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LIMIT OF APPROXIMATE INVERSE SYSTEM OF TOTALLY REGULAR CONTINUA IS TOTALLY REGULAR

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Abstract. It is known that the limit of an inverse system of totally regular continua is a totally regular continuum. In this paper we shall prove that this is true for approximate limit of an approximate inverse system in the sense of S. Mardešić (Theorem 14).

1. Introduction

In this paper we shall use the notion of *inverse systems* $\mathbf{X} = \{X_a, p_{ab}, A\}$ and their limits in the usual sense [**1**, p. 135].

The cardinality of a set X will be denoted by $\text{card}(X)$. The cofinality of a cardinal number m will be denoted by $\text{cf}(m)$. $\text{Cov}(X)$ is the set of all normal coverings of a topological space X. If $\mathcal{U}$, $\mathcal{V} \in \text{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} \leq \mathcal{U}$. For two mappings $f,\ g : Y \to X$ which are $\mathcal{U}$-near (for every $y \in Y$ there exists a $U \in \mathcal{U}$ with $f(y),\ g(y) \in U$), we write $(f,\ g) \leq \mathcal{U}$. A basis of (open) normal coverings of a space X is a collection $\mathcal{C}$ of normal coverings such that every normal covering $\mathcal{U} \in \text{Cov}(X)$ admits a refinement $\mathcal{V} \in \mathcal{C}$. We denote by $\text{cw}(X)$ (*covering weight*) the minimal cardinal of a basis of normal coverings of X [**9**, p. 181].

Lemma 1. [**9**, Example 2.2]. *If X is a compact Hausdorff space, then* $\text{cw}(X) = w(X)$.

The notion of *approximate inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ will be used in the sense of S. Mardešić [**11**].

Definition 1. An *approximate inverse system* is a collection $\mathbf{X} = \{X_a, p_{ab}, A\}$, where $(A, \leq)$ is a directed preordered set, X_a, $a \in A$, is a topological space and $p_{ab} : X_b \to X_a, a \leq b$, are mappings such that $p_{aa} = \text{id}$ and the following condition (A2) is satisfied:

(A2) For each $a \in A$ and each normal cover $\mathcal{U} \in \text{Cov}(X_a)$ there is an index $b \geq a$ such that $(p_{ac}p_{cd}, p_{ad}) \leq \mathcal{U}$ whenever $a \leq b \leq c \leq d$.

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An approximate map $\mathbf{p} = \{p_a : a \in A\} : X \to \mathbf{X}$ into an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a collection of maps $p_a : X \to X_a$, $a \in A$, such that the following condition holds

(AS) For any $a \in A$ and any $U \in \mathrm{Cov}(X_a)$ there is $b \geq a$ such that $(p_{ac}p_c, p_a) \leq U$ for each $c \geq b$. (See [**10**]).

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate system and let $\mathbf{p} =\{p_a : a \in A\} : X \to \mathbf{X}$ be an approximate map. We say that $\mathbf{p}$ is a *limit* of $\mathbf{X}$ provided it has the following universal property:

(UL) For any approximate map $\mathbf{q} = \{q_a : a \in A\} : Y \to \mathbf{X}$ of a space Y there exists a unique map $g : Y \to X$ such that $p_a g = q_a$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate system. A point $x = (x_a) \in \prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ provided it satisfies the following condition:

(L) $(\forall a \in A)(\forall U \in \mathrm{Cov}(X_a))(\exists b \geq a)(\forall c \geq b)p_{ac}(x_c) \in \mathrm{st}(x_a, U)$.

If X_a is a $\mathrm{T}_{3.5}$ space, then the sets $\mathrm{st}(x_a,\mathcal{U})$, $\mathcal{U} \in \mathrm{Cov}(X_a)$, form a basis of the topology at the point x_a. Therefore, for an approximate system of Tychonoff spaces condition (L) is equivalent to the following condition:

(L*) $(\forall a \in A) \lim\{p_{ac}(x_c) : c \geq a\} = x_a$.

Some other properties of approximate systems and their subsystems are given in Appendix.

Let τ be an infinite cardinal. We say that a partially ordered set A is τ*-directed* if for each $B \subseteq A$ with $\mathrm{card}(B) \leq \tau$ there is an $a \in A$ such that $a \geq b$ for each $b \in B$. If A is $\aleph_0$-directed, then we will say that A is σ-directed. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be τ*-directed* if A is τ-directed. An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be σ*-directed* if A is σ-directed.

The proof of the following theorem is similar to the proof of Theorem 1.1 of [**4**].

Theorem 1. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed approximate inverse system of compact spaces with surjective bonding mappings and limit X. Let Y be a metric compact space. For each surjective mapping $f : X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b : X_b \to Y$ such that $f = g_b p_b$.*

Theorem 2. *Let X be a compact spaces. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of compact metric spaces X_a and surjective bonding mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Theorem 3. [**8**, p. 163, Theorem 2.]. *If X is a locally connected compact space, then there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metric locally connected compact space, each p_{ab} is a monotone surjection and X is homeomorphic to* $\lim \mathbf{X}$. *Conversely, the inverse limit of such system is always a locally connected compact space.*

Remark 1. *We may assume that $\mathbf{X} = \{X_a, p_{ab}, A\}$ in Theorem 3 is σ-directed* [**12**, Theorem 9.5].

Theorem 4. [**13**, Corollary 2.9]. *If X is a hereditarily locally connected continuum, then there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is a metrizable hereditarily locally connected continuum, each p_{ab} is a monotone surjection and X is homeomorphic to* $\lim \mathbf{X}$.

Theorem 5. [**3**, Corollary 3]. *Let $\boldsymbol{X} = \{X_a,\ p_{ab},\ A\}$ be a σ-directed inverse system of hereditarily locally connected continua X_a. Then $X = lim\boldsymbol{X}$ is hereditarily locally connected.*

The following theorem is Theorem 1.7 from [**5**].

