Acta Math. Univ. Comenianae
Vol. LXXIV, 1(2005), pp. 59–70

SOME CHANGE OF VARIABLE FORMULAS IN INTEGRAL REPRESENTATION THEORY

L. MEZIANI

Abstract. Let X, Y be Banach spaces and let us denote by $C(S,X)$ the space of all X-valued continuous functions on the compact Hausdorff space S, equipped with the uniform norm. We shall write $C(S,X) = C(S)$ if $X = \mathbb{R}$ or $\mathbb{C}$. Now, consider a bounded linear operator $T : C(S,X) \to Y$ and assume that, due to the effect of a change of variable performed by a bounded operator $V : C(S,X) \to C(S)$, the operator T takes the product form $T = \theta \cdot V$, with $\theta : C(S) \to Y$ linear and bounded. In this paper, we prove some integral formulas giving the representing measure of the operator T, which appeared as an essential object in integral representation theory. This is made by means of the representing measure of the operator θ which is generally easier. Essentially the estimations are of the Radon-Nikodym type and precise formulas are stated for weakly compact and nuclear operators.

1. Introduction

Let S be a compact Hausdorff space and $\mathcal{B}_S$ the σ-field of the Borel sets of S. In all what follows, X and Y will be fixed Banach spaces and we consider the Banach space $C\left(S,X\right)$ of all X-valued continuous functions on S, with the uniform norm; we write $C\left(S,X\right) = C\left(S\right)$ when $X = \mathbb{R}$ or $\mathbb{C}$. In this work, we will be concerned with the integral analysis of bounded operators $T : C\left(S,X\right) \to Y$, taking the form:

$$T = \theta \cdot V \tag{1.1}$$

due to the effect of a change of variable performed by a bounded operator $V : C\left(S,X\right) \to C\left(S\right)$; θ being a bounded operator on $C\left(S\right)$ with values into Y. When the operators T and V are given, we will show how to get the operator $\theta : C\left(S\right) \to Y$, satisfying the product form (1.1). Then we determine the structure of the additive operator valued measure $G : \mathcal{B}_S \to \mathcal{L}\left(X, Y^{**}\right)$ attached to the operator T via the integral representation:

$$f \in C\left(S,X\right), \quad Tf = \int_S f\, dG. \tag{1.2}$$

Received May 9, 2003.
2000 *Mathematics Subject Classification.* Primary 28C05; Secondary 46G10.
Key words and phrases. Change of variable in bounded operators, vector measures, Weakly compact & Nuclear Operators.

According to the Theorem of Dinculeanu [**2**, §19], $\mathcal{L}(X, Y^{**})$ is the Banach space of all bounded operators from X into the second conjugate space Y^{**} of Y. In doing the computations, we shall make use of the integral form

$$g \in C(S), \quad \theta g = \int_S g \, d\mu \tag{1.3}$$

of the operator θ, given by Bartle-Dunford-Schwartz, [**3**, VI-7]; in this context μ is a vector set function on $\mathcal{B}_S$ with values in Y^{**} (resp. a vector measure with values in Y, if θ is weakly compact). As we will see, the relations between G and μ are, in some sense, of the Radon-Nikodym type. We shall compute explicitly the derivatives arising from these relations. The most precise results about the vector measure G are obtained for weakly compact and nuclear operators T.

The paper is organized as follows. In Section 2 we will make precise the change of variable $V : C(S, X) \to C(S)$ leading to the product form (1.1). Also we recall some facts from integral representation theory giving (1.2) and (1.3). In Section 3 we give a general estimation formula for the measure G by means of the set function μ. We examine in section 4 the case of weakly compact operators T, which allows an improvement of the estimation made in Section 3. We consider nuclear operators T in Section 5. If T takes the form (1.1) by a change of variable $V : C(S, X) \to C(S)$, we show how we can recover the nuclear property for the component θ. Then we prove that the measure G is a Bochner integral with respect to a bounded scalar measure. A simple example is given in Section 6, where all computations of Sections 2 – 5 are performed explicitly. Finally, Section 7 is intended to a remark about another estimation of G made elsewhere [**5**, §5].

2. The change of variable $V : C(S, X) \to C(S)$.

In all what follows, we will always assume that $C(S, X)$ is mapped onto $C(S)$ by the operator

$$C(S, X) \text{ is mapped onto } C(S) \text{ by the operator } V. \tag{2.1}$$

We need this hypothesis in constructing the component $\theta : C(S) \to Y$ as a bounded operator giving the product form $T = \theta \cdot V$. The operator V in (2.1) may be considered as performing a change of variable from the space $C(S, X)$ to the space $C(S)$.

