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THE QUADRATIC CONTROL FOR LINEAR DISCRETE-TIME SYSTEMS WITH INDEPENDENT RANDOM PERTURBATIONS IN HILBERT SPACES CONNECTED WITH UNIFORM OBSERVABILITY

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Abstract. The optimal control problem for linear discrete-time, time-varying systems with state dependent noise and quadratic control is considered. The asymptotic behavior of the solution of the related discrete-time Riccati equation is investigated. The existence of an optimal control, under stabilizability and uniform observability (respectively detectability) conditions, for the given quadratic cost function is proved.

1. Introduction

We consider the quadratic control problem for linear, time-varying, discrete-time systems, with control and independent random perturbations in real, separable Hilbert spaces. The existence of an optimal control is connected with the asymptotic behavior of the solution of the discrete-time Riccati equation associated with this problem. We will establish that, under stabilizability and uniform observability (respectively stabilizability and detectability) conditions, this Riccati equation has a unique, uniformly positive (respectively nonnegative) solution, which is bounded on $\mathbf{N}^*$ and stabilizing for the considered stochastic system with control. Using this result, we obtained the control, which minimize the given quadratic cost function. In 1974 J. Zabczyk treated (see [**11**]) the same problem for time-homogeneous systems. In connection with this problem, he introduced a notion of stochastic observability, which is equivalent, in the finite-dimensional case, with the one considered in this paper. He obtained similar results to those mentioned above, but the results which involved the uniform observability are obtained only for finite dimensional Hilbert spaces. The case of time-varying systems in finite dimensions has been investigated by T. Morozan (see [**6**]). In this paper we generalize the results of T. Morozan and J. Zabczyk. We also establish that, in the stochastic case, the uniform observability cannot imply the detectability. Consequently, the results obtained under detectability conditions (see the results obtained by J. Zabczyk in[**11**]) are different to those obtained using the uniform

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observability property (see Theorems 28, 29 in this paper or Corollary 3.4 in [**11**], for finite dimensional case and time-homogenous stochastic systems). Moreover, we note that the uniform observability property is easier to verify than the detectability condition. The results of this paper are the discrete-time versions of those obtained in [**9**] for the continuous case.

2. Notations and the statement of the problem

Let H, V, U be separable, real Hilbert spaces and let us denote by $L(H,V)$ the Banach space of all bounded linear operators which transform H into V. If $H=V$ we put $L(H,V)=L\,(H)$. We write $\langle\cdot,\cdot\rangle$ for the inner product and $\|\cdot\|$ for norms of elements and operators. If $A\in L(H)$ then A^* is the adjoint operator of A. The operator $A\in L(H)$ is said to be nonnegative and we write $A\geq 0$, if A is self-adjoint and $\langle Ax,x\rangle\geq 0$ for all $x\in H$.

We denote by $\mathcal{H}$ the Banach subspace of $L(H)$ formed by all self-adjoint operators, by $\mathcal{K}$ the cone of all nonnegative operators on $\mathcal{H}$ and by I the identity operator on H. $\mathbf{N}$ is the set of all natural numbers and $\mathbf{N}^*=\mathbf{N}-\{0\}$.

The sequence $\{L_n\}_{n\in\mathbf{N}^*}$, $L_n\in L(H,V)$, is bounded on $\mathbf{N}^*$ if $\widetilde{L}=\sup\limits_{n\in\mathbf{N}^*}\|L_n\|<\infty$.

Let $(\Omega,\mathcal{F},P)$ be a probability space and ξ be a real or H- valued random variable on Ω. We write $E(\xi)$ for mean value (expectation) of ξ. We will use the notation $\mathcal{B}(H)$ for the Borel σ-field of H. We recall that a mapping $\eta:(\Omega,\mathcal{F},P)\rightarrow(H,\mathcal{B}(H))$ is a H valued random variable if and only if, for arbitrary $x\in H$, $\langle\eta,x\rangle:(\Omega,\mathcal{F},P)\rightarrow\mathbf{R}$ is a real random variable (see [**1**]). It is easy to see that if $\xi,\eta:(\Omega,\mathcal{F},P)\rightarrow(H,\mathcal{B}(H))$ are two random variables, then $\langle\xi,\eta\rangle$ is a real random variable.

Let $\xi_n, n\in\mathbf{N}$ be real independent random variables, which satisfy the conditions $E(\xi_n)=0$ and $E\left|\xi_n\right|^2=b_n<\infty$ and let $\mathcal{F}_n$, $n\in\mathbf{N}^*$ be the $\sigma-$ algebra generated by $\{\xi_i, i\leq n-1\}$. We will denote by $L^2_n(H)=L^2(\Omega,\mathcal{F}_n,P,H)$ the space of all equivalence class of H-valued random variables η (i.e. η is a measurable mapping from $(\Omega,\mathcal{F}_n)$ into $(H,\mathcal{B}(H))$) such that $E\left\|\eta\right\|^2<\infty$.