Theorem 6. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of compact metrizable spaces and surjective bonding mappings. Then $X = \lim \mathbf{X}$ is metrizable if and only if there exists an $a \in A$ such that $p_b : X \to X_b$ is a homeomorphism for each $b \geq a$.*

2. Limit of approximate inverse system of totally regular continua

We shall say that a non-empty compact space is *perfect* if it has no isolated point.

A continuum is said to be *totally regular* [**12**, p. 47] if for each $x \neq y$ in X there is a positive integer n and perfect subsets $A_1, \dots, A_n, \dots$ of X such that $x_i \in A_i$ for $i = 1, \dots, n$ implies that $\{x_1, \dots, x_n\}$ separates x from y in X.

Lemma 2. [**12**, Proposition 7.4]. *Each totally regular continuum is hereditarily locally connected and rim-finite.*

The following theorem is a part of [**12**, Theorem 7.15].

Theorem 7. *If X is a continuum then the following conditions are equivalent:*

1. *X is totally regular,*
2. *X is homeomorphic to* $\lim\{X_a, f_{ab}, \Gamma\}$ *such that each X_a is a totally regular continuum and each f_{ab} is a monotone surjection.*

Theorem 8. [**12**, Theorem 7.7]. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of totally regular continua X_a and monotone surjective mappings p_{ab}. Then $X =$* $\lim \mathbf{X}$ *is totally regular.*

Theorem 9. *Let X be a non-metric totally regular continuum. There exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that each X_a is totally regular, each f_{ab} is a monotone surjection and X is homeomorphic to* $\lim \mathbf{X}$.

Proof. Apply [**12**, Theorem 9.4], Theorem 8 and Lemma 3.5 of [**14**]. □

Now we consider approximate inverse systems of totally regular continua. We start with the following theorem.

Theorem 10. *Let $\mathbf{X} = \{X_n, p_{nm}, \mathbb{N}\}$ be an approximate inverse sequence of totally regular metric continua. If the bonding mappings are monotone and surjective, then $X = \lim \mathbf{X}$ is totally regular.*

Proof. There exists a usual inverse sequence $\mathbf{Y} = \{Y_i, q_{ij}, M\}$ such that $Y_i = X_{n_i}$, $q_{ij} = p_{n_i n_{i+1}} p_{n_{i+1} n_{i+2}} \dots p_{n_{j-1} n_j}$ for each $i, j \in \mathbb{N}$ and a homeomorphism $H : \lim \mathbf{X} \to \lim \mathbf{Y}$ [**2**, Proposition 8]. Each mapping q_{ij} as a composition of the monotone mappings is monotone. This means that $\mathbf{Y}$ is a usual inverse sequence of totally regular continua with monotone bonding mappings q_{ij}. By virtue of Theorem 8 $\lim \mathbf{Y}$ is totally regular. We infer that $X = \lim \mathbf{X}$ is totally regular since there exists a homeomorphism $H : \lim \mathbf{X} \to \lim \mathbf{Y}$. □

Theorem 11. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular continua such that* $\operatorname{card}(A) = \aleph_0$. *Then* $X = \lim \mathbf{X}$ *is totally regular.*

Proof. By virtue of Lemma 6 of Appendix there exists a countable well-ordered subset B of A such that the collection $\{X_b, p_{bc}, B\}$ is an approximate inverse sequence and $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$. From Theorem 10 it follows that $\lim\{X_b, p_{bc}, B\}$ is totally regular. Hence $X = \lim \mathbf{X}$ is totally regular. □

Theorem 12. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular continua and monotone bonding mappings. If* $w(X_a) \leq \tau < \operatorname{card}(A)$ *for each* $a \in A$, *then* $X = \lim \mathbf{X}$ *is totally regular continuum.*

Proof. By virtue of Theorem 15 (for $\lambda = \aleph_0$) of Appendix there exists a σ-directed inverse system $\{X_\alpha, q_{\alpha\beta}, T\}$, where each X_α is a limit of an approximate inverse subsystem $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$, $\operatorname{card}(\Phi) = \aleph_0$. From Theorem11 it follows that every X_α is totally regular. Theorem 8 completes the proof. □

Theorem 13. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of totally regular metric continua and monotone bonding mappings. Then* $X = \lim \mathbf{X}$ *is totally regular continuum.*

Proof. If $\operatorname{card}(A) = \aleph_0$, then we apply Theorem 11. If $\operatorname{card}(A) \geq \aleph_1$, then from Theorem 12 it follows that X is totally regular. □

A directed preordered set $(A, \leq)$ is said to be *cofinite* provided each $a \in A$ has only finitely many predecessors. If $a \in A$ has exactly n predecessors, we shall write $p(a) = n + 1$. Hence, $a \in A$ is the first element of $(A, \leq)$ if and only if $p(a) = 1$.

Lemma 3. *If* $(A, \leq)$ *is cofinite, then it satisfies the following principle of induction:*

Let $B \subset A$ *be a set such that:*

(i) *B contains all the first elements of A,*
(ii) *if B contains all the predecessors of $a \in A$, then $a \in B$.*

Then $B = A$.

Lemma 4. [**15**, Lemma 1]. *Let* $q = (q_a) : Y \to \mathcal{Y} = \{Y_b, \mathcal{V}_b, q_{ab'}, B\}$ *be an approximate map (approximate resolution) of a space* Y. *Then there exists an approximate map (approximate resolution)* $q = (q_a) : Y \to \mathcal{Y} = \{Y_c^{'}, \mathcal{V}_c^{'}, q_{cc'}, C\}$ *of the space* Y *and an increasing surjection* $t : C \to B$ *satisfying the following conditions:*

(i) *C is directed, unbounded, antisymmetric and cofinite set,*
(ii) $(\forall c \in C)(\forall b \in B)(\exists c' > c) \;\; t(c') > b$;
(iii) $(\forall c \in C) \;\; Y'_c = Y_{t(c)}$, $\mathcal{V}'_c = \mathcal{V}_{t(c)}$, $q'_c = q_{t(c)}$ *and* $q'_{cc'} = q_{t(c)t(c')}$, *whenever* $c < c'$.

