\tau_h}\int_p \left(\sum_{n=0}^{m-1}\left(u(t_n,x_p)-u(t_n,x)\right)\left(e_p^n - e_p^{n+1}\right)\right) \\
&+\ 2\sum_{p\in\tau_h}\int_p \left(\left(u(t_m,x_p)-u(t_m,x)\right)e_p^m - \left(u(0,x_p)-u(0,x)\right)e_p^0\right) \\
\le& \sqrt{2}\sum_{n=0}^{m-1}\int_\Omega h|\nabla u(t_n,\cdot)||\overline{e}^{n+1}-\overline{e}^n| + \sqrt{2}\int_\Omega h|\nabla u(t_m,\cdot)||\overline{e}^m| \\
&+ \sqrt{2}\int_\Omega h|\nabla u(0,\cdot)||\overline{e}^0|.
\end{aligned}
$$

After using Young's inequality we get

$$\mathrm{I} \le h^2 \sum_{n=0}^{m-1} \int\limits_{\Omega} |\nabla u(t_n,\cdot)|^2 + \frac{1}{2} \sum_{n=0}^{m-1} \int\limits_{\Omega} |\overline{e}^{n+1} - \overline{e}^{n}|^2 + \ h^2 \int\limits_{\Omega} |\nabla u(t_m,\cdot)|^2$$
$$+ \frac{1}{2} \int\limits_{\Omega} |\overline{e}^{m}|^2 + h^2 \int\limits_{\Omega} |\nabla u(0,\cdot)|^2 + \frac{1}{2} \int\limits_{\Omega} |\overline{e}^{0}|^2.$$

Finally

$$I \le \frac{1}{2} \sum_{n=0}^{m-1} \int\limits_{\Omega} |\overline{e}^{n+1} - \overline{e}^{n}|^2 + \frac{1}{2} \sum_{\Omega} |\overline{e}^{m}|^2$$
$$+ \frac{1}{2} \int\limits_{\Omega} |\overline{e}^{0}|^2 + \left(\frac{h^2 T}{k} + 2h^2 \right) ||\nabla u||_{L_\infty(I, L_2(\Omega))}$$

We estimate the second term in the following way:

$$\mathrm{II} = 2 \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq} I} \left(g_{pq}^{\sigma,n} - g(|J(u(x))|) \right) \frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} (e_p^{n+1} - e_q^{n+1}) dx.$$

First we estimate

$$|g_{pq}^{\sigma,n} - g(|J(u)|)| = |g(|\nabla G_\sigma * \tilde{u}(t_n, x_{pq})|) - g(|\nabla G_\sigma * \tilde{u}(t,x)|)|$$
$$\le L_g | \int_{I\!R^d} \nabla G_\sigma(x_{pq} - \eta) \tilde{u}_{h,k}(t_n,\eta) d\eta - \int_{I\!R^d} \nabla G_\sigma(s - \eta) \tilde{u}(t,\eta) d\eta |$$
$$\le L_g \int_{I\!R^d} |\nabla G_\sigma(x_{pq} - \eta) - \nabla G_\sigma(s-\eta)| |\tilde{u}_{h,k}(t_n,\eta)| d\eta$$
$$+ L_g \int_{I\!R^d} |\nabla G_\sigma(s-\eta)| |\tilde{u}_{h,k}(t_n,\eta) - \tilde{u}(t,\eta)| d\eta.$$

We obtain

$$|g_{pq}^{\sigma,n} - g(|J(u)|)|$$
$$\le \frac{L_g B}{\sqrt{2}} \cdot h ||D^2 G_\sigma||_{L_\infty(\Omega)} ||u_{h,k}||_{L_\infty(Q_T)} m(K_\sigma) + L_g B ||\nabla G_\sigma||_{L_\infty(\Omega)}$$
$$\cdot \left(\left(\int\limits_{\Omega} |\overline{e}^{n}|^2 dx \right)^{\frac{1}{2}} + \int\limits_{\Omega} \int\limits_{t}^{t_n} |\partial_t u(s,x)| ds dx + \sum_{p \in \tau_h} \int\limits_{p} \int\limits_{x_p}^{x} |\frac{\partial u(t,y)}{\partial \mathbf{n}}| dy dx \right),$$

where $m(K_\sigma)$ is measure of the compact support K_σ, σ is fixed, B is the estimation for mirror reflexion function. We denote

$C_3 = 2L_g B ||D^2 G_\sigma||_{L_\infty(\Omega)} ||u_{h,k}||_{L_\infty(Q_T)} m(K_\sigma)$,
$C_4 = 2L_g B ||\nabla G_\sigma||_{L_\infty(\Omega)}$,
C_g is such that $g(|J(u)|) \ge C_g$.