In order to solve the quadratic control problem we need the following hypothesis:

H_1: The sequences A_n, $B_n\in L(H)$, $D_n\in L(U,H)$, $C_n\in L(H,V)$, $K_n\in L(U)$ and $b_n=E\left|\xi_n\right|^2$ are bounded on $\mathbf{N}^*$ and

$$K_n\geq\delta I,\delta>0,\text{ for all }n\in\mathbf{N}^*. \tag{1}$$

Assume that H_1 holds. Under the above notations, we consider the system with control

$$\begin{cases} x_{n+1}=A_nx_n+\xi_nB_nx_n+D_nu_n\\ x_k=x,\ k\in\mathbf{N}^*,\end{cases} \tag{2}$$

denoted $\{A:D,B\}$, where the control $u=\{u_k,u_{k+1},\ldots\}$ belongs to the class $\widetilde{U}_{k,x}$, defined by the properties that u_n, $n\geq k$ is an U-valued random variable, $\mathcal{F}_n$-measurable and $\sup\limits_{n\geq k}E\left\|u_n\right\|^2<\infty$.

To the system (2), we associate the following quadratic cost

$$I_k(x,u) = E\sum_{n=k}^{\infty}[\|C_nx_n\|^2 + < K_nu_n, u_n >], \tag{3}$$

where x_n is the solution of (2), for all $n \in \mathbf{N}^*, n \geq k$ and the control $u = \{u_k, u_{k+1}, ...\}$ belongs to the class $U_{k,x}$, formed by all infinite sequences $u \in \widetilde{U}_{k,x}$ with the property $I_k(x,u) < \infty$.

We study the following problem: For every $k \in \mathbf{N}^*$ and $x \in H$, we look for an optimal control u, which belongs to the class $U_{k,x}$ and minimizes the above quadratic cost. We will prove (Theorem 29) that, under stabilizability and either uniform observability or detectability conditions, there exists an optimal control, which minimize the cost function (3).

3. Properties of the solutions of the linear discrete time systems

We consider the stochastic system $\{A, B\}$ associated with (2)

$$\begin{cases} x_{n+1} = A_nx_n + \xi_nB_nx_n \\ \quad x_k = x, \ k \in \mathbf{N}^*, \end{cases} \tag{4}$$

where A_n, $B_n \in L(H)$ for all $n \in \mathbf{N}^*$, $n \geq k$ and ξ_n are introduced above.

The random evolution operator associated to (4) is the operator $X(n,k)$, $n \geq k > 0$, where $X(k,k) = I$ and

$$X(n,k) = (A_{n-1} + \xi_{n-1}B_{n-1})\dots(A_k + \xi_kB_k),$$

for all $n > k$. If $x_n = x_n(k,x)$ is a solution of the system (4) with the initial condition $x_k = x, \ k \in \mathbf{N}^*$, then it is unique and $x_n = X(n,k)x$.

Lemma 1. *Under the above notations $X(n,k), n \geq k, n, k \in \mathbf{N}^*$ is a linear and bounded operator from $L^2_k(H)$ into $L^2_n(H)$, which has the following properties $E\|X(k,k)(\xi)\|^2 = E\|\xi\|^2$ and*

$$\begin{aligned} E\|X(n,k)(\xi)\|^2 \leq (\|A_{n-1}\|^2 + b_{n-1}\|B_{n-1}\|^2)\dots \\ \|A_k\|^2 + b_k\|B_k\|^2)E\|\xi\|^2 \end{aligned} \tag{5}$$

for all $n > k > 0$ and $\xi \in L^2_k(H)$.

Proof. We use the induction to prove that $X(n,k)$ is a bounded linear operator from $L^2_k(H)$ into $L^2_n(H)$ for all $p = n-k$, $p \in \mathbf{N}$.

For $p = 0$ we have $n = k$ and the conclusion follows immediately. We assume that the statement of the lemma holds for all $n \geq k \geq 0$ such that $n - k = p$ and we will prove it for all $n > k \geq 0$, $n - k = p + 1$.

Let $n, k \in \mathbf{N}^*$ such that $n - k = p + 1$. If $\eta \in L^2_k(H)$, then $X(n,k)(\eta) = A_{n-1}(Y) + \xi_{n-1}B_{n-1}(Y)$, where $Y = X(n-1,k)(\eta) \in L^2_{n-1}(H)$ (by the induction hypothesis).

For every $x \in H$ we have $\langle A_{n-1}(Y), x\rangle = \left\langle Y, A^*_{n-1}x\right\rangle$. Since $\left\langle Y, A^*_{n-1}x\right\rangle$ is a real $\mathcal{F}_{n-1}$-measurable random variable, then $A_{n-1}(Y)$ is a H-valued, $\mathcal{F}_{n-1}$-measurable random variable.

Analogously, we deduce that $B_{n-1}(Y)$ is a H-valued, $\mathcal{F}_{n-1}$-measurable random variable. Since $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n$, it is clear that $A_{n-1}(Y) + \xi_{n-1}B_{n-1}(Y)$ is $\mathcal{F}_n$-measurable. Hence $X(n,k)(\eta)$ is $\mathcal{F}_n$- measurable.