One usefull fact about V is:

Proposition 1. *There exists a constant $K > 0$, such that for every $h \in C(S)$, there is a solution $f \in C(S, X)$ of $h = Vf$, satisfying $\|f\| \leq K \|h\|$.*

Proof. Since V is onto, then by the open mapping Theorem, the open unit ball B of $C(S, X)$ maps onto a set VB which contains some relative open ball $\{u \in VB : \|u\| < \alpha\}$, with $\alpha > 0$. Thus, for $0 \neq h \in VC(S, X) = C(S)$, the vector $\frac{\alpha}{2} \frac{h}{\|h\|}$ is the image under V of a vector g, with $\|g\| < 1$. Hence if we put $f = \frac{2\|h\| \cdot g}{\alpha}$, we have $Vf = h$ and $\|f\| \leq \frac{2}{\alpha} \|h\|$, which proves the proposition with $K = \frac{2}{\alpha}$. $\square$

The effect of a change of variable $V : C(S,X) \to C(S)$ is given by:

Theorem 1. *A bounded operator $T : C(S,X) \to Y$ factors as $T = \theta \cdot V$, where $\theta : C(S) \to Y$ is a bounded operator, if and only if the following condition is satisfied:*

$$\operatorname{Ker} V \subset \operatorname{Ker} T \tag{2.2}$$

Proof. The necessity of the condition is clear. To see that it is sufficient, we first proceed to the construction of θ. Let $h \in C(S)$, then citing (2.1) gives an $f \in C(S,X)$ such that $h = Vf$; let us put $\theta h = Tf$. Then θ is a well defined mapping; for, if $Vf_1 = Vf_2 = h$, where $f_1,\ f_2 \in C(S,X)$, then we have $f_1 - f_2 \in \operatorname{Ker} V$ which implies $f_1 - f_2 \in \operatorname{Ker} T$ by (2.2); so $Tf_1 = Tf_2$. It is clear that θ is linear and that we have $Tf = \theta \cdot Vf$, for all $f \in C(S,X)$. We must show that θ is bounded. By Proposition 1 there exists $K > 0$ such that for every $h \in C(S)$ we can choose a solution f of $h = Vf$ so that $\|f\| \leq K\|h\|$. Therefore we have $\|\theta h\| = \|Tf\| \leq \|T\|\,\|f\| \leq \|T\|\,K\,\|h\|$, which gives the boundedness of θ. □

Remark 1. It is noteworthy that we may relax the assumption (2.1) if we require from V to be of closed range. In this case we still have the validity of both Proposition 1 and Theorem 1, but with θ defined and bounded on the range of V.

Before stating the Theorems we need in the context of vector integration, let us put some preliminaries and facts for later use.

Definition 1. Let $G : \mathcal{B}_S \to \mathcal{L}(X, Y^{**})$ be a finitely additive vector measure on $\mathcal{B}_S$ with values in the Banach space $\mathcal{L}(X, Y^{**})$. For each $y^* \in Y^*$, let us define the set function $G_{y^*} : \mathcal{B}_S \to X^*$ by:

$$E \in \mathcal{B}_S, x \in X : \ G_{y^*}(E)(x) = y^* G(E)(x) \tag{2.3}$$

that is, the functional $G(E)(x)$ of Y^{**} applied to the vector $y^* \in Y^*$. Then it is a simple fact that G_{y^*} is for each $y^* \in Y^*$ a finitely additive X^*-valued measure on B_S. The family of measures $\{G_{y^*},\ y^* \in Y^*\}$ induces in turn a family of scalar finitely additive measures $\left\{M^x_{y^*} : x \in X\ ,\ y^* \in Y^*\right\}$ defined by:

$$E \in \mathcal{B}_S,\ \ x \in X,\ \ y^* \in Y^* : M^x_{y^*}(E) = \ G_{y^*}(E)(x)\,. \tag{2.4}$$

Let us recall also the notions of variation and semivariation of a measure:

Definition 2. Let Z be a Banach space and $\mu : \mathcal{B}_S \to Z$ a vector measure (note that μ may be scalar). Then
(a) The variation of μ is the set function $v(\mu,\cdot)$ of $\mathcal{B}_S$ in $[0,+\infty]$ defined by:

$$E \in \mathcal{B}_S : v(\mu, E) = \sup_{\pi} \sum_{A \in \pi} \|\mu(A)\| \tag{2.5}$$

the sup is over all finite partitions π of E by sets in $\mathcal{B}_S$. Call $v(\mu, S) = v(\mu)$, the variation of μ.

(b) The semivariation of μ is the set function $\|\mu\| : \mathcal{B}_S \to [0, +\infty]$ defined by the formula:

$$E \in \mathcal{B}_S : \ \|\mu\|(E) = \sup\{v(z^*\mu, E) : z^* \in Z^*, \ \|z^*\| \leq 1\} \tag{2.6}$$

note that $z^*\mu$ is scalar for each $z^* \in Z^*$.