The last estimate can be established due to the properties of the solution u. Hence the whole term II can be estimated as follows:

$$
\begin{aligned}
\mathrm{II} \leq{}& C_3 h \cdot \left(\sum_{n=0}^{m-1} k \sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{m-1} k \sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|e_p^{n+1} - e_q^{n+1}|^2}{d_{pq}} \right)^{\frac{1}{2}} \\
&+ C_4 \cdot \sum_{n=0}^{m-1} k \left(\sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \right)^{\frac{1}{2}} \left(\sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|e_p^{n+1} - e_q^{n+1}|^2}{d_{pq}} \right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\overline{e}^n|^2 \right)^{\frac{1}{2}} \\
&+ C_4 \cdot \sum_{n=0}^{m-1} \int\limits_{I_n} \left(\sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \right)^{\frac{1}{2}} \left(\sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|e_p^{n+1} - e_q^{n+1}|^2}{d_{pq}} \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int\limits_{\Omega} \int\limits_{t}^{t_n} |\partial_t u(s,x)| ds dx + \left(\sum_{p \in \tau_h} \int\limits_{p} \int\limits_{x_p}^{x} |\frac{\partial u(t,y)}{\partial n}| dy dx \right) \right) \\
\mathrm{II} \leq{}& \frac{4 C_2 C_3^2 h^2}{C_g^2} + \frac{1}{2} \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq}I} g(|J(u)|) \frac{\left(e_p^{n+1} - e_q^{n+1}\right)^2}{d_{pq}} \\
&+ \frac{4 C_4^2}{C_g^2} \cdot \sum_{n=0}^{m-1} k \sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \int\limits_{\Omega} |\overline{e}^n|^2 \\
&+ \frac{4 C_4^2}{C_g^2} \cdot \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \\
&\qquad \cdot \left(\int\limits_{\Omega} \int\limits_{t}^{t_n} |\nabla \cdot (g(|J(u)| \nabla u) \, | ds dx + \sum_{p \in \tau_h} \int\limits_{p} \int\limits_{x_p}^{x} |\frac{\partial u(t,y)}{\partial \mathbf{n}}| dy dx \right) \\
={}& \frac{4 C_2 C_3^2 h^2}{C_g^2} + \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3,
\end{aligned}
$$

where the inequalities (14), (ii) and the equation (1) has been used. The last term can be estimated using the properties of the exact solution:

$$
\begin{aligned}
\mathrm{II}_3 \leq{}& \left(\frac{8 C_4^2 L_g C_2}{C_g^2} ||D^2 G_\sigma||_{L_\infty(\Omega)} ||\nabla u||_{L_\infty(I, L_2(\Omega))} + \frac{4 C_4^2 C_2}{C_g^2} ||\Delta u||_{L_2(I, L_2(\Omega))} \right) \cdot k \\
&+ \left(\frac{8 C_4^2 L_g C_2}{C_g^2} ||D G_\sigma||_{L_\infty(\Omega)} ||\nabla u||_{L_\infty(I, L_2(\Omega))} \right) \cdot h.
\end{aligned}
$$

Finally the third term can be estimated:

$$\mathrm{III} = 2\sum_{n=0}^{m-1}\int_{I_n}\sum_{\mathcal{E}}\int_{e_{pq}I} g(|J(u)|)\left(\frac{u(t_{n+1},x_q)-u(t_{n+1},x_p)}{d_{pq}} - \nabla u\cdot\mathbf{n_{pq}}\right)$$
$$\cdot\,(e_p^{n+1}-e_q^{n+1})$$