Since ξ_n is $\mathcal{F}_n$-independent, then $X(n,k)(\eta)$ and ξ_n are independent for all $n-k = p+1$. From the above considerations it follows that $\{\xi_{n-1}, \langle B_{n-1}(Y), A_{n-1}(Y)\rangle\}$, $\{\xi_{n-1}, \langle B_{n-1}(Y), B_{n-1}(Y)\rangle\}$ are independent. Thus

$$\begin{aligned} E\left\|X(n,k)(\eta)\right\|^2 &= E\left\|A_{n-1}(Y)\right\|^2 + b_{n-1}E\left\|B_{n-1}(Y)\right\|^2 \\ &\le (\left\|A_{n-1}\right\|^2 + b_{n-1}\left\|B_{n-1}\right\|^2)E\left\|Y\right\|^2 . \end{aligned}$$

From the induction assumption we get (5) for all $\eta \in L_k^2(H)$ and $n > k > 0$. Since $X(n,k)$ is linear, we use (5) to obtain the conclusion. The proof is finished. $\square$

Corollary 2. *If A_n, B_n and b_n are bounded on $\mathbf{N}^*$ and we use the notations $\widetilde{A}, \widetilde{B}, \widetilde{b}$ introduced in the last section, then*

$$\|X(n,k)\| \le \max\{1, (\widetilde{A}^2 + \widetilde{b}\widetilde{B}^2)^{(n-k)/2}\},$$

where $\left\|X(n,k)\right\|^2 = \sup\limits_{\eta \in L_k^2(H)} E\left\|X(n,k)\eta\right\|^2$ *for all* $n \ge k > 0$.

Remark 3. If $x_n = X(n,k)x$ is the solution of (4), with the initial condition $x_k = x, k \in \mathbf{N}^*$, then it follows from Lemma 1 that $E\left\|x_n\right\|^2 < \infty$, x_n is $\mathcal{F}_n$-measurable and x_n, ξ_n (or equivalently, $X(n,k)x, \xi_n$) are independent for all $n \ge k > 0$. It is obviously true that x_n is $\mathcal{F}$-measurable for all $n \ge k > 0$.

If $S \in \mathcal{H}$ and A_n, B_n, respectively $b_n = E\left|\xi_n\right|^2 < \infty$, $n \in \mathbf{N}^*$ are introduced as above, then we define the operators $Q_n, T(n,k) : \mathcal{H} \to \mathcal{H}$

$$Q_n(S) = A_n^* S A_n + b_n B_n^* S B_n, \tag{6}$$

$T(n,k)(S) = Q_k\left(Q_{k+1}\left(\ldots\left(Q_{n-1}(S)\right)\right)\right)$, for all $n-1 \ge k$ and $T(k,k)(S) = S$. It is clear that Q_n and $T(n,k)$ are linear and bounded operators (see [**7**] for the finite dimensional case).

Lemma 4. [**13**] *Let $X \in L(\mathcal{H})$. If $X(\mathcal{K}) \subset \mathcal{K}$ then $\|X\| = \|X(I)\|$, where I is the identity operator, $I \in \mathcal{H}$.*

The following proposition is known (see [**10**]), but I present the proof for the readers' convenience.

Theorem 5. *If $X(n,k)$ is the random evolution operator associated to (4), then $T(n,k)(\mathcal{K}) \subset \mathcal{K}$ (that is $T(n,k)$ satisfies the hypotheses of the above Lemma) and we have*

$$\langle T(n,k)(S)x, y\rangle = E\left\langle SX(n,k)x, X(n,k)y\right\rangle \tag{7}$$

for all $S \in \mathcal{H}$ and $x, y \in H$.

Proof. Let $S \in \mathcal{H}$ and $x, y \in H$. Since ξ_{n-1} is $\mathcal{F}_{n-1}-$ independent, we deduce by Lemma 1 that ξ_{n-1} and $\langle AX(n-1,k)x, BX(n-1,k)y\rangle$ (respectively ξ_{n-1}^2

and $\langle AX(n-1,k)x, BX(n-1,k)y\rangle)$ are independent on $(\Omega, \mathcal{F}, P)$ for all $A, B \in L(H)$. Using the relation $X(n,k) = (A_{n-1} + \xi_{n-1}B_{n-1})X(n-1,k)$, we get

$$\begin{aligned} E\langle SX(n,k)x, X(n,k)y\rangle =& E\left\langle A^*_{n-1}SA_{n-1}X(n-1,k)x, X(n-1,k)y\right\rangle \\ &+ b_{n-1}E\left\langle B^*_{n-1}SB_{n-1}X(n-1,k)x, X(n-1,k)y\right\rangle \end{aligned}$$

and

$$E\langle SX(n,k)x, X(n,k)y\rangle = E\langle Q_{n-1}(S)X(n-1,k)x, X(n-1,k)y\rangle \tag{8}$$

for all $x, y \in H$. Let us consider the operator $V(n,k) : \mathcal{H} \to \mathcal{H}$,

$$\langle V(n,k)(S)x, y\rangle = E\langle SX(n,k)x, X(n,k)y\rangle \tag{9}$$

for all $S \in \mathcal{H}$ and $x, y \in H$. It is easy to see that $V(n,k)$ is well defined because the right member of this equality is a symmetric bilinear form, which also defines a unique linear, bounded and self-adjoint operator on H.