Definition 3. We say that a vector measure $\mu : \mathcal{B}_S \to Z$ is regular if for each $E \in \mathcal{B}_S$ and $\varepsilon > 0$ there exist an open set O and a compact set K such that, $K \subset E \subset O$ and $\|\mu\|(O \setminus K) < \varepsilon$. If the measure μ is scalar this inequality may be replaced by $v(\mu, O \setminus K) < \varepsilon$ [**1**, Chapter 1] for all relations between the set functions $v(\mu, \cdot)$ and $\|\mu\|$).

With the ingredients above, we have:

Proposition 2. *Suppose that the measure G_{y^*} is bounded and regular for some $y^* \in Y^*$ then we have*
(i) *G_{y^*} is countably additive.*
(ii) *The scalar measures $M_{y^*}^x$ are countably additive and regular for each $x \in X$.*

Proof. Let $E \in \mathcal{B}_S$ and $\varepsilon > 0$, then there exist an open set O and a compact set K such that, $K \subset E \subset O$ and $\|G_{y^*}\|(O \setminus K) < \varepsilon$. Since G_{y^*} is X^*-valued, we have

$$\|G_{y^*}\|(O \setminus K) = \sup\{v(x^{**}G_{y^*}, O \setminus K) : \ x^{**} \in X^{**}, \ \|x^{**}\| \leq 1\} < \varepsilon$$

by (2.6). This implies that the family of scalar set functions

$$\{x^{**}G_{y^*} : x^{**} \in X^{**}, \ \|x^{**}\| \leq 1\}$$

is uniformly regular; since they are additive, we deduce, by the Theorem III.5.13 in [**3**], that $x^{**}G_{y^*}$ is countably additive for each $x^{**} \in X^{**}$, $\|x^{**}\| \leq 1$ and then also for all $x^{**} \in X^{**}$. Consequently G_{y^*} is countably additive by the Orlicz-Pettis Theorem. To see part (ii), let $\gamma : X \to X^{**}$ denote the canonical isomorphism of X into X^{**}, and let us observe that $M_{y^*}^x = \gamma(x)\, G_{y^*}$, by formula (2.4); therefore we deduce that the scalar measure $M_{y^*}^x$ is countably additive and regular for each $x \in X$. ☐

Now we turn to the integral representation Theorems we shall need in the sequel.

Theorem 2. *Let $T : C(S, X) \to Y$ be a linear bounded operator. Then there exists a unique additive operator valued measure $G : \mathcal{B}_S \to \mathcal{L}(X, Y^{**})$ such that:*

$$Tf = \int_S f(s)\ dG \tag{2.7}$$

(we call G the representing measure of the operator T).

Moreover, for each $y^* \in Y^*$, G_{y^*} is a regular countably additive bounded X^*-valued measure and we have

$$T^*y^* = G_{y^*} \tag{2.8}$$

where T^* is the adjoint of T and where the identification, between the dual space $C\,(S,X)^*$ and the Banach space $rcab\,(\mathcal{B}_S, X^*)$ of X^*-valued measures on $\mathcal{B}_S$ is used.

Because of the equation (2.8) we shall call the family of measures $\{G_{y^*}, y^* \in Y^*\}$, the adjoint family of G or of T. For the proof see reference [**2**, § 19].

Theorem 3. *Let $\theta : C(S) \to Y$ be a bounded linear operator. Then there exists a unique set function $\mu : B_S \to Y^{**}$ such that*
(a) *$\mu(\cdot)y^*$ is a regular countably additive scalar measure on $\mathcal{B}_S$ for all $y^* \in Y^*$ (in symbols $\mu(\cdot)y^* \in rca(S)$).*
(b) $y^*\theta f = \int_S f\,(s)\; d\mu\,(s)\,y^*$ *for all $y^* \in Y^*$ and $f \in C(S)$.*
We call μ the representing measure of θ.
Moreover, if the operator θ is weakly compact, then μ is a true countably additive measure with values in Y such that
(a′) *$y^*\mu$ is a regular scalar measure for all $y^* \in Y^*$.*
(b′) $\theta f = \int_S f\,(s)\;\; d\mu\,(s)$ *for all $f \in C(S)$.*
On the other hand, if $\theta^* : Y^* \to C^*(S)$ is the adjoint of θ then we have $\theta^* y^* = y^*\mu$ for all $y^* \in Y^*$.

For the proof see [**3**, VI.7.2 and VI.7.3].