Corollary 1. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of compact spaces. Then there exists an approximate inverse system $\mathbf{Y} = \{Y_c, p_{cc'}, C\}$ such that: a) each Y_c is some X_a, b) each $p_{cc'}$ is some p_{ab}, c) C is directed, unbounded, antisymmetric and cofinite set and* $\lim \mathbf{X}$ *is homeomorphic to* $\lim \mathbf{Y}$*.*

Proof. By virtue of Theorem 4.2 of [**10**] an approximate map $p : X \to \mathbf{X}$ is an approximate resolution if and only if it is a limit of $\mathbf{X} = \{X_a, p_{ab}, A\}$. Apply Lemma 4. □

Now we shall prove the main theorem of this paper.

Theorem 14. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of totally regular continua with monotone surjective bonding mappings p_{ab}. Then $X = \lim X$ is totally regular.*

Proof. If every X_a is a metric totally regular continuum, then we apply Theorem 13. Now, suppose that each X_a is a non-metric totally regular continuum. The proof consists of several steps. In the Steps $0 - 11$ we shall define a usual inverse system $\mathbf{X}_D = \{X_d, F_{de}, D\}$ whose inverse limit X_D is homeomorphic to $X = \lim \mathbf{X}$. In *Step* 12 we shall use Theorem 8 which completes the proof.

Step 0.

From Corollary 1 it follows that we may assume that A is cofinite.

Step 1.

By virtue of Theorem 9 for each X_a there exists a σ directed inverse system

$$\mathbf{X}(\mathbf{a}) = \{X_{(a,\gamma)},\ f_{(a,\gamma)(a,\delta)},\ \Gamma_a\} \tag{2.1}$$

such that each $X_{(a,\gamma)}$ is a totally regular metric continuum, each $f_{(a,\gamma)(a,\delta)}$ is monotone and surjective and X_a is homeomorphic to $\lim \mathbf{X}(\mathbf{a})$. Now we have the following diagram

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\gamma_a)}"] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_c)}"] & X_c \arrow[l, "p_{bc}"'] \arrow[d, "f_{(c,\gamma_c)}"] & X \arrow[l, "p_d"'] \\
X_{(a,\gamma_a)} \arrow[d, "f_{(a,\gamma_a)(a,\delta_a)}"] & X_{(b,\gamma_b)} \arrow[d, "f_{(b,\gamma_b)(b,\delta_b)}"] & X_{(c,\gamma_c)} \arrow[d, "f_{(c,\gamma_c)(c,\delta_c)}"] & \\
X_{(a,\delta_a)} \arrow[d] & X_{(b,\delta_b)} \arrow[d] & X_{(c,\delta_c)} \arrow[d] & \\
{} & {} & {} &
\end{tikzcd} \tag{2.2}$$

Step 2.

Put B = $\{$(a , γ_a) : a $\in$ A, $\gamma_a \in \Gamma_a\}$ and put C to be the set of all subsets c of B of the form

$$c = \{(a, \gamma_a) : a \in A\}, \tag{2.3}$$

where every γ_a is the fixed element of Γ_a.
Step 3.

Let D be a subset of C containing all $c \in C$ for which there exist the mappings

$$g_{(a,\gamma_a)(b,\gamma_b)} : X_{(b,\gamma_b)} \to X_{(a,\gamma_a)}, b \geq a, \tag{2.4}$$

such that

$$\{X_{(a,\gamma_a)},\ g_{(a,\gamma_a)(b,\gamma_b)}, A\} \tag{2.5}$$

is an approximate inverse system and each diagram

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\gamma_a)}"] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_b)}"] \\
X_{(a,\gamma_a)} & X_{(b,\gamma_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"']
\end{tikzcd} \tag{2.6}$$

commutes, where $f_{(a,\gamma_a)} : X_a \to X_{(a,\gamma_a)}$ is the canonical projection.
Step 4.

The set D is non empty. Moreover, for each countable subset $S_a \subset \Gamma_a$, $a \in A$, there exists a $d \in D$ such that $d = \{(a, \gamma_a) : a \in A\}$, $\gamma_a \geq \gamma$ for every $\gamma \in S_a$. Let $a \in A$ be some first element of A and let $\gamma_a \in \Gamma_a$ such that $\gamma_a \geq \gamma$ for every $\gamma \in S_a$. The space $X_{(a,\gamma_a)}$ is a metric compact space and there exist the mappings $f_{(a,\gamma_a)}p_{ab} : X_b \to X_{(a,\gamma_a)}, b \geq a$. By virtue of Theorem 1 for each $b \geq a$ there exist a $\gamma_b^1 \in \Gamma_b$ such that for each $\gamma_b \geq \gamma_b^1, \gamma$, where $\gamma \in S_b$, there exists a monotone surjective mapping $g_{(a,\gamma_a)(b,\gamma_b)} : X_{(b,\gamma_b)} \to X_{(a,\gamma_a)}$ with $f_{(a,\gamma_a)}p_{ab} = g_{(a,\gamma_a)(b,\gamma_b)}f_{(b,\gamma_b)}$, i.e., the diagram

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\gamma_a)}"] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_b)}"] \\
X_{(a,\gamma_a)} & X_{(b,\gamma_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"']
\end{tikzcd} \tag{2.7}$$

commutes. Suppose that $(a, \gamma_b^1), (a, \gamma_b^2), \ldots, (a, \gamma_b^{n-1})$ are defined for each $a \in A$ with $p(a) \leq n - 1$ such that the each diagram (2.6) commutes. Let $a \in A$ be a member of A with $p(a) = n$. This means that $(a, \gamma_b^1), (a, \gamma_b^2), \ldots, (a, \gamma_b^{n-1})$ are defined. From the cofinitness of A it follows that the set of γ_a^j which are defined in Γ_a is finite. Hence there exists $\gamma_a^n \geq \gamma_a^{n-1}, \ldots, \gamma_a^1$. We define $\gamma_b^n \in \Gamma_b$ considering the space $X_{(a,\gamma_a^n)}$ and the mappings $f_{(a,\gamma_a^n)}p_{ab} : X_b \to X_{(a,\gamma_a^n)}$. Again, by Theorem 1 for each $b \geq a$ there exists an $\gamma_b^n \in \Gamma_b$ such that for each $\gamma_b \geq \gamma_b^n, \gamma_b^{n-1}, \ldots, \gamma_b^1$ and there is a mapping $g_{(a,\gamma_b)(b,\gamma_b)} : X_{(b,\gamma_b)} \to X_{(a,\gamma_a^n)}$ with $f_{(a,\gamma_a^n)}p_{ab} = g_{(a,\gamma_b)(b,\gamma_b)}f_{(b,\gamma_b)}$, i.e., the diagram