From (8) and (9) we obtain $V(n,k)(S) = V(n-1,k)Q_{n-1}(S)$ if $n-1 \geq k$ and $V(k,k) = I$, where I is the identity operator from $L(\mathcal{H})$. Now, it is easy to see that $V(n,k) = T(n,k)$ and it follows (7).

Since $Q_p(\mathcal{K}) \subset \mathcal{K}$ for all $p \in \mathbf{N}^*$ we deduce that $T(n,k)(\mathcal{K}) \subset \mathcal{K}$. □

Remark 6. Since the set of all simple random variables (see [**1**]), which belongs to $L^2_k(H)$ is dense in $L^2_k(H)$ and $\{\xi, X(n,k)x\}$ are independent random variables for all $\xi \in L^2_k(H), x \in H$, it is not difficult to see that $E\langle T(n,k)(S)\xi, \xi\rangle = E\langle SX(n,k)\xi, X(n,k)\xi\rangle$ for all $S \in \mathcal{H}$ and $\xi \in L^2_k(H)$.

Lemma 7. *The solution $x_j = x_j(x,k;u)$ of (2) satisfies the equation*

$$x_j(x,k;u) = X(j,k)x + \sum_{i=k}^{j-1} X(j,i+1)D_iu_i \tag{10}$$

for $j \geq k+1$. Moreover, x_j is $\mathcal{F}_j-$ measurable and ξ_j- independent.

Proof. It is not difficult to verify, by induction, that (10) holds.

Indeed, for $j = k+1$, (10) is obviously true. Suppose that (10) holds for $j = n$, $n > k$ and let us prove the statement for $j = n+1$.

Then $x_n = X(n,k)x + \sum\limits_{i=k}^{n-1} X(n,i+1)D_iu_i$ and we have

$$\begin{aligned} x_{n+1} &= (A_n + \xi_nB_n)\left[X(n,k)x + \sum_{i=k}^{n-1} X(n,i+1)D_iu_i\right] + D_nu_n \\ &= X(n+1,k)x + \sum_{i=k}^{n} X(n+1,i+1)D_iu_i. \end{aligned}$$

Hence (10) holds for $j = n+1$. The induction is complete. By the hypothesis, u_i is $\mathcal{F}_i$-measurable. Using Lemma 1, we deduce that $X(j,i+1)D_iu_i$, $i = k, \ldots, j-1$ and $X(j,k)x$ are $\mathcal{F}_j$-measurable. Consequently x_j is $\mathcal{F}_j$-measurable. Since ξ_j is $\mathcal{F}_j-$ independent we see that x_j and ξ_j are independent. The proof is finished. □

4. UNIFORM EXPONENTIAL STABILITY, UNIFORM OBSERVABILITY AND DETECTABILITY

4.1. Mean square stability

Definition 8. We say that the stochastic system $\{A, B\}$ is uniformly exponentially stable iff there exist $\beta \geq 1$, $a \in (0,1)$ and $n_0 \in \mathbf{N}^*$ such that we have

$$E\left\|X(n,k)x\right\|^2 \leq \beta a^{n-k}\left\|x\right\|^2 \tag{11}$$

for all $n \geq k \geq n_0$ and $x \in H$.

It is not difficult to see that if the stochastic system $\{A, B\}$ is uniformly exponentially stable, then there exist $\beta \geq 1$, $a \in (0,1)$ such that (11) holds for all $n \geq k > 0$.

Using the Theorem 5 and Lemma 4 we obtain the following result:

Proposition 9. [**10**] *The following statements are equivalent:*

a) *the system* (4) *is uniformly exponentially stable;*

b) *there exist* $\beta \geq 1$, $a \in (0,1)$ *and* $n_0 \in \mathbf{N}^*$ *such that we have*

$$\left\|T(n,k)\left(I\right)\right\| = \left\|T(n,k)\right\| \leq \beta a^{n-k} \tag{12}$$

for all $n \geq k \geq n_0$*, where* I *is the identity operator on* H.

Corollary 10. *Assume that* $A_n = A$, $B_n = B$, $b_n = b$ *for all* $n \in \mathbf{N}^*$ *and* H *has a finite dimension. The stochastic system* $\{A, B\}$ *is uniformly exponentially stable if and only if the stochastic system* $\{A^*, B^*\}$ *is uniformly exponentially stable.*

Proof. By the above proposition it follows that the stochastic system $\{A, B\}$ (respectively $\{A^*, B^*\}$) is uniformly exponentially stable if and only if there exist $\beta \geq 1$, $a \in (0,1)$ (respectively $\beta_1 \geq 1$, $a_1 \in (0,1)$) such that we have $\left\|Q^n(I)\right\| \leq \beta a^n$ (respectively $\left\|\widehat{Q}^n(I)\right\| \leq \beta_1 a_1^n$) for all $n \in \mathbf{N}$, where $Q_n = Q$ is given by (6) (respectively $\widehat{Q}(S) = ASA^* + bBSB^*$). Since $TrQ^n(I) = Tr\widehat{Q}^n(I)$ and, for all $S \in \mathcal{K}$, we have $\left\|S\right\| \leq TrS \leq Tr\left(I\right)\left\|S\right\|$ (I is the identity operator on H and $Tr(I) < \infty$) it is clear that $\left\|Q^n(I)\right\| \leq \beta a^n$ if and only if $\left\|\widehat{Q}^n(I)\right\| \leq \beta_1 a_1^n$ for all $n \in \mathbf{N}$. □

4.2. Uniform observability, detectability and stabilizability

We consider the discrete-time stochastic system $\{A, B; C\}$ formed by the system (4) and the observation relation $z_n = C_n x_n$, where $C_n \in L(H, V)$, $n \in \mathbf{N}^*$ is a bounded sequence on $\mathbf{N}^*$.