3. General estimation of the representing measures

Let $T : C\,(S,X) \to Y$ and $V : C\,(S,X) \to C\,(S)$ be bounded operators and suppose that T factors as $T = \theta \cdot V$, where $\theta : C(S) \to Y$ is bounded. In this section, we will prove a general formula between the representing measures G and μ of the operators T and θ. We will see that the resulting relations between G and μ are of the Radon-Nikodym type and we will give the expression of the derivatives by means of the operator V. To make the estimation tractable we shall impose on the operator V the following condition

$$\forall g \in C(S), \forall h \in C\,(S,X) :\; V\,(g \cdot h) = g \cdot V\,(h)\,. \tag{3.1}$$

In the computations below, we need condition (3.1) to be satisfied only for the constant functions $h \in C\,(S,X)$. Here is an example of a non trivial bounded $V : C\,(S,X) \to C\,(S)$ satisfying (3.1):

Example 1. Let $K : S \times S \to R$ be a continuous function and let μ be a measure with bounded variation on B_S. Let us consider the operator $\phi : C(S) \to C(S)$, defined by: $\phi\,(g)\,(s) = \int_S K\,(s,t)\,g\,(t)\;d\mu\,(t)$. The fact that K is continuous and μ of bounded variation makes it easy to prove that $\phi\,(g)$ is in $C(S)$. Now take $X = C(S)$ and define $V : C(S,X) \to C(S)$, by

$$h \in C(S,X),\;\; V\,(h)\,(r) = \phi\,(h_r)\,(r)\,, \quad \text{for } r \in S.$$

Let us note that the value h_r, of the function h at the point r, is in $C(S)$ because $h \in C(S,X)$, and $X = C(S)$. Note also, from the definition of ϕ, that we have $V\,(h)\,(r) = \int_S K\,(r,t)\,h_r\,(t)\;d\mu\,(t)$. It is not difficult to show that the function

$r \to V(h)(r)$ is continuous and that $V : C(S,X) \to C(S)$ is a linear bounded operator with $\|V\| \le M_K \cdot v(\mu)$, where $M_K = \sup\{|K(s,t)|, (s,t) \in S \times S\}$. We prove that V satisfies (3.1).

Let $g \in C(S)$, $h \in C(S,X)$, then we have

$$\begin{aligned} V(g \cdot h)(r) &= \int_S K(r,t) g(r)\, h_r(t)\, d\mu(t) \\ &= g(r) \int_S K(r,t) h_r(t)\, d\mu(t) \\ &= g(r)\, V(h)(r) . \end{aligned}$$

(For an other example of operator satisfying (3.1), see Section 6 below.)

We now state and prove the general estimation Theorem. Recall the measures G_{y^*}, $M^x_{y^*}$ in (2.3) and (2.4), and $\mu(\cdot)y^*$ in Theorem 3(a).

Theorem 4. *Under* (2.1), (2.2), (3.1), *the operator* T *factors as* $T = \theta \cdot V$ *and we have*

$$G_{y^*}(E)(x) = \int_E V(c_x)(t) \;\; d\mu(t)\, y^* . \tag{3.2}$$

for all $E \in \mathcal{B}_S$, $y^* \in Y^*$ *and* $x \in X$, *where* $c_x \in C(S,X)$ *is the constant function* $S \to X$ *given by* $c_x(t) \equiv x$, x *being fixed in* X.
In other words the measure $M^x_{y^*}$ *is absolutely continuous with respect to* $\mu(\cdot)y^*$, *with Radon-Nikodym derivatives given by* $\frac{d\, M^x_{y^*}}{d\, \mu(\cdot) y^*} = V(c_x)$, *so we may write* (3.2) *as* $d\, M^x_{y^*} = V(c_x) \cdot \; d\, \mu(\cdot) y^*$.

Proof. First let us apply the integral (2.7) to the function $f \in C(S,X)$ of the form $f(t) = g(t) \cdot c_x(t)$, with $g \in C(S)$ and x fixed in X. We obtain $Tg \cdot c_x = \int_S g \cdot c_x \; dG$, and for $y^* \in Y^*$

$$y^* T g \cdot c_x = \int_S g \cdot c_x \; dG_{y^*} = \int_S g \; dM^x_{y^*},$$

where the second equality results from (2.8) and the third one from standard integration tools, starting with (2.4). Recall that G_{y^*} is X^*-valued and then, for $E \in \mathcal{B}_S$, and $x \in X$, we have

$$\int_E x \; dG_{y^*} = G_{y^*}(E)(x) = M^x_{y^*}(E) .$$

On the other hand, since $T = \theta \cdot V$, we have $Tg \cdot c_x = \theta \cdot V(g \cdot c_x) = \theta \cdot (g \cdot V(c_x))$, where we are appealing to (3.1) for the identity $V(g \cdot c_x) = g \cdot V(c_x)$. By the first part of Theorem 3, it is clear that

$$y^* \theta \cdot (g \cdot V(c_x)) = \int_S g \cdot V(c_x)(t) \;\; d\,\mu(t)\, y^*, \quad \text{for each } y^* \in Y^* .$$

Now, comparing this integral to the one computed above for $y^* T g \cdot c_x$, we get

$$\int_S g \cdot V(c_x) \;\; d\,\mu(\cdot) y^* = \int_S g \; dM^x_{y^*}, \quad \text{for all } g \in C(S) .$$

Since the scalar measures $\mu(\cdot)y^*$, $M_{y^*}^x$ are regular (the first one by Theorem 3 and the second by Proposition 2), it results from the classical Riesz representation Theorem that $M_{y^*}^x(E) = \int_E V(c_x)(t)\; d\mu(t)\, y^*$, which is exactly (3.2). □

In the sequel, we want to improve the estimation formula (3.2), by suppressing its dependance with respect to the functional y^*. We will reach an improvement with the help of the second part of Theorem 3, since the formulas given there are more tractable in vector integration calculus. To achieve this program we must impose a weak compactness assumption on the operator T.

4. Weakly compact Operators

Let $T : E \to F$ be a bounded operator of the Banach space E into the Banach space F and let B be the closed unit ball of E. The operator T is said to be weakly compact if the weak closure of TB is compact in the weak topology of F. If $T : C(S,X) \to Y$ factors as $T = \theta \cdot V$, (see section 2), then we have the following interesting property:

Proposition 3. *The operator T is weakly compact iff the operator θ is weakly compact.*

Proof. Assume θ weakly compact. Since B is bounded VB is bounded and then $TB = \theta \cdot VB$ has a weakly compact closure, so T is weakly compact. More important for us is the converse.
Assume T weakly compact. To prove that the same is true for θ, it is sufficient, by the Eberlein-Šmulian Theorem [**3**, Theorem V 6.1.], to show that θA is weakly sequentially compact for every bounded set $A \subset C(S)$. Let h_n be a sequence in A, and let $f_n \in C(S,X)$ be such that $h_n = Vf_n$; then, citing Proposition 1, for some $K > 0$ we may choose f_n so that $\|f_n\| \leq K\|h_n\|$ for all n. This shows that f_n is uniformly bounded. Since T is weakly compact, the Eberlein-Šmulian Theorem just cited, allows the extraction of a subsequence f_{n_i} of f_n such that Tf_{n_i} will be weakly convergent. But $Tf_{n_i} = \theta h_{n_i}$, thus the sequence θh_n contains a convergent subsequence, proving that θA is weakly sequentially compact. □

Remark 2. *It is proved in* [**3**, VI.4.5]*, that for every weakly compact θ and every bounded V, the product $\theta \cdot V$ is weakly compact. In the preceding Proposition we were able to get the converse, that is, θ is weakly compact provided that $\theta \cdot V$ is weakly compact and V is onto.*

While Theorem 4 gives the structure of the adjoint family $\{G_{y^*},\ y^* \in Y^* \cdot\}$, via formula (3.2), we now state an improvement of this formula by imposing on the operator T a condition of weak compactness. Let $\gamma : Y \to Y^{**}$ denote the canonical isomorphism of Y into its bidual Y^{**}.

Theorem 5. *Let $T : C(S,X) \to Y$ be a bounded operator and assume that T is weakly compact and factors as $T = \theta \cdot V$. Then there exists a unique countably* *additive vector measure* μ *on* $\mathcal{B}_S$ *with values in* Y*, such that the representing measure* G *of* T *has the following consolidated form:*

$$G(E)(x) = \int_E V(c_x)(t)\ d\gamma\mu(t) \tag{4.1}$$

for all $E \in \mathcal{B}_S$ *and all* $x \in X$*.*

Proof. From Proposition 4, the operator θ is weakly compact since T is weakly compact. Therefore, by the second part of Theorem 3, μ is a true vector measure on $\mathcal{B}_S$ with values in Y. With this in mind, we proceed as in the proof of Theorem 4 to get

$$y^*Tg \cdot c_x = y^*\theta \cdot (g \cdot V(c_x)) = \int_S g \cdot V(c_x)(t)\ d\,y^*\mu(t),$$

where the second equality is from (b$'$) of Theorem 3. But $y^*Tg{\cdot}c_x = \displaystyle\int_S g{\cdot}c_x\ dG_{y^*}$, thus we conclude that

$$G(E)(x)(y^*) = \int_E V(c_x)(t)\ d\,y^*\mu(t), \tag{$*$}$$

since g is arbitrary in $C(S)$ (see the proof of Theorem 4). Let us put α for the right hand side of this last formula; we have by Theorem IV.10.8(f), in [**3**], $\alpha = y^* \displaystyle\int_S V(c_x)(t)\ d\,\mu(t)$, and since the integral $\displaystyle\int_S V(c_x)(t)\ d\,\mu(t)$ is in Y, we get $\alpha = \gamma(\displaystyle\int_S V(c_x)(t)\ d\,\mu(t))(y^*)$; now let us replace the integral in $(*)$ by this value, we obtain $G(E)(x)(y^*) = \gamma(\displaystyle\int_S V(c_x)(t)\ d\,\mu(t))(y^*)$, for each $y^* \in Y^*$, and consequently $G(E)(x) = \gamma(\displaystyle\int_S V(c_x)(t)\ d\,\mu(t))$. But the last transformed integral is exactly $\displaystyle\int_S V(c_x)(t)\ d\gamma\mu(t)$, by the Theorem just cited. This achieves the proof of (4.1). □

There is an interesting class of operators for which formula (4.1) has a stronger meaning, because the integrals will be of Bochner type. It is the class of nuclear operators which we consider in the following section.

5. NUCLEAR OPERATORS

Definition 4. Let E, F be Banach spaces. We say that a bounded linear operator $T : E \to F$, from E into F, is nuclear if there exist sequences (x_n^*) in E^* and (y_n) in F such that $\sum_n \|x_n^*\|\,\|y_n\| < \infty$ and such that $T(x) = \sum_n x_n^*(x)\,y_n$ for all $x \in X$.

The following Theorem gives an integral representation for a nuclear operator $\theta : C(S) \to Y$:

Theorem 6. (i) *Every nuclear operator is compact and thus weakly compact.* (ii) *A bounded linear operator* $\theta : C(S) \to Y$ *is nuclear if and only if its representing measure* μ *is of bounded variation and has a Bochner integrable derivative* g *with respect to its variation* $v(\mu, \cdot)$, *that is* $\mu(E) = \displaystyle\int_E g(s)\; v(\mu, ds)$. *(Recall the variation of a measure in* (2.5).*)*

For the proof see reference [**1**, p. 173].

We now turn to nuclear operators $T : C(S,X) \to Y$ which have the product form $T = \theta \cdot V$. We first give the link with the nuclear property of the component θ.

Theorem 7. (a) *Assume that* θ *is nuclear. Then there are sequences* $(\mu_n) \subset C(S,X)^*$, $(y_n) \subset Y$ *such that* $\sum\limits_n \|\mu_n\| \, \|y_n\| < \infty$ *and* $Tf = \sum\limits_n \mu_n(f)\, y_n$ *for all* $f \in C(S,X)$, *so* T *is nuclear. Moreover we have*

$$Vf = 0 \Longrightarrow \mu_n(f) = 0, \text{ for all } n.$$

(b) *Assume that the operator* $T = \theta \cdot V$ *is nuclear and write* T *as:* $Tf = \sum\limits_n \mu_n(f)\, y_n$, *where* $f \in C(S,X)$, $(\mu_n) \subset C(S,X)^*$, $(y_n) \subset Y$ *and* $\sum\limits_n \|\mu_n\| \, \|y_n\| < \infty$.

If the condition

$$(\mathcal{N}) \qquad Tf = 0 \Longrightarrow \mu_n(f) = 0, \text{ for all } n.$$

is satisfied then the operator θ *is nuclear.*

Proof. (a) Assume that θ is nuclear and let us write θ as $\theta h = \sum\limits_n \theta_n(h) y_n$, where $(\theta_n) \subset C(S)^*$, $(y_n) \subset Y$, $h \in C(S)$ and $\sum\limits_n \|\theta_n\| \, \|y_n\| < \infty$. If $f \in C(S,X)$ then $Vf = h \in C(S)$ and $Tf = \theta h = \sum\limits_n \theta_n Vf \; y_n = \sum\limits_n \mu_n(f) y_n$, where we define the bounded linear operator μ_n on $C(S,X)$ by $\mu_n(f) = \theta_n Vf$. Since we have $\sum\limits_n \|\theta_n\| \, \|y_n\| < \infty$, it follows that $\sum\limits_n \|\mu_n\| \, \|y_n\| < \infty$ and T is nuclear. On the other hand it is clear that: $Vf = 0 \Longrightarrow \mu_n(f) = 0$, for all n.

(b) The condition imposed to the μ_n and T reads $\operatorname{Ker} T \subset \bigcap\limits_n \operatorname{Ker} \mu_n$. Then $\operatorname{Ker} V \subset \bigcap\limits_n \operatorname{Ker} \mu_n$ and by Theorem 1, Section 2, with $Y = \mathbb{R}$, for each n there exists a bounded operator $\theta_n : C(S) \to \mathbb{R}$ such that $\mu_n(f) = \theta_n \cdot Vf$ for all $f \in C(S,X)$. Let $h \in C(S)$ and $f \in C(S,X)$ be such that $Vf = h$; then $Tf = \theta h = \sum\limits_n \mu_n(f) y_n$, but $\mu_n(f) = \theta_n \cdot Vf = \theta_n h$. Thus $\theta h = \sum\limits_n \theta_n(h) y_n$. Since $\sum\limits_n \|\mu_n\| \, \|y_n\| < \infty$ it follows that $\sum\limits_n \|\theta_n\| \, \|y_n\| < \infty$ and θ is nuclear. □

Theorem 8. *Let* $T : C(S,X) \to Y$ *be a nuclear operator such that* $T = \theta \cdot V$. *Assume that for all* $f \in C(S,X)$, $Tf = \sum\limits_n \mu_n(f) y_n$, *where* $(\mu_n) \subset C(S,X)^*$, $(y_n) \subset Y$ *and* $\sum\limits_n \|\mu_n\| \, \|y_n\| < \infty$. *If condition* $(\mathcal{N})$ *is satisfied for the* μ_n *and*

T then the representing measure G of T is a Bochner integral with respect to a bounded scalar measure.

Proof. By Theorem 6 (i) T is weakly compact and so we have by (4.1):

$$G(E)(x) = \int_E V(c_x)(t) d\gamma\mu(t), \tag{$**$}$$

where μ is the representing measure of θ. From the condition imposed on T, we deduce that θ is nuclear (Theorem 7) (b) and then $\mu(E) = \int_E g(s)\ v(\mu, ds)$, for a $v(\mu, ds)$-Bochner integrable function $g : S \to Y$, (Theorem 6 (ii)). Applying the bounded operator γ to the preceding equality gives $\gamma\mu(E) = \int_E \gamma g(s)\ v(\mu, ds)$. On the other hand, by a simple argument of integration theory, we have

$$\int_E u(s)\ d\gamma\mu(s) = \int_E u(s)\gamma g(s)\ v(\mu, ds),$$

for every bounded scalar measurable function u on S. Therefore, taking $u(s) = V(c_x)(s)$ in formula $(**)$, we get

$$G(E)(x) = \int_E V(c_x)(s)\ \gamma g(s)\ v(\mu, ds) \tag{5.1}$$

which is the conclusion of the Theorem. □

6. EXAMPLES

We give now an example of a bounded operator $V : C(S, X) \to C(S)$, that meets condition (2.1) and then we factorize under condition (2.2) a bounded operator $T : C(S, X) \to Y$. In this context we will perform explicitly the computations made in all of Sections 2 – 5.

Let z^* be a fixed functional in the conjugate space X^* of X. Then consider the operator $W_{z^*} : C(S, X) \to C(S)$, given by $(W_{z^*} f)(s) = z^*(f(s))$, $f \in C(S, X)$, $s \in S$. It is a simple fact that W_{z^*} is bounded and that $\|W_{z^*}\| = \|z^*\|$. Moreover we have:

Lemma 1. *The operators W_{z^*} are onto for all $z^* \neq 0$.*

Proof. Let $\alpha \in X$ be fixed such that $z^*(\alpha) \neq 0$. Let $h \in C(S)$ and let us put $f(s) = h(s) \cdot \frac{\alpha}{z^*(\alpha)}$, $s \in S$. Then it is clear that $f \in C(S, X)$ and we have

$$(W_{z^*} f)(s) = z^*(f(s)) = h(s) z^*(\frac{\alpha}{z^*(\alpha)}) = h(s),$$

thus

$$W_{z^*} f = h.$$

It is noteworthy that, in general the vector f given above is not unique. □

Consider now a bounded operator $T : C(S, X) \to Y$; to factorize T through W_{z^*}, with a bounded $\theta : C(S) \to Y$, we must assume condition (2.2) of Theorem 1. In this case, for each $g \in C(S)$, T has the constant value θg on the fiber

$W_{z^*}^{-1}(g)$ of $C(S,X)$. As a simple example of this situation take $X = \mathbb{R}^n$ and $z^*(y_1, y_2, \ldots, y_n) = y_1 + y_2 + \ldots + y_n$. Then (2.2) reads: $f = (f_1, f_2, \ldots, f_n) \in C(S, \mathbb{R}^n)$, $f_1 + f_2 + \ldots + f_n \equiv 0 \Longrightarrow Tf = 0$, and we have

$$Tf = \theta(f_1 + f_2 + \ldots + f_n), \text{ for all } f \in C(S, \mathbb{R}^n).$$

Note also that W_{z^*} satisfies condition (3.1). Now, if we want to compute the representing measure G of T, all what we have to do , in view of (3.2), (4.1), and (5.1), is to compute the function $V(c_x)$ for $V = W_{z^*}$. This is a trivial matter since c_x is a constant function with value x on $S : V(c_x)(s) = (W_{z^*} c_x)(s) = z^*(c_x(s)) = z^*(x)$. Thus formulas (3.2), (4.1), (5.1), become respectively

Proposition 4. *Let T and W_{z^*} be as above and such that $T = \theta . W_{z^*}$ where θ is bounded. Then we have:*
(a) *$G_{y^*}(E) = (\mu(E) \cdot y^*) \cdot z^*$, for all $E \in \mathcal{B}_S$ and $y^* \in Y^*$, that is the X^*-valued measure G_{y^*} is generated by the unique functional $z^* \in X^*$.*
(b) *If T is weakly compact then $G(E) = (\gamma\mu(E)) \cdot z^*$, for all $E \in \mathcal{B}_S$.*
(c) If T is nuclear then $G(E) = (\int_E \gamma g(s)\ v(\mu, ds)) \cdot z^*$, *for all $E \in \mathcal{B}_S$.*

Now we give an example of a nuclear operator which satisfies condition $(\mathcal{N})$ of Theorem 7(b). To this end, let us recall that if Y is finite dimensional then every linear operator $T : C(S,X) \to Y$ is said to be degenerate.

Proposition 5. *If $T : C(S,X) \to Y$ is a bounded degenerate operator, then T is nuclear and satisfies condition $(\mathcal{N})$ of Theorem 7(b).*

Proof. By the [**4**, Theorem 2.13.3], a bounded degenerate operator $T : C(S,X) \to Y$ has a representation of the form $T(x) = \sum_1^n \mu_k(x)\ y_k$ where $\{y_k\ ,\ 1 \le k \le n\ \}$ and $\{\mu_k\ ,\ 1 \le k \le n\ \}$ are sets of linearly independent elements in Y and $C(S,X)^*$, respectively. Therefore T is nuclear and by the representation above it satisfies condition $(\mathcal{N})$ of Theorem 7(b). □

If $\dim Y = \infty$, the question arises whether there exist nuclear operators $T : C(S,X) \to Y$ which satisfy condition $(\mathcal{N})$ of Theorem 7(b). In this context, Proposition 5 allows the following conjecture:

Conjecture 1. *If Y is a separable Hilbert space, then every nuclear operator $T : C(S,X) \to Y$ satisfies condition $(\mathcal{N})$.*

7. Remark

In this work we attempted to give some information about the representing measure G, which had occured in the context of the integral representation (2.7). We obtained results for the class of factorizable Banach valued operators on $C(S,X)$. Let us point out that similar results had been obtained in [**5**, §5] for another special class of operators, and we may summarize as follows. Consider a bounded operator $T : C(S,X) \to X$ which satisfies the following condition: for $x^*, y^* \in X^*, f, g \in C(S,X)$, if $x^* \circ f = y^* \circ g$, then $x^* \circ Tf = y^* \circ Tg$. Then there exists a unique bounded scalar regular measure on S, $\mathcal{B}_S$ such that $Tf = \int_S f\ d\mu$

for all $f \in C(S,X)$; that is the operator T is a Bochner integral on the function space $C(S,X)$, (See [**5**, §5] for more details). Now, according to the integral form (2.7), the operator T has a representing vector measure G with values in the Banach space $\mathcal{L}(X, X^{**})$. A comparison made by the author in [**5**, §5], between the measures G and μ, allowed the following rather precise relation on the structure of the measure G :

$$\forall E \in \mathcal{B}_S \quad G(E) = \mu(E) \cdot \gamma \tag{7.1}$$

where γ is the canonical isomorphism of X into X^{**}.

Acknowledgement. I would like to thank the referee for valuable suggestions leading to the final version of the paper.

References

1. Diestel J. and Uhl J. J., *Vector Measures,* Mathematical Surveys number **15**, AMS, 1977.
2. Dinculeanu N., *Vector Measures*, Pergamon Press 1967.
3. Dunford N. and Schwartz J., *Linear Operators,* Wiley Classics Edition Published 1988.
4. Hille E. and Phillips R. S., *Fonctional Analysis and Semigroups,* Colloq. Publ. AMS 1958
5. Meziani L., *Integral representation for a class of vector valued operators*, Proc. Amer. Math. Soc. **130**(7) (2002), 2067–2077.

L. Meziani, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O Box 80203 Jeddah, 21589, Saudi Arabia, *e-mail*: `mezianilakhdar@hotmail.com`.