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\gamma_a^n)}"] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_b)}"] \\
X_{(a,\gamma_a^n)} & X_{(b,\gamma_b)} \arrow[l, "g_{(a,\gamma_a^n)(b,\gamma_b)}"']
\end{tikzcd} \tag{2.8}$$

commutes. By induction on A (Lemma 3) the set D is defined. It remains to prove that $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ is an approximate inverse system. Let $\mathcal{U}$ be a normal cover of $X_{(a,\gamma_a)}$. Then $\mathcal{V} = f_{(a,\gamma_a)}^{-1}(\mathcal{U})$ is a normal cover of X_a. By virtue of (A2) there exists a $b \geq a$ such that for each $c \geq d \geq b$ we have $(p_{ad}, p_{ca}p_{cd}) \leq \mathcal{V}$. By virtue of the commutativity of the diagrams of the form (2.8) it follows that

$$(g_{(a,\gamma_a)(d,\gamma_d)},\ g_{(a,\gamma_a)(c,\gamma_c)}g_{(c,\gamma_c)(d,\gamma_d)}) \leq \mathcal{V}. \tag{2.9}$$

Thus, $\{X_{(a,\gamma_a)}, g_{(a,\gamma_a)(b,\gamma_b)}, A\}$ is an approximate inverse system.

Step 5.

We define a partial order on D as follows. Let d_1, d_2 be a pair of members of D such that $d_1 = \{(a,\gamma_a) : a \in A,\ \gamma_a \in \Gamma_a\}$ and $d_2 = \{(a,\delta_a) : a \in A,\ \delta_a \in \Gamma_a\}$. We write $d_2 \leq d_1$ if and only if $\delta_a \leq \gamma_a$ for each $a \in\ A$. From *Step* 4 it follows that $(D, \leq)$ is σ-directed.

Step 6.

For each $d \in D$ a limit space X_d of the inverse system (2.5) is a totally regular continuum (Theorem 13). Moreover, there exists a mapping $F_d : X \to X_d$. The existence of F_d follows from the commutativity of the diagram (2.6). The following diagram illustrates the construction of $d \in D$ and the space X_d.

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\delta_a)}"] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\delta_c)}"] & X_c \arrow[l, "p_{bc}"'] \arrow[d, "f_{(c,\delta_c)}"] & X \arrow[l, "p_d"'] \\
X_{(a,\delta_a)} \arrow[d, "f_{(a,\gamma_a)(a,\delta_a)}"] & X_{(b,\delta_b)} \arrow[d, "f_{(b,\gamma_b)(b,\delta_b)}"] & X_{(c,\delta_c)} \arrow[d, "f_{(c,\gamma_c)(c,\delta_c)}"] & \\
X_{(a,\gamma_a)} \arrow[d] & X_{(b,\gamma_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"'] \arrow[d] & X_{(c,\gamma_c)} \arrow[l, "g_{(b,\gamma_b)(c,\gamma_c)}"'] \arrow[d] & X_d \arrow[l, "g_{(c,\gamma_c)}"'] \\
{} & {} & {} &
\end{tikzcd} \tag{2.10}$$

Step 7.

If d_1, d_2 is a pair of members of D such that $d_1 = \{(a,\gamma_a) : a \in A,\ \gamma_a \in \Gamma_a\}$, $d_2 = \{(a,\delta_a) : a \in A,\ \delta_a \in \Gamma_a\}$ and $d_2 \geq d_1$, then for each $a \in A$ the following diagram commutes

$$\begin{tikzcd}
X_{(a,\delta_a)} \arrow[d, "f_{(a,\gamma_a)(a,\delta_a)}"] & X_{(b,\delta_b)} \arrow[l, "g_{(a,\delta_a)(b,\delta_b)}"'] \arrow[d, "f_{(b,\gamma_b)(b,\delta_b)}"] \\
X_{(a,\gamma_a)} & X_{(b,\gamma_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"']
\end{tikzcd} \tag{2.11}$$

This follows from the commutativity of the diagrams of the form (2.6) for d_1 and d_2, i.e., from the commutativity of the diagrams

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\gamma_a)}"] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\gamma_b)}"] \\
X_{(a,\gamma_a)} & X_{(b,\gamma_b)} \arrow[l, "g_{(a,\gamma_a)(b,\gamma_b)}"']
\end{tikzcd} \tag{2.12}$$

and

$$\begin{tikzcd}
X_a \arrow[d, "f_{(a,\delta_a)}"] & X_b \arrow[l, "p_{ab}"'] \arrow[d, "f_{(b,\delta_b)}"] \\
X_{(a,\delta_a)} & X_{(b,\delta_b)} \arrow[l, "g_{(a,\delta_a)(b,\delta_b)}"']
\end{tikzcd} \tag{2.13}$$

Step 8.

From *Step* 7 it follows that for $d_1, d_2 \in D$ with $d_2 \geq d_1$ there exists a mapping $F_{d_1d_2} : X_{d_2} \to X_{d_1}$ (see [**1**, p. 138]) such that $F_{d_1} = F_{d_1d_2}F_{d_2}$.

Proof of Step 8. Let $d_1, d_2, d_3 \in D$ and let $d_1 \leq d_2 \leq d_3$. Then $F_{d_1d_3} = F_{d_1d_2}F_{d_2d_3}$. This follows from *Step* 7 and the commutativity condition in each inverse system $\mathbf{X}(\mathbf{a}) = \{X_{(a,\gamma)}, f_{(a,\gamma)(a,\delta)}, \Gamma_a\}$ (see (2.1) of *Step* 1).

Step 9.

The collection $\{X_d, F_{de}, D\}$ *is a usual inverse system of totally regular metric continua.*

Apply *Steps* 1 – 8.

Step 10.

There is a mapping $F : X \to X_D$ *which is* $1-1$.

By *Step* 6 and *Step* 8 for each $d \in D$ there is a mapping $F_d : X \to X_d$ such that $F_{d_1} = F_{d_1d_2}F_{d_2}$ for $d_2 \geq d_1$. This means that there exists a mapping $F : X \to X_D$ [**1**, p. 138]. Let us prove that F is $1-1$. Take a pair x, y of distinct points of X. There exists an $a \in A$ such that $x_a = p_a(x)$ and $y_a = p_a(y)$ are distinct points of X_a. Now, there exists an (a, γ_a) such that $f_{(a,\gamma_a)}(x_a)$ and $f_{(a,\gamma_a)}(y_a)$ are distinct points of $X_{(a,\gamma_a)}$. From *Step* 4 it follows that there is a $d \in D$ such that $F_d(x)$ and $F_d(y)$ are distinct points of X_d. Thus, F is $1-1$.

Step 11.

The mapping F *is a homeomorphism onto* X_D. Let y be a point of X_D. Let us prove that there exists a point $x \in X$ such that $F(x) = y$. For each $d \in D$ we have a point $y_d = F_d(y)$. Now, we have the points $g_{(a,\gamma_a)}F_d(y)$ in $X_{(a,\gamma_a)}$ and the subsets $Y_a = f^{-1}_{(a,\gamma_a)}(g_{(a,\gamma_a)}F_d(y))$ of X_a. Let U be an open neighborhood Y_a. There exists an open neighborhood V of $g_{(a,\gamma_a)}F_d(y)$ such that $f^{-1}_{(a,\gamma_a)}(V) \subseteq U$. We infer that $\mathrm{Ls}\{g_{(b,\gamma_b)}(Y_b) : b \geq a\} \subseteq Y_a$ since $g_{(a,\gamma_a)}F_d(y) = \lim\{g_{(a,\gamma_a)(b,\gamma_b)}g_{(b,\gamma_b)}F_d(y) : b \geq a\}$ and the diagrams (2.6) commute. By virtue of [**6**, Lemma 2.1] it follows that there exists a non-empty closed subset C_d of $\lim \mathbf{X}$ such that $p_b(C_d) \subseteq Y_b$. The family $\{C_d : d \in D\}$ has the finite intersection property. This means that $X' = \bigcap\{C_d : d \in D\}$ is non-empty. For each $x \in X'$ we have $F_d(x) = F_d(y), d \in D$. Thus, $F(y) = x$. The proof of this *Step* is completed.

Step 12.

By virtue of Theorem 8 it follows that $X_D = \lim\{X_d, F_{de}, D\}$ is totally regular. We infer that X is totally regular since the mapping F is a homeomorphism of X onto X_D (*Step* 11). □

3. Appendix

In this Appendix we investigate the approximate subsystem of an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$. We start with the following definition.

Definition 2. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system and let B be a directed subset of A such that $\{X_b, p_{bc}, B\}$ is an approximate inverse system. We say that $\{X_b, p_{bc}, B\}$ is an approximate subsystem of $\mathbf{X} = \{X_a, p_{ab}, A\}$ if there exists a mapping $q : \lim \mathbf{X} \to \lim\{X_b, p_{bc}, B\}$ such that

$$p_b q = P_b, \quad b \in B,$$

where $p_b : \lim\{X_b, p_{bc}, B\} \to X_b$ and $P_b : \lim \mathbf{X} \to X_b, b \in B$, are natural projections.

We say that an approximate system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *irreducible* if for each $B \subset A$ with $\operatorname{card}(B) < \operatorname{card}(A)$ it follows that B is not cofinal in A.

Lemma 5. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system. There exists a cofinal subset B of A such that $\mathbf{X} = \{X_a, p_{ab}, B\}$ is irreducible.*

Proof. Consider the family $\mathcal{B}$ of all cofinal subset of B of A. The set $\{\operatorname{card}(B) : B \in \mathcal{B}\}$ has a minimal element b since each $\operatorname{card}(B)$ is some initial ordinal number. Let $B \in \mathcal{B}$ be such that $\operatorname{card}(B) = b$. It is clear that $\{X_a, p_{ab}, B\}$ is irreducible. □

In the sequel we will assume that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is irreducible.

Lemma 6. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an approximate inverse system of compact spaces such that* $\operatorname{card}(A) = \aleph_0$. *Then there exists a countable well-ordered subset B of A such that the collection $\{X_b, p_{bc}, B\}$ is an approximate inverse sequence and* $\lim \mathbf{X}$ *is homeomorphic to* $\lim\{X_b, p_{bc}, B\}$.

Proof. Let ν be any finite subset of A. There exists a $\delta(\nu) \in A$ such that $\delta \leq \delta(\nu)$ for each $\delta \in \nu$. Since A is infinite, there exists a sequence $\{\nu_n : n \in \mathbb{N}\}$ such that $\nu_1 \subseteq \ldots \nu_n \subseteq \ldots$ and $A = \bigcup\{\nu_n : n \in \mathbb{N}\}$. Recursively, we define the sets $A_1, \ldots, A_n, \ldots$ by

$$A_1 = \nu_1 \bigcup \{\delta(\nu_1)\},$$

and

$$A_{n+1} = A_n \bigcup \nu_{n+1} \bigcup \{\delta(A_n \bigcup \nu_{n+1})\}.$$

It follows that there exists a sequence

$$A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \ldots$$

of finite sets A_n such that $A = \bigcup\{A_n : n \in \mathbb{N}\}$. Using a $\delta(A_n)$ for each A_n, we obtain a sequence $B = \{b_n : n \in \mathbb{N}\}$ such that B is cofinal in A. Let us prove that $\{X_b, p_{bc}, B\}$ is an approximate inverse system, i.e., that (A2) is satisfied for $\{X_b, p_{bc}, B\}$. For each X_b and each normal cover of X_b there exists an $a' \in A$ such that (A2) is satisfied for $b \leq a' \leq c \leq d$ since (A2) is satisfied for $\mathbf{X} = \{X_a, p_{ab}, A\}$. There exists a b' such that $b' \in B, b' \geq a'$, since B is cofinal in A. It is obvious that (A2) is satisfied for each $c, d \in B$ such that $b \leq b' \leq c \leq d$. By virtue of [**10**, Theorem 1.19] it follows that $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$. □

Now we consider irreducible approximate inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ with $\operatorname{card}(A) \geq \aleph_1$.

Lemma 7. *Let A be a directed set. For each subset B of A there exists a directed set $F_\infty(B)$ such that* $\operatorname{card}(F_\infty(B)) = \operatorname{card}(B)$.

Proof. For each $B \subseteq A$ there exists a set $F_1(B) = B \bigcup \{\delta(\nu) : \nu \in B\}$, where ν is a finite subset of B and $\delta(\nu)$ is defined as in the proof of Lemma 6. Put

$$F_{n+1} = F_1(F_n(B),$$

and

$$F_\infty(B) = \bigcup \{F_n(B) : n \in \mathbb{N}\}.$$

It is clear that

$$F_1(B) \subseteq F_2(B) \subseteq \ldots \subseteq F_n(B) \subseteq \ldots$$

The set $F_\infty(B)$ is directed since each finite subset ν of $F_\infty(B)$ is contained in some $F_n(B)$ and, consequently, $\delta(\nu)$ is contained in $F_\infty(B)$.

If B is finite, then $\mathrm{card}(F_\infty(B)) = \aleph_0$. If $\mathrm{card}(B) \geq \aleph_0$, then we have

$$\mathrm{card}(\{\delta(\nu) : \nu \in B\}) \leq \mathrm{card}(B)\aleph_0.$$

We infer that $\mathrm{card}(F_1(B)) \leq \mathrm{card}(B)\aleph_0$. Similarly, $\mathrm{card}(F_n(B)) \leq \mathrm{card}(B)\aleph_0$. This means that $\mathrm{card}(F_\infty(B)) \leq \mathrm{card}(B)\aleph_0$. Thus

$$\mathrm{card}(F_\infty(B)) \leq \mathrm{card}(B)\aleph_0, \qquad \text{if} \qquad \mathrm{card}(B) < \mathrm{card}(A).$$

The proof is completed. □

Lemma 8. *Let* $\{X_a, p_{ab}\ , A\}$ *be an approximate inverse system such that* $\mathrm{cw}(X_a) < \mathrm{card}(A)$*,* $a \in A$*. For each subset* B *of* A *with* $\mathrm{card}(B) < \mathrm{card}(A)$*, there exists a directed set* $G_\infty(B) \supseteq B$ *such that the collection* $\{X_a, p_{ab}\ , G_\infty(B)\}$ *is an approximate system and* $\mathrm{card}(G_\infty(B)) = \mathrm{card}(B)$.

Proof. Let $\mathcal{B}_a$ be a base of normal coverings of X_a. Let $\mathcal{U}_a$ be a normal covering of $\mathcal{B}_a$. By virtue of (A2) there exists an $a(\mathcal{U}_a) \in A$ such that $(p_{ad}, p_{ac}p_{cd}) \leq \mathcal{U}_a$, $a \leq a(\mathcal{U}_a) \leq c \leq d$. For each subset B of A we define $G_\infty(B)$ by induction as follows:

a) Let $G_1(B) = F_\infty(B)$. From Lemma 7 it follows that $\mathrm{card}(G_1(B)) = \mathrm{card}(F_\infty(B)) = \mathrm{card}(B)$.
b) For each $n > 1$ we define $G_n(B)$ as follows:
   1) If n is odd then $G_n(B) = F_\infty(G_{n-1}(B))$,
   2) If n is even, then $G_n(B) = G_{n-1}(B) \cup \{a(U_a) : U_a \in B_a, a \in G_{n-1}(B)\}$.
      Since $\mathrm{card}(\mathcal{B}_a) < \mathrm{card}(A)$ the set $G_n(B)$ has the cardinality $< \mathrm{card}(A)$.
      Now we define $G_\infty(B) = \cup\{G_n(B) : n \in \mathbb{N}\}$. It is obvious that $\mathrm{card}(G_\infty(B)) < \mathrm{card}(A)$.

The set $G_\infty(B)$ *is directed.* Let a, b be a pair of the elements of $G_\infty(B)$. There exists a $n \in \mathbb{N}$ such that $a, b \in G_n(B)$. We may assume that n is odd. Then $a, b \in F_\infty(G_{n-1}(B))$. Thus there exists a $c \in F_\infty(G_{n-1}(B))$ such that $c \geq a, b$. It is clear that $c \in G_\infty(B)$. The proof of directedness of $G_\infty(B)$ is completed.

The collection $\{X_a, p_{ab}\ , G_\infty(B)\}$ *is an approximate system.* It suffices to prove that the condition (A2) is satisfied. Let a be any member of $G_\infty(B)$. There exists a $n \in N$ such that $a \in G_n(B)$. We have two cases.

1) If n is odd then $G_n(B) = F_\infty(G_{n-1}(B))$. This means that $a \in F_\infty(G_{n-1}(B))$. By definition of $F_\infty(G_{n-1}(B))$ we infer that $a(\mathcal{U}_a) \in F_\infty(G_{n-1}(B))$. Thus (A2) is satisfied.

2) If n is even, then $G_n(B) = G_{n-1}(B) \cup \{a(\mathcal{U}_a) : \mathcal{U}_a \in \mathrm{Cov}(X_a),\ a \in G_{n-1}(B)\}$. In this case $a \in G_{n+1}(B) \subseteq G_\infty(B)$. Arguing as in the case 1, we infer that (A2) is satisfied.

□

Theorem 15. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces. If* $\lambda \leq w(X_a) \leq \tau <$card$(A)$ *for each* $a \in A$, *then* $\lim \mathbf{X}$ *is homeomorphic to a limit of a* λ*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, T\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$, $\mathrm{card}(\Phi) = \lambda$.

Proof. The proof consists of several steps.
Step 1.

Let $\mathcal{B} = \{B_\mu : \mu \in M\}$ be a family of all subsets B_a of A with $\mathrm{card}(B_\alpha) = \lambda$. Put $A_\mu = G_\infty(B_\mu)$ (Lemma 8) and let $\Delta = \{A_\mu : \mu \in M\}$ be ordered by inclusion $\subseteq$.

Step 2.

If Φ *and* Ψ *are in* Δ *such that* $\Phi \subset \Psi$, *then there exists a mapping* $q_{\Phi\Psi} : \lim\{X_\alpha, p_{\alpha\beta}, \Psi\} \to \lim\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$.

Namely, if $x = (x_\alpha, \alpha \in \Psi) \in \lim\{X_\alpha, p_{\alpha\beta}, \Psi\}$, then by definition of the threads of $\{X_\alpha, p_{\alpha\beta}, \Psi\}$ the condition (L) is satisfied. If (L) is satisfied for $x = (x_\alpha, \alpha \in \Psi) \in \lim\{X_\alpha, p_{\alpha\beta}, \Psi\}$, then it is satisfied for $(x_\gamma, \gamma \in \Phi)$ since the required a' in (L) lies – by definition of the set Φ – in the set Φ. This means that $(x_\gamma, \gamma \in \Phi) \in \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$. Now we define $q_{\Phi\Psi}(x) = (x_\gamma, \gamma \in \Phi)$.

Step 3.

The collection $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ *is a usual inverse system.* It suffices to prove the transitivity, i.e., if $\Phi \subseteq \Psi \subseteq \Omega$, then $q_{\Phi\Psi} q_{\Psi\Omega} = q_{\Phi\Omega}$. This easily follows from the definition of $q_{\Phi\Psi}$.

Step 4.

The space $\lim X$ *is homeomorphic to* $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$, *where* $X_\Phi = \lim\{X_\gamma, p_{\alpha\beta}, \Phi\}$. We shall define a homeomorphism $H : \lim X \to \lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. Let $x = (x_a : a \in A)$ be any point of $\lim \mathbf{X}$. Each collection $\{x_a : a \in \Phi \in \Delta\}$ is a point x_Φ of X_Φ since $X_\Phi = \lim\{X_a, p_{ab}, \Phi\}$. Moreover, from the definition of $q_{\Phi\Psi}$ (*Step* 2) it follows that $q_{\Phi\Psi}(x_\Psi) = x_\Phi$, $\Psi \supseteq \Phi$. Thus, the collection $\{x_\Phi : \Phi \in \Delta\}$ is a point of $\lim\{X_\Phi, q_{\Phi\Psi}, \Delta\}$. Let $H(x) = \{x_\Phi, \Phi \in\in \Delta\}$. Thus, H is a continuous mapping of $\lim \mathbf{X}$ to $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. In order to complete the proof it suffices to prove that H is $1-1$ and onto. Let us prove that H is $1-1$. Let $x = (x_a : a \in A)$ and $y = (y_a : a \in A)$ be a pair of points of $\lim \mathbf{X}$. This means that there exists an $a \in A$ such that $y_a \neq x_a$. There exists a $\Phi \in \Delta$ such that $a \in \Phi$. Thus, the collections $\{x_a : a \in \Phi\}$ and $\{x_a : a \in \Phi\}$ are different. From this we conclude that $x_\Phi \neq y_\Phi, x_\Phi, y_\Phi \in X_\Phi = \lim\{X_a, p_{ab}, \Phi\}$. Hence H is $1-1$. Let us prove that H is onto. Let $y = (y_\Phi : \Phi \in \Delta)$ be any point of $\lim\{X_\Psi, q_{\Phi\Psi}, \Delta\}$. Each y_Φ is a collection $\{x_a : a \in \Phi\}$ and if $\Psi \supseteq \Phi$, then the collection $\{x_a : a \in \Phi\}$ is the restriction of the collection $\{x_a : a \in \Psi\}$ on Φ. Let x be the collection which is the union of all collections $\{x_a : a \in \Phi\}$, $\Phi \in \Delta$. Hence x is a collection $(x_a : a \in A)$ which is a point of $\lim \mathbf{X}$ and $H(x) = y$.

Step 5.

Inverse system $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ *is a* λ*-directed inverse system.* Let $\{\{X_\gamma, p_{\alpha\beta}, \Phi_\kappa\} : \kappa \leq \lambda\}$ be a collection of approximate subsystems $\{X_\gamma, p_{\alpha\beta}, \Phi_\kappa\}$. The set $\Phi = \cup\{\Phi_\kappa : \kappa \leq \lambda\}$ has the cardinality $\leq \lambda$ since $\mathrm{card}(\Phi_\kappa) \leq \lambda$. By virtue of *Steps* $1-4$ there exists an approximate subsystem $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$, $\mathrm{card}(\Phi) = \lambda$. This means that $\{X_\Phi, q_{\Phi\Psi}, \Delta\}$ is a λ-directed inverse system. □

If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an approximate inverse system of compact metric spaces, then $\mathrm{w}(X_a) = \aleph_0$, for each $a \in A$. It follows that $\lambda = \aleph_0$ if $\mathrm{card}(A) \geq \aleph_1$. Hence we have the following theorem.

Corollary 2. *Let* $X = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact metric spaces such that* $\mathrm{card}(A) \geq \aleph_1$*. Then* $\lim \mathbf{X}$ *is homeomorphic to the limit of a* σ*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, \Delta\}$*, where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$*,* $\mathrm{card}(\Phi) = \aleph_0$.

Lemma 9. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate system such that* $X_a, a \in A$*, are compact locally connected spaces and* p_{ab} *are monotone surjections. If* $\mathbf{Y} = \{X_b, p_{cd}, B\}$ *is an approximate subsystem of* $\mathbf{X}$*, then the mapping* $q_{AB} : \lim \mathbf{X} \to \lim \mathbf{Y}$ *(defined in Step 2 of the proof of Theorem 15) is a monotone surjection.*

Proof. Let $P_a : \lim \mathbf{X} \to X_a, a \in A$, be the natural projection. Similarly, let $p_a : \lim \mathbf{Y} \to X_a, a \in B$, be the natural projection. From the definition of q_{AB} (Step 2 of the proof of Theorem 15) it follows that $p_a q_{AB} = P_a$ for each $a \in B$. By virtue of [**10**, Corollary 4.5] and [**7**, Corollary 5.6] it follows that P_a and p_a are monotone surjections. Let us prove that q_{AB} is a surjection. Let $y = (y_a : a \in B) \in \lim \mathbf{Y}$. The sets $P_a^{-1}(y_a), a \in B$, are non-empty since P_a is surjective for each $a \in A$. From the compactness of $\lim \mathbf{X}$ it follows that a limit superior $Z =\mathrm{Ls}\{P_a^{-1}(y_a), a \in B\}$ is a non-empty subset of $\lim \mathbf{X}$. We shall prove that for each $z = (z_a : a \in A) \in Z$ we have $P_a(z) = y_a$. Suppose that $P_a(z) \neq y_a$. There exists a pair U, V of open disjoint subsets of X_a such that $y_a \in U$ and $P_a(z) \in V$. For a sufficiently large $b \in B$ the set $P_a(P_b^{-1}({}_b))$ is in U because (AS). This means that $P_a^{-1}(V) \bigcap P_b^{-1}(y_b) = \emptyset$ for a sufficiently large $b \in B$. This contradicts the assumption $z \in\mathrm{Ls}\{P_a^{-1}(y_a), a \in B\}$. Hence q_{AB} is a surjection. In order to complete the proof it suffices to prove that q_{AB} is monotone. Take a point $y \in \lim \mathbf{Y}$ and suppose that $q_{AB}^{-1}(y)$ is disconnected. There exists a pair U, V of disjoint open sets in $\lim \mathbf{X}$ such that $q_{AB}^{-1}(y) \subseteq U \bigcup V$. From the compactness of $\lim \mathbf{X}$ it follows that q_{AB} is closed. This means that there exists an open neighborhood W of y such that $q_{AB}^{-1}(y) \subseteq q_{AB}^{-1}(W) \subseteq U \bigcup V$. From the definition of the basis in $\lim \mathbf{Y}$ it follows that there exists an open set W_a in some $X_a, a \in B$, such that $y \in p_a^{-1}(W_a) \subseteq W$. Moreover, we may assume that W_a is connected since X_a is locally connected. Then $P_a^{-1}(W_a)$ is connected since P_a is monotone [**7**, Corollary 5.6]. Moreover, $q_{AB}^{-1}(y) \subseteq P_a^{-1}(W_a)$ and $P_a^{-1}(W_a) \subseteq U \bigcup V$ since $P_a = p_a q_{AB}$. This is impossible since U and V are disjoint open sets and $P_a^{-1}(W_a)$ is connected. □

Theorem 16. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an approximate inverse system of compact spaces such that* $\lambda \leq w(X_a) < \operatorname{card}(A)$ *for each* $a \in A$. *If* $cf(\operatorname{card}(A)) \neq \lambda$, *then* $X = \lim \mathbf{X}$ *is homeomorphic to a limit of a* λ*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, T\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$, $\operatorname{card}(\Phi) = \lambda$. *Moreover, if* $\operatorname{card}(A)$ *is a regular cardinal, then* $X = \lim \mathbf{X}$ *is homeomorphic to a limit of a* λ*-directed usual inverse system* $\{X_\alpha, q_{\alpha\beta}, T\}$, *where each* X_α *is a limit of an approximate inverse subsystem* $\{X_\gamma,\ p_{\alpha\beta}, \Phi\}$, $\operatorname{card}(\Phi) = \lambda$.

REFERENCES

1. Engelking R., *General Topology*, PWN, Warszawa, 1977.

2. Charalambous M.G., *Approximate inverse systems of uniform spaces and an application of inverse systems*, Comment. Math. Univ. Carolinae, *32*(33) (1991), 551–565.

3. Gordh G. R., Jr. and Mardešić S., *Characterizing local connectedness in inverse limits*, Pacific Journal of Mathematics **58** (1975), 411–417.

4. Lončar I., *A note on hereditarily locally connected continua*, Zbornik radova Fakulteta organizacije i informatike Varaždin **1** (1998), 29–40.

5. ———, *Inverse limit of continuous images of arcs,* Zbornik radova Fakulteta organizacije i informatike Varaždin **2** (23) (1997), 47–60.

6. ———, *A note on approximate limits*, Zbornik radova Fakulteta organizacije i informatike Varaždin **19** (1995), 1–21.

7. ———, *Set convergence and local connectedness of approximate limits*, Acta Math. Hungar. **77** (3) (1997), 193–213.

8. Mardešić S., *Locally connected, ordered and chainable continua*, Rad JAZU Zagreb **33**(4) (1960), 147–166.

9. Mardešić S. and Uglešić N., *Approximate inverse systems which admit meshes*, Topology and its Applications **59** (1994), 179–188.

10. Mardešić S. and Watanabe T., *Approximate resolutions of spaces and mappings*, Glasnik Matematički **24**(3) (1989), 587–637.

11. Mardešić S., *On approximate inverse systems and resolutions*, Fund. Math. **142** (1993), 241–255.

12. Nikiel J., Tuncali H. M. and Tymchatyn E. D., *Continuous images of arcs and inverse limit methods*, Mem. Amer. Math. Soc. **104** (1993).

13. Nikiel J., *The Hahn-Mazurkiewicz theorem for hereditarily locally connected continua*, Topology and Appl. **32** (1989), 307–323.

14. Nikiel J., *A general theorem on inverse systems and their limits*, Bulletin of the Polish Academy of Sciences, Mathematics, **32** (1989), 127–136.

15. Uglešić N., *Stability of gauged approximate resolutions*, Rad Hravatske akad. znan. umj. mat. **12** (1995), 69-85.

16. Whyburn G. T., *Analytic Topology*, Amer. Math. Soc. **28** (1971).

17. Wilder R. L., *Topology of manifolds*, Amer. Math. Soc. **32** (1979).

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