Definition 11. [**6**, Definition 6] We say that $\{A, B; C\}$ is uniformly observable if there exist $n_0 \in \mathbf{N}^*$ and $\rho > 0$ such that

$$\sum_{n=k}^{k+n_0} E\left\|C_n X(n,k)x\right\|^2 \geq \rho\left\|x\right\|^2 \tag{13}$$

for all $k \in \mathbf{N}^*$ and $x \in H$.

If the stochastic perturbation is missing, that is $B_n = 0$ for all $n \in \mathbf{N}^*$, we will use the notation $\{A, \_; C\}$ for the observed, deterministic system. We have the following definition of the deterministic uniform observability (see [**3**] and [**4**]).

Definition 12. We say that $\{A, \_; C\}$ is uniformly observable iff there exist $n_0 \in \mathbf{N}$ and $\rho > 0$ such that $\sum\limits_{n=k+1}^{k+n_0} \|C_n A_{n-1} A_{n-2} \dots A_k x\|^2 + \|C_k x\|^2 \geq \rho \|x\|^2$ for all $k \in \mathbf{N}^*$ and $x \in H$.

Remark 13. [**3**] It is known that, in the time-invariant case and for finite dimensional spaces, the deterministic system $\{A, \_; C\}$ is uniformly observable iff $\operatorname{rank}(C^*, A^*C^*, \dots, (A^*)^{n-1} C^*) = n$, where n is dimension of H.

The following proposition is a consequence of the Theorem 5.

Proposition 14. *The stochastic system $\{A, B; C\}$ is uniformly observable if and only if there exist $n_0 \in \mathbf{N}$ and $\rho > 0$ such that, for all $k \in \mathbf{N}^*$,*

$$\sum_{n=k}^{k+n_0} T(n,k)(C_n^* C_n) \geq \rho I, \tag{14}$$

where I is the identic operator on H.

Conclusion 15. *From the above proposition it follows that if the deterministic system $\{A, \_; C\}$ is uniformly observable, then the stochastic system $\{A, B; C\}$ is uniformly observable.*

Definition 16. The system $\{A, B, C\}$ is detectable if there exists a bounded on $\mathbf{N}^*$ sequence $P = \{P_n\}_{n \in \mathbf{N}^*}, P_n \in L(U, H)$ such that the stochastic system $\{A + PC, B\}$ is uniformly exponentially stable.

Definition 17. [**6**] The system (2) is stabilizable if there exists a bounded on $\mathbf{N}^*$ sequence $F = \{F_n\}_{n \in \mathbf{N}^*}, F_n \in L(H, U)$ such that the stochastic system $\{A + DF, B\}$ is uniformly exponentially stable.

Using Corollary 10 and the above definition we deduce the following:

Remark 18. In the time invariant case and for finite dimensional spaces, the system $\{A, B, C\}$ is detectable if and only if the stochastic system with control $\{A^* : C^*, B^*\}$ is stabilizable.

4.3. The stochastic observability doesn't imply detectability

Let us consider the time invariant case, when $A_n = A$, $B_n = B$, $b_n = b$ and $C_n = C$. We introduce the Riccati equation

$$R_n = \mathcal{A}(R_{n+1}) \tag{15}$$

associated to the system $\{A, B, C\}$, where $\mathcal{A} : \mathcal{K} \to \mathcal{K}$,

$$\mathcal{A}(S) = ASA^* + bBSB^* + I - ASC^*(I + CSC^*)^{-1} CSA^*.$$

It is not difficult to see [**12**, Lemma 3.1] that, since $I + CSC^*$ is invertible, $I + C^*CS$ is invertible and $C^*(I + CSC^*)^{-1} = (I + C^*CS)^{-1} C^*$. Moreover $S(I + C^*CS)^{-1} \geq 0$. Thus $\mathcal{A}(S) = bBSB^* + I + AS(I + C^*CS)^{-1} A^*$ and the mapping $\mathcal{A}$ is well defined.

Lemma 19. *If the Riccati equation* (15) *has a bounded on* $\mathbf{N}^*$, *nonnegative solution then the algebraic Riccati equation*

$$R = \mathcal{A}(R) \tag{16}$$

has a nonnegative solution.

Proof. From Theorem 3.1 in [**11**] we deduce that
a) $\mathcal{A}$ is monotonic (that is $\mathcal{A}(S_1) \geq \mathcal{A}(S_2)$ for all $S_1 \geq S_2, S_1, S_2 \in \mathcal{K}$.
b) if $S_n \nearrow S$ (strongly) then $\mathcal{A}(S_n) \nearrow \mathcal{A}(S)$ (strongly).

Therefore, if R_n is a nonnegative, bounded on $\mathbf{N}^*$ solution of (15), then there exists $\alpha > 0$ such as $\alpha I \geq R_n \geq \mathcal{A}^p(R_{n+p}) \geq \mathcal{A}^p(0) \geq 0$ for all $n,\ p \in \mathbf{N}^*$. Then $\mathcal{A}^n(0) \nearrow \overline{A}$ (strongly) and it is clear that $\overline{A}$ is a solution of the Riccati equation (16). □

Remark 20. Let us consider the finite dimensional case. Assume that the system $\{A, B, C\}$ is detectable. Using Remark 18 we deduce that the stochastic system with control $\{A^* : C^*, B^*\}$ is stabilizable. Thus, it follows by Proposition 25 that the Riccati equation (17), where we replace the operators A with A^*, B with B^*, C with I (I is the identic operator on H), and D with C^* admits a nonnegative bounded on $\mathbf{N}^*$ solution (that is the Riccati equation (15)) has a nonnegative bounded on $\mathbf{N}^*$ solution.

The following counter-example prove that the stochastic observability doesn't imply detectability.

Counter-example Let us consider $H = \mathbf{R}^2$, $V = \mathbf{R}$ ($\mathbf{R}^2$ is the real 2-dimensional space), $A_n = A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, C_n = C = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $b_n = 1$ and $B_n = B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for all $n \in \mathbf{N}^*$. Since $\operatorname{rank}(C^*, A^*C^*) = 2$, then the deterministic system $\{A, \_; C\}$ (see Remark 13) is observable. Therefore (see Conclusion 15) the stochastic system $\{A, B; C\}$ is uniformly observable. We will prove that the equation (16) has not a nonnegative solution and consequently the equation (15) cannot have a nonnegative bounded on $\mathbf{N}^*$ solution. Then, from Remark 20, we deduce that the system $\{A, B, C\}$ cannot be detectable.

If we seek a solution $K = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$ of (16) which satisfies the condition $x_1 x_3 \geq x_2^2$, $x_1 \geq 0$, then (16) is equivalent with the following system

$$\begin{cases} (x_1 + x_2)\left(\dfrac{x_2}{2} - 1\right) & = & x_1 + 1 \\ x_1 + x_2)\left(\dfrac{x_2}{2} + \dfrac{x_3 - 1}{4}\right) & = & \dfrac{x_2}{2} \\ (x_2 + x_3)\left(\dfrac{x_2}{2} - 1\right) & = & \dfrac{x_2}{2} \\ (x_2 + x_3)(\dfrac{x_2}{2} + \dfrac{x_3 - 1}{4}) & = & \dfrac{3x_3 + 1}{4} \end{cases}$$

We deduce $\left(\frac{x_2}{2}\right)^2 = (x_1+1)\left(\frac{3x_3+1}{4}\right)$. Then $x_2^2 = 3x_3x_1+3x_3+x_1+1 \geq 3x_2^2+1$, that is impossible. The conclusion follows.

5. The discrete-time Riccati equation of stochastic control

Throughout this section we suppose that H_1 holds. We define the transformation

$$\mathcal{G}_n : \mathcal{K} \to \mathcal{K}, \mathcal{G}_n(S) = A_n^* S D_n (K_n + D_n^* S D_n)^{-1} D_n^* S A_n.$$

It is easy to see that $D_n^* S D_n \in \mathcal{K}$ for all $S \in \mathcal{K}$ and from (1) we have $K_n + D_n^* S D_n \geq K_n \geq \delta I$. Thus $K_n + D_n^* S D_n$ is invertible and $(K_n + D_n^* S D_n)^{-1} \geq 0$. Therefore $\mathcal{G}_n$ is well defined. We consider the following Riccati equation

$$R_n = A_n^* R_{n+1} A_n + b_n B_n^* R_{n+1} B_n + C_n^* C_n - \mathcal{G}_n(R_{n+1}) \tag{17}$$

on $\mathcal{K}$, connected with the quadratic cost (3).

Definition 21. A sequence $R_n \in \mathcal{K}$, $n \in \mathbf{N}^*$ such that (17) holds is said to be a solution of the Riccati equation (17).

Definition 22. [**6**] A solution $(R_n)_{n\in\mathbf{N}^*}$ of (17) is said to be stabilizing for (2) if the stochastic system $\{A + DF, B\}$ with

$$F_n = -(K_n + D_n^* R_{n+1} D_n)^{-1} D_n^* R_{n+1} A_n, n \in \mathbf{N}^* \tag{18}$$

is uniformly exponentially stable.

Proposition 23. *The Riccati equation* (17) *has at most one stabilizing and bounded on* $\mathbf{N}^*$ *solution.*

Proof. Let $R_{n,1}$ and $R_{n,2}$ be two stabilizing and bounded on $\mathbf{N}^*$ solutions of equation (17). We introduce the systems

$$\begin{cases} x_{n+1,i} = (A_n + D_n F_{n,i}) x_{n,i} + \xi_n B_n x_{n,i} \\ \qquad\qquad x_{k,i} = x \in H \end{cases} \tag{19}$$

for all $n \geq k$, $n, k \in \mathbf{N}^*$, where $F_{n,i} = -(K_n + D_n^* R_{n+1,i} D_n)^{-1} D_n^* R_{n+1,i} A_n$, $i = 1, 2$. We see that $Q_n = R_{n,1} - R_{n,2}$ is the solution of the equation

$$\begin{aligned} Q_n = {} & A_n^* Q_{n+1} A_n + b_n B_n^* Q_{n+1} B_n + A_n^* R_{n+1,1} D_n F_{n,1} \\ & - F_{n,2}^* D_n^* R_{n+1,2} A_n. \end{aligned} \tag{20}$$

Since $x_{n,i}$ and ξ_n, $i = 1, 2$ are independent random variables for all $n \in \mathbf{N}^*$ (see Lemma 1), it follows that

$$\begin{aligned} & E\left\langle Q_{n+1} x_{n+1,1}, x_{n+1,2} \right\rangle \\ & \quad = b_n E\left\langle B_n^* Q_{n+1} B_n x_{n,1}, x_{n,2} \right\rangle \\ & \qquad + E\left\langle (A_n + D_n F_{n,2})^* Q_{n+1} (A_n + D_n F_{n,1}) x_{n,1}, x_{n,2} \right\rangle \\ & \quad = E\left\langle A_n^* Q_{n+1} A_n x_{n,1}, x_{n,2} \right\rangle + b_n E\left\langle B_n^* Q_{n+1} B_n x_{n,1}, x_{n,2} \right\rangle \\ & \qquad + E\left\langle F_{n,2}^* D_n^* Q_{n+1} A_n x_{n,1}, x_{n,2} \right\rangle + E\left\langle A_n^* Q_{n+1} D_n F_{n,1} x_{n,1}, x_{n,2} \right\rangle \\ & \qquad + E\left\langle F_{n,2}^* D_n^* Q_{n+1} D_n F_{n,1} x_{n,1}, x_{n,2} \right\rangle. \end{aligned}$$

From (20) we deduce that

$$\begin{aligned}
&E\left\langle Q_{n+1}x_{n+1,1},x_{n+1,2}\right\rangle\\
&\qquad=E\left\langle Q_nx_{n,1},x_{n,2}\right\rangle-E\left\langle A_n^*R_{n+1,1}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle\\
&\qquad\quad+E\left\langle F_{n,2}^*D_n^*R_{n+1,2}A_nx_{n,1},x_{n,2}\right\rangle+E\left\langle F_{n,2}^*D_n^*Q_{n+1}A_nx_{n,1},x_{n,2}\right\rangle\\
&\qquad\quad+E\left\langle A_n^*Q_{n+1}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle+E\left\langle F_{n,2}^*D_n^*Q_{n+1}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle.
\end{aligned}$$

Now we obtain

$$\begin{aligned}
&E\left\langle Q_{n+1}x_{n+1,1},x_{n+1,2}\right\rangle\\
&\qquad=E\left\langle Q_nx_{n,1},x_{n,2}\right\rangle-E\left\langle A_n^*R_{n+1,2}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle\\
&\qquad\quad+E\left\langle F_{n,2}^*D_n^*R_{n+1,1}A_nx_{n,1},x_{n,2}\right\rangle+E\left\langle F_{n,2}^*D_n^*Q_{n+1}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
&E\left\langle F_{n,2}^*D_n^*Q_{n+1}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle\\
&\quad=E\left\langle F_{n,2}^*D_n^*R_{n+1,1}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle-E\left\langle F_{n,2}^*D_n^*R_{n+1,2}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle\\
&\quad=-E\left\langle F_{n,2}^*D_n^*R_{n+1,1}A_nx_{n,1},x_{n,2}\right\rangle-E\left\langle F_{n,2}^*K_nF_{n,1}x_{n,1},x_{n,2}\right\rangle\\
&\qquad+E\left\langle A_n^*R_{n+1,2}D_nF_{n,1}x_{n,1},x_{n,2}\right\rangle+E\left\langle F_{n,2}^*K_nF_{n,1}x_{n,1},x_{n,2}\right\rangle,
\end{aligned}$$

we get $E\left\langle Q_{n+1}x_{n+1,1},x_{n+1,2}\right\rangle=E\left\langle Q_nx_{n,1},x_{n,2}\right\rangle$ for all $n\geq k$.

It is easy to see that for all $n\geq k$, $x\in H$, we have

$$E\left\langle Q_{n+1}x_{n+1,1},x_{n+1,2}\right\rangle=E\left\langle Q_kx_{k,1},x_{k,2}\right\rangle=\left\langle Q_kx,x\right\rangle.$$

Let $M_i>0$, $i=1,2$ be such as $\|R_{n+1,i}\|\leq M_i$, for all $n\in\mathbf{N}$. Thus,

$$\begin{aligned}
0\leq|\langle Q_kx,x\rangle|&\leq\|Q_{n+1}\|\,E(\|x_{n+1,1}\|\,\|x_{n+1,2}\|)\\
&\leq(M_1+M_2)\sqrt{E\left\|x_{n+1,1}\right\|^2E\left\|x_{n+1,2}\right\|^2}.
\end{aligned}$$

From the hypothesis and from the Definition 17, it follows that the systems (19) are uniformly exponentially stable and $E\left\|x_{n+1,i}\right\|^2\underset{n\to\infty}{\to}0$, $i=1,2$.

As $n\to\infty$ in the last inequality, we obtain $\langle Q_kx,x\rangle=0$ for all $x\in H$. Therefore $Q_k=0$ and $R_{k,1}=R_{k,2}$ for all $k\in\mathbf{N}^*$. The proof is complete. □

By $U_{k,M}$, M, $k\in\mathbf{N}^*$, $M>k$ we denote the set of all finite sequences $u_k^M=\{u_k,u_{k+1},\dots u_{M-1}\}$ of U-valued and $\mathcal{F}_i$-measurable random variables u_i, $i=k,\dots,M-1$ with the property $E\left\|u_i\right\|^2<\infty$.

Let $x_n, n<M$ be the solution of system (2) with the control u_k^M. We introduce the performance

$$V(M,k,x,u_k^M)=E\sum_{n=k}^{M-1}[\|C_nx_n\|^2+<K_nu_n,u_n>].$$

Let us consider the sequence $R(M,M)=0\in\mathcal{K}$,

$$\begin{aligned}
R(M,n)=A_n^*R(M,n+1)A_n+b_nB_n^*R(M,n+1)B_n\\
+C_n^*C_n-\mathcal{G}_n(R(M,n+1))
\end{aligned}$$

for all $n\leq M-1$.

Lemma 24. *The sequence* $R(M,n) \in \mathcal{K}$ *for all* $0 < n \leq M$ *and*

$$0 \leq R(M-1,n) \leq R(M,n) \qquad \textit{for all } 0 < n \leq M-1.$$

Proof. We will prove the first assertion by induction. For $n = M$, $R(M,n) = 0 \in \mathcal{K}$. Let us assume that $R(M,n) \in \mathcal{K}$ for all $n \in \mathbf{N}^*$, $k < n \leq M$, $k \in \mathbf{N}^*$.

We prove that $R(M,k) \in \mathcal{K}$. Let x_n be the solution of system (2) with the initial condition $x_k = x$ and with the control u_k^M, introduced above. Using Lemma 7 we get, for all $k \leq n \leq M-1$.

$$\begin{aligned}
&E\left\langle R(M,n+1)x_{n+1}, x_{n+1}\right\rangle \\
&\quad = E\left\langle R(M,n+1)(A_n x_n + \xi_n B_n x_n + D_n u_n), (A_n x_n + \xi_n B_n x_n + D_n u_n)\right\rangle \\
&\quad = E\left\langle A_n^* R(M,n+1)A_n x_n, x_n\right\rangle + b_n\left\langle B_n^* R(M,n+1)B_n x_n, x_n\right\rangle \\
&\qquad + E\left\langle D_n^* R(M,n+1)A_n x_n, u_n\right\rangle + E\left\langle A_n^* R(M,n+1)D_n u_n, x_n\right\rangle \\
&\qquad + E\left\langle D_n^* R(M,n+1)D_n u_n, u_n\right\rangle .
\end{aligned}$$

Further, we obtain by the definition of $R(M,n)$

$$\begin{aligned}
&E\left\langle R(M,n+1)x_{n+1}, x_{n+1}\right\rangle \\
&\quad = E(\left\langle R(M,n)x_n, x_n\right\rangle - \left\langle C_n^* C_n x_n, x_n\right\rangle + \left\langle \mathcal{G}_n(R(M,n+1))x_n, x_n\right\rangle) \\
&\qquad + 2E\left\langle D_n^* R(M,n+1)A_n x_n, u_n\right\rangle + E\left\langle D_n^* R(M,n+1)D_n u_n, u_n\right\rangle .
\end{aligned}$$

If $z_n = u_n + (K_n + D_n^* R(M,n+1)D_n)^{-1} D_n^* R(M,n+1)A_n x_n$ we get

$$\begin{aligned}
&E\left\langle [K_n + D_n^* R(M,n+1)D_n]z_n, z_n\right\rangle \\
&\qquad = E\left\langle [K_n + D_n^* R(M,n+1)D_n]u_n, u_n\right\rangle + 2E\left\langle u_n, D_n^* R(M,n+1)A_n x_n\right\rangle \\
&\qquad\quad + E\left\langle \mathcal{G}_n(R(M,n+1))x_n, x_n\right\rangle .
\end{aligned}$$

Thus we may write

$$\begin{aligned}
&E\left\langle R(M,n+1)x_{n+1}, x_{n+1}\right\rangle \\
&\qquad = E\left\langle R(M,n)x_n, x_n\right\rangle - E\left\langle C_n^* C_n x_n, x_n\right\rangle - E\left\langle K_n u_n, u_n\right\rangle \\
&\qquad\quad + E\left\langle (K_n + D_n^* R(M,n+1)D_n)z_n, z_n\right\rangle
\end{aligned}$$

Now, we consider the last equality for $n = k, k+1, \ldots, M-1$ and, summing, we obtain

$$\begin{aligned}
E\left\langle R(M,M)x_M, x_M\right\rangle = {}& E\left\langle R(M,k)x_k, x_k\right\rangle - V(M,k,x,u_k^M) \\
& