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CONVERGENCE OF BANACH LATTICE VALUED STOCHASTIC PROCESSES WITHOUT THE RADON-NIKODYM PROPERTY

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Abstract. We obtain almost sure convergence theorems for stochastic processes consisting of Bochner integrable functions taking values in a Banach lattice without assuming the Radon-Nikodym property. It is shown that if the limit exists in a weak sense then the almost sure convergence follows.

1. Introduction

For Banach lattice valued subpramarts the Radon-Nikodym property is equivalent to the convergence a. e. (see [**4**], [**11**] and [**6**]). If the Radon-Nikodym property is not assumed it is natural to ask how small can be the class T of functionals f such that the a.s. convergence of fX_n to fX for $f \in T$ implies the convergence of X_n to X in some stronger sense. In case of Banach valued processes it was established that T can be a total set. In particular in [**8**] it was proved that an amart (X_n) converges scalarly almost surely to a random variable X if fX_n converges to fX a. s for each f in a total subset of the dual. In [**3**], under the same assumption, the strong a.s. convergence for martingales follows. Analogous results has been obtained also for weak amarts and uniform amarts in [**1**].

In §3 we obtain similar results for subpramarts taking values in a Banach lattice (see Theorem 2).

In §4, under a suitable covering condition (Vitali condition V), we generalize the subpramarts result to directed sets.

2. Definitions and notations

Throught this note $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_n)_{n\in I\!N}$ a family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_m \subset \mathcal{F}_n$ if $m < n$. Moreover, without loss of generality, we will assume that $\mathcal{F}$ is the completion of $\sigma(\cup_n \mathcal{F}_n)$. From now on E will denote a Banach lattice with norm $\|\cdot\|$ and E^* its dual. A subset T of E^* is called a *total set* over E if $f(x) = 0$ for each $f \in T$ implies $x = 0$. For an element $x \in E$

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we denote by x^+ the least upper bound between x and 0. The Banach lattice E is said *to have the order continuous norm* or, briefly, to be *order continuous*, if for every downward directed set $\{x_\alpha\}_\alpha$ in E with $\wedge_\alpha x_\alpha = 0$, then $\lim_\alpha \|x_\alpha\| = 0$. The norm on E has the *Kadec-Klee property with respect to a set* $D \subset E^*$ if whenever $\lim_n f(x_n) = f(x)$ for every $f \in D$ and $\lim_n \|x_n\| = \|x\|$, then $\lim_n x_n = x$ strongly. If $D = E^*$ we say that the norm has the *Kadec-Klee property.* It was proved in [**2**] the following renorming theorem for Banach lattices.

Theorem 1. *A Banach lattice E is order continuous if and only if there is an equivalent lattice norm on E with the Kadec-Klee property.*

It is obvious that if E is separable, the equivalent norm has the Kadec-Klee property with respect to a countable set of functionals.
A *stopping time* is a map $\tau : \Omega \to I\!N \cup \{\infty\}$ such that, for each $n \in I\!N$, $\{\tau \leq n\} = \{\omega \in \Omega : \tau(\omega) \leq n\} \in \mathcal{F}_n$. We denote by Γ the collection of all simple stopping times (i.e. taking finitely many values and not taking the value ∞). Then Γ is a set filtering to the right.
We recall that a stochastic process $(X_n, \mathcal{F}_n)$ is called

- (i) a *submartingale* if $X_n \leq E(X_{n+1}|\mathcal{F}_n)$ a.s. for each $n \in I\!N$, or equivalently if
$$\int_A X_n \leq \int_A X_{n+1},$$
for each $A \in \mathcal{F}_n$ and for each $n \in I\!N$;
- (ii) a *subpramart* if for each $\varepsilon > 0$ there exists $\tau_0 \in \Gamma$ such that for all τ and σ in Γ, $\tau > \sigma > \tau_0$ then
$$P(\{\|(X_\sigma - E(X_\tau|\mathcal{F}_\sigma))^+\| > \varepsilon\}) \leq \varepsilon.$$

We remind that if $(X_n, \mathcal{F}_n)$ is a positive subpramart (i.e. $X_n(\omega) \geq 0$ for each $n \in I\!N$ and $\omega \in \Omega$), then for each $f \in (E^*)^+$, where $(E^*)^+$ denotes the nonnegative cone in E^*, $(fX_n, \mathcal{F}_n)$ and $(\|X_n\|, \mathcal{F}_n)$ are real valued positive subpramarts [**5**, Lemma viii.1.12].

3. CONVERGENCE THEOREMS FOR PROCESSES INDEXED BY $I\!N$

We will need the following Propositions.

Proposition 1. [**5**, p. 303] *Let E be a Banach space and let $(X_n, \mathcal{F}_n)$ be a L^1-bounded stochastic process. Then there exists a subsequence $(n_k)_k$ in $I\!N$ such that for every $k \in I\!N$*
$$X_{n_k} = Y_{n_k} + Z_{n_k}$$
where Y_{n_k} and Z_{n_k} are $\mathcal{F}_{n_k}$-measurable, $(Y_{n_k})_k$ is uniformly integrable and $\lim_k Z_{n_k} = 0$ *a.s..*

Proposition 2. [**5**, p. 298] *Let $(X_n^m, \mathcal{F}_n)_n$ be a sequence of real valued positive subpramarts for which for each $\varepsilon > 0$ there exists $\tau_0 \in \Gamma$ such that for all τ and σ in Γ, $\tau > \sigma > \tau_0$ then*
$$P(\{\sup_m(X_\sigma^m - E(X_\tau^m|\mathcal{F}_\sigma)) \leq \varepsilon\}) \geq 1 - \varepsilon.$$

Suppose, moreover, that there is a subsequence $(n_k)_k$ *such that*

$$\sup_k \int \sup_m X^m_{n_k} < \infty.$$

Then each subpramart $(X^m_n, \mathcal{F}_n)_n$ *converges a.s. to an integrable function* X^m *and we have*

$$\lim_n (\sup_m X^m_n) = \sup_m X^m a.s..$$

We are able to prove the following theorem.

Theorem 2. [**9**, Theorem 3.8] *Let* E *be an order continuous Banach lattice, which is weakly sequentially complete and let* T *be a total subset of* E^**. Let* $(X_n, \mathcal{F}_n)$ *be a positive subpramart with an* L^1*-bounded subsequence and let* X *be a strongly measurable random variable. Assume that, for each* $f \in T$*,* fX_n *converges to* fX *a.s. (the null depends on* f*). Then* X_n *converges to* X *strongly, a.s..*

Proof. Since (X_n) and X are strongly measurable it is possible to assume that E is separable. Using Proposition 1 and the fact that a subsequence of $(X_n)_n$, still denoted by $(X_n)_n$, is L^1-bounded we can also assume that

$$X_{n_k} = Y_{n_k} + Z_{n_k}$$

where Y_{n_k} and Z_{n_k} are $\mathcal{F}_{n_k}$-measurable, $(Y_{n_k})_k$ is uniformly integrable and

$$\lim_k Z_{n_k} = 0 \ a.s..$$

For each $f \in (E^*)^+$, $(fX_n)_n$ is a real valued subpramart with a L^1-bounded subsequence, then it converges a.s. to a real random variable X_f. Also fY_{n_k} converges to X_f a.s. and in L^1. In particular for each $f \in T$, $\lim_k fY_{n_k} = fX$. So for $A \in \mathcal{F}$

$$\lim_k \int_A fY_{n_k}$$

exists in $I\!R$. Hence $(\int_A Y_{n_k})_k$ is weakly Cauchy. Since the Banach lattice E is weakly sequentially complete, let for every $A \in \mathcal{F}$

$$\mu(A) = w - \lim_k \int_A Y_{n_k}.$$

Then μ is a measure of bounded variation and it is absolutely contionuous with respect to P. For each $f \in T$ we have

$$f(\mu(A)) = \lim_k \int_A fY_{n_k} = \int_A fX.$$

Let $A_n = \{\|X\| \leq n\}$, then XI_{A_n} is Bochner integrable and

$$f(\mu(A_n)) = \int_{A_n} fX = f \int_{A_n} X.$$

Since T is a total set it follows that

$$\mu(A_n) = \int_{A_n} X.$$

Moreover the uniform integrability of $(Y_{n_k})_k$ implies that

$$\int_{A_n} \|X\| = \|\mu\|(A_n) \leq \sup_k \int_\Omega Y_{n_k}, \tag{1}$$

and since X is strongly measurable, $P(\cup_n(\|X\| \leq n)) = 1$. Letting $n \to \infty$ in (1), we get that X is Bochner integrable and for each $A \in \mathcal{F}$

$$\mu(A) = \int_A X.$$

It follows that

$$\int_A fX = f(\mu(A)) = \lim_k \int_A fY_{n_k} = \int_A X_f,$$

for each $f \in (E^*)^+$ and $A \in \bigcup \mathcal{F}$. Hence $fX = X_f$ a.s. and for each $f \in (E^*)^+$, fX_n converges to fX a.s.. Let $|\!|\!| \cdot |\!|\!|$ denote the Kadec-Klee norm equivalent to $\| \cdot \|$, as in Theorem 1, and let $D \in (E^*)^+$ be a countable norming subset. Applying Proposition 2 to the sequence $\{(fX_n, \mathcal{F}_n), n \in I\!N, f \in D\}$ it follows that $\lim_n |\!|\!| X_n |\!|\!| = |\!|\!| X |\!|\!|$, a.s.. Now invoking again Theorem 1 we get the strong convergence of X_n to X and the assertion follows. □

The following corollary holds.

Corollary 1. *Let E be a Banach lattice not containing c_0 as an isomorphic copy and let T be a total subset of E^*. Let $(X_n, \mathcal{F}_n)$ be a positive subpramart with a L^1-bounded subsequence and let X be a strongly measurable random variable. Assume that, for each $f \in T$, fX_n converges to fX a.s. (the null set depends on f). Then X_n converges to X strongly a.s..*

Proof. If E does not contain c_0, E is an order continuous Banach lattice which is weakly sequentially complete [**7**, p. 34] and the assertion follows from Theorem 2. □

Since a submartingale is a subpramart we get

Corollary 2. [**3**, Proposition 11] *Let E be a Banach lattice not containing c_0 as an isomorphic copy and let T be a total subset of E^*. Let $(X_n, \mathcal{F}_n)$ be a L^1-bounded positive submartingale and let X be a strongly measurable random variable. Assume that, for each $f \in T$, fX_n converges to fX a.s. (the null set depends on f). Then X_n converges to X strongly a.s..*

4. A CONVERGENCE THEOREM FOR SUBPRAMARTS INDEXED BY A DIRECTED SET

In this section we will consider stochastic processes indexed by a directed set. Let J be a directed set filtering to the right. Throughout this section we assume that there is an increasing cofinal sequence (t_n) in J. Let $(\mathcal{F}_t)$ be a filtration, that is an increasing family of sub-σ-algebras of $\mathcal{F}$. A filtration $(\mathcal{F}_t)$ is said to satisfy the *Vitali condition V* if for every adapted family of sets (A_t) and for every $\varepsilon > 0$ there exists a simple stopping time $\tau \in \Gamma$ such that $P(\limsup_J A_t \setminus A_\tau) < \varepsilon$. Even in the real-valued case the Vitali condition on the filtration is necessary for the convergence of classes of random variables. Under the condition V, the analogue of Theorem 2 holds for subpramarts indexed by directed sets.

Theorem 3. *Let the filtration satisfy the condition V and let E be a separable order continuous Banach lattice, which is weakly sequentially complete. Let* $(X_t, \mathcal{F}_t)$ *be a* L^1*-bounded positive subpramart and let* X *be a strongly measurable random variable. Let* T *be a total subset of* E^* *and assume that, for each* $f \in T$, fX_t *converges to* fX *a.s.. Then* X_t *converges to* X *strongly a.s..*

Proof. Let (t_n) be an increasing cofinal sequence in J. Set $X_{t_n} = Y_n$ and $\mathcal{F}_{t_n} = \mathcal{G}_n$. We first show that $(Y_n, \mathcal{G}_n)$ is a subpramart sequence. Since (X_t) is a subpramart, for every $\varepsilon > 0$ there exists $\tau_o \in \Gamma$ such that if $\tau > \sigma > \tau_o$ then

$$P(\{||(X_\sigma - E(X_\tau|\mathcal{F}_\sigma))^+|| > \varepsilon\}) \leq \varepsilon.$$

Now if σ is a stopping time for $\mathcal{G}$ then t_σ is a stopping time for $\mathcal{F}_t$. Thus choose σ_o such that $t_{\sigma_o} \geq \tau_o$. Now for each $\tau > \sigma > \sigma_o$ it follows

$$P(\{\|(Y_\sigma - E(Y_\tau|\mathcal{G}_\sigma))^+\| > \varepsilon\}) = P(\{\|(X_{t_\sigma} - E(X_{t_\tau}|\mathcal{F}_{t_\sigma}))^+\| > \varepsilon\}) \leq \varepsilon.$$

Then Y_n is a subpramart sequence. For each $f \in T$, fY_n converges to fX a.s.. Therefore by Theorem 2, Y_n converges to X a.s. and also scalarly. As E is a separable Banach lattice there exists a countable norming subset D of $(E^*)^+$ (i.e. $\|x\| = \sup\{|x^*(x)| : x^* \in D \cap \mathcal{B}(X^*)\}$). Now, for each $f \in D$, fX_t is a L^1-bounded real valued subpramart and since the filtration satisfies V, by [**10**] Theorem 4.3, fX_t converges to X_f a.s.. Since fX_{t_n} converges to fX, it follows that $fX = X_f$. As in Theorem 1, we denote by $|||\cdot|||$ the Kadec-Klee norm equivalent to $\|\cdot\|$. Applying [**6**] Lemma 2.3 to the sequence $\{(fX_t, \mathcal{F}_t), t \in T, f \in D\}$ it follows that $\lim_t |||X_t||| = |||X|||$, a.s.. Now invoking again Theorem 1 we get the strong convergence of X_t to X and the assertion follows. ☐

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