We denote

$$R_{pq} = \frac{1}{m(e_{pq})}\left(-\int_{13454568e_{pq}I}\nabla u\cdot\mathbf{n_{pq}} + \frac{u(t_{n+1},x_q)-u(t_{n+1},x_p)}{d_{pq}}m(e_{pq})\right).$$

Applying the properties of function g, this term can be estimated as

$$|\mathrm{III}| \le 2\sum_{n=0}^{m-1}\int_{I_n}\sum_{\mathcal{E}}\int_{e_{pq}I}|R_{pq}||e_p^{n+1}-e_q^{n+1}|.$$

Now using the regularity of a weak solution and the estimates well known in the finite volume method see, [**5**, Chapter 3.1.6], we get

$$\begin{aligned}|\mathrm{III}| \le & \frac{C}{C_g}h^2\sum_{n=0}^{m-1}\int_{I_n}\int_{\Omega}(H(u)(z))^2\\ &+\frac{1}{4}\sum_{n=0}^{m-1}\int_{I_n}\sum_{\mathcal{E}}\int_{e_{pq}I}g(|J(u)|)\frac{\left(e_p^{n+1}-e_q^{n+1}\right)^2}{d_{pq}}.\end{aligned}$$

Here $|H(u)(z)|^2 = \sum_{i,j=1}^{d}|D_iD_ju(z)|^2$ and D_i denote the weak derivatives with respect to the component z_i. Since $u \in L_2(I, W^{2,2}(\Omega))$ we can denote

$$C_5 = \frac{C}{C_g}||H(u)||^2_{L_2(Q_T)}$$

and we have

$$\mathrm{III} \le C_5h^2 + \frac{1}{4}\sum_{n=0}^{m-1}\int_{I_n}\sum_{\mathcal{E}}\int_{e_{pq}I}g(|J(u)|)\frac{\left(e_p^{n+1}-e_q^{n+1}\right)^2}{d_{pq}}.$$

Putting all these estimates together, we obtain:

$$\int\limits_{\Omega} |\overline{e}^m|^2 + \sum_{n=0}^{m-1} \int\limits_{\Omega} |\overline{e}^{n+1} - \overline{e}^n|^2 + \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{\mathcal{E}} \int\limits_{e_{pq}I} g(|J(u)|) \frac{\left(e_p^{n+1} - e_q^{n+1}\right)^2}{d_{pq}}$$
$$\leq 4 \int\limits_{\Omega} |\overline{e}^0|^2 + 2(\frac{h^2 T}{k} + 2h^2) ||\nabla u||_{L_\infty(I,L_2(\Omega))} + \left(\frac{4C_2 C_3}{C_g} + 2C_5\right) h^2$$
$$+ \left(\frac{8C_4^2 L_g}{C_g^2} ||D^2 G_\sigma||_{L_\infty(\Omega)} ||\nabla u||_{L_\infty(I,L_2(\Omega))} + \frac{4C_4^2 C_2}{C_g^2} ||\Delta u||_{L_2(I,l_2(\Omega))}\right) k$$
$$+ \left(\frac{8C_4^2 L_g C_2}{C_g^2} ||DG_\sigma||_{L_\infty(\Omega)} ||\nabla u||_{L_\infty(I,L_2(\Omega))}\right) \cdot h$$
$$+ \frac{4C_4}{C_g} \cdot \left(\sum_{n=0}^{m-1} k \sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \int\limits_{\Omega} |\overline{e}^n|^2 \right).$$

If we realize, that the first term on the right hand side with $\overline{e}^0$ is less than Ch because of the properties of the exact solution and the definition of u_p^0, we are prepared to use Gronwall's lemma in the form:

Lemma 3.1. *If $u(t)$ and $v(t)$ are non-negative measurable functions for $t \leq 0$ and u_0 is a non-negative constant, then the inequality $u(t) \leq u_0 + \int\limits_0^t v(s)\ u(s) ds$ implies that $u(t) \leq u_0 \exp\left(\int\limits_0^t v(s) ds\right)$.*

To estimate the last term of the previous inequality let us denote for $t \in I_n = \langle t_{n-1}, t_n)$

$$v(t) = \sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}}, \qquad u(t) = \int\limits_{\Omega} |\overline{e}^n|^2 dx.$$

If we use the properties of function v then we can obtain

$$\int\limits_{\Omega} |\overline{e}^m|^2 \leq C(h^2 + h + \frac{h^2}{k} + k) \cdot \exp(\left(\sum_{n=0}^{m-1} k \sum_{\mathcal{E}} \int\limits_{e_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \right)$$
$$\leq C \cdot \exp(C_2)(h^2 + k + h + \frac{h^2}{k}),$$

where C is a generic constant depending only on domain Ω, time T and some norms of the exact solution. To obtain convenient error estimate result we can choose

$$k = Ch, \tag{18}$$

Theorem 3.1. *Let the relation between scale and space discretization be chosen as in* (18). *Then for the error estimates for Perona-Malik weak solution and numerical solution obtained via finite volume method it holds*

$$\sum_{n=0}^{N_{max}} \int\limits_{I_n} \int\limits_{\Omega} |u(t_{n+1},x) - \overline{u}_{h,k}(t_{n+1},x)|^2 \leq Ch$$

and

$$\sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{e_{pq}I} m(e_{pq}) d_{pq} \left(\frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} - \frac{1}{m(e_{pq})} \int\limits_{e_{pq}} \nabla u \cdot \mathbf{n_{pq}} \right)^2 \leq Ch.$$

Proof. It is easy to see that

$$\begin{aligned} &\sum_{n=0}^{N_{max}} \int\limits_{I_n} \int\limits_{\Omega} |u(t_{n+1},x) - \overline{u}_{h,k}(t_{n+1},x)|^2 \\ &\qquad \leq 2h^2 \|\nabla u\|_{L_2(I,L_2(\Omega))} + 2 \sum_{n=0}^{N_{max}} \int\limits_{I_n} \int\limits_{\Omega} |\overline{e}^{n+1}|^2 \leq Ch \end{aligned}$$

and the first inequality is proved. Now

$$\begin{aligned} &\sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{e_{pq}I} m(e_{pq}) d_{pq} \left(\frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} - \frac{1}{m(e_{pq})} \int\limits_{e_{pq}} \nabla u \cdot \mathbf{n_{pq}} \right)^2 \\ &\qquad \leq C \sum_{n=0}^{m-1} \int\limits_{I_n} \sum_{e_{pq}I} \int\limits_{e_{pq}} g(J(u)) \frac{\left(e_p^{n+1} - e_q^{n+1}\right)^2}{d_{pq}} \\ &\qquad\quad + C \sum_{n=0}^{m-1} \int\limits_{I_n} d_{pq} \sum_{e_{pq}I} \left(\frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} - \frac{1}{m(e_{pq})} \int\limits_{e_{pq}} \nabla u \cdot \mathbf{n_{pq}} \right)^2 \\ &\qquad \leq Ch, \end{aligned}$$

where we have again used the estimate of the finite volume method for the second term. □

4. Numerical experiments

In this section we present experiments with some artificial images perturbed by various types of noise. We want to confirm the relation between scale step, mesh size and the data coefficients obtained in the previous theorem. In simulations, we use the function

$$g(s) = \frac{1}{1 + Ks^2}$$

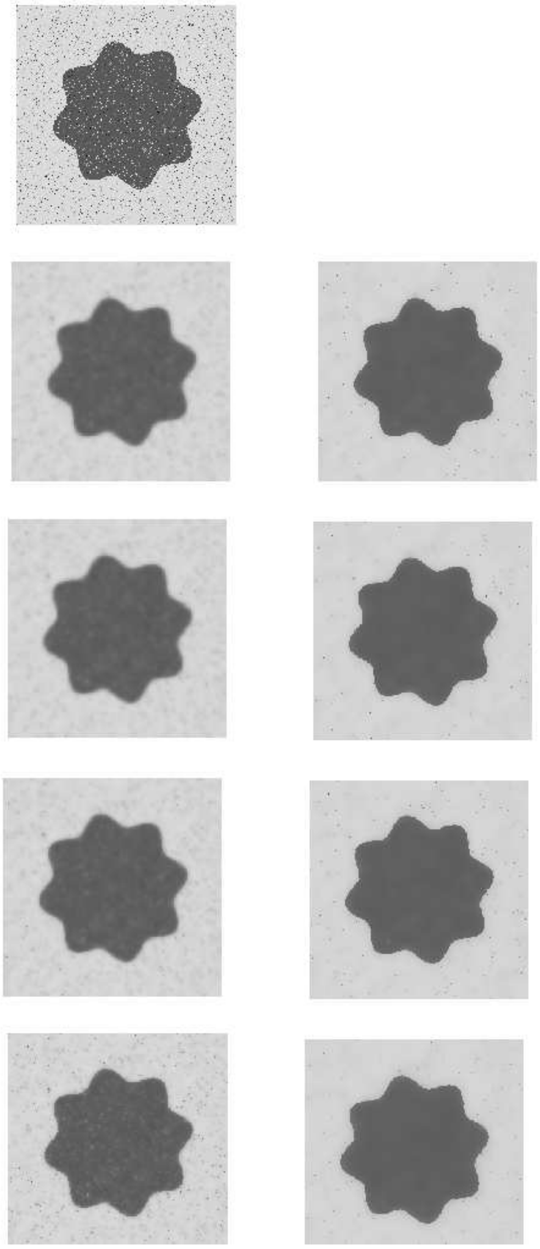

Figure 1. The first column shows the work for parameter $K = 10$ for different scale steps $k = 0.1, 0.5, 1, 2.5.$, the second column shows the work for parameter $K = 100$ for scale steps $k = 0.5, 1, 4, 10.$ for Example 1.

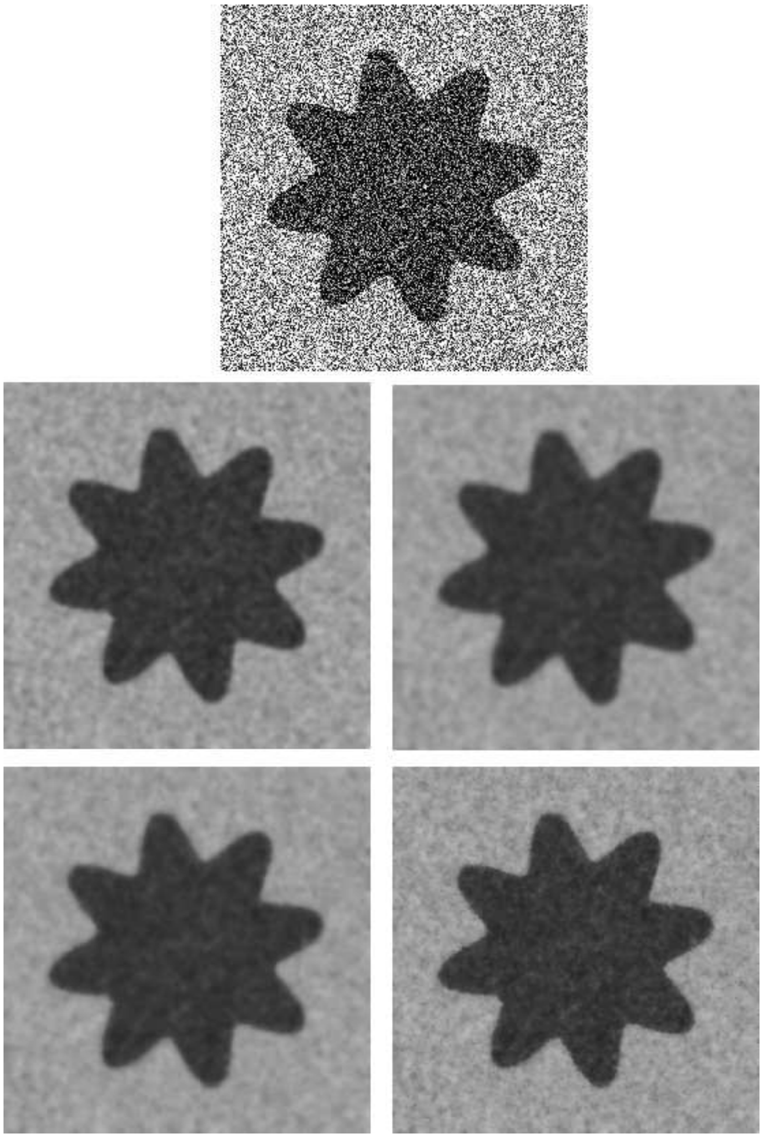

Figure 2. Pictures shows the work for parameter $K = 10$ for different scale steps $k = 0.2, 1, 5, 10$ (from the top to bottom, from left to right) for Example 2.

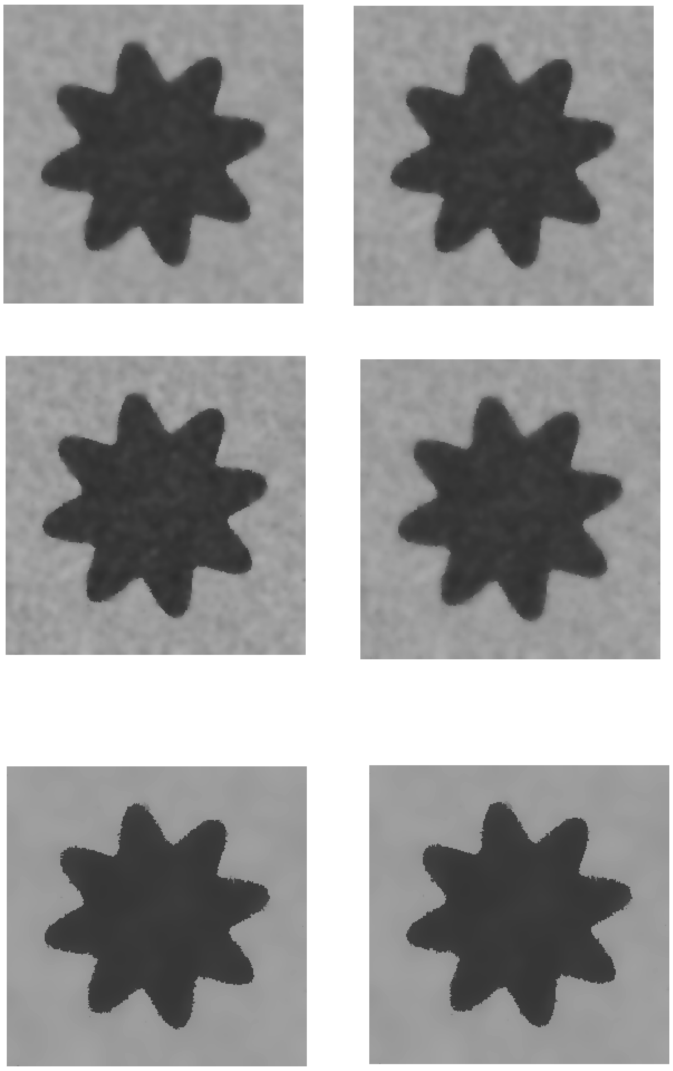

Figure 3. Pictures shows the work for parameter $K = 100$ (top) for different scale steps $k = 1, 2, 5, 10$, and for parameter $K = 1000$ for scale parameter $k = 10, 20$ for Example 2.

and the convolution is realized with the kernel

$$G_\sigma(x) = \frac{1}{Z} e^{\frac{|x|^2}{|x|^2-\sigma^2}},$$

where the constant Z is chosen so that G_σ has unit mass.

Example 1. To every position of the initial image we apply a 10% salt and pepper noise.

Example 2. We have chosen another type of picture with a noise function f defined as follows: if $\psi(x)$ is a functiongenerating random values in $[0, 2C]$, then for every position x

$$f(u_0(x)) = \mathrm{MIN}(255, \mathrm{MAX}(0, u_0(x) - C + \psi).$$

$C = 100$ and the difference in intensity between the two values of the initial image is 200.

In both examples the size of one finite volume corresponds to the size of one pixel. We computed the same example for different scale steps. In both figures we choose the best visual result for every parameter K in function g which plays an important role in smoothing effect. For the best cases it seems that the relation between grid mesh, scale step and parameter K remains is constant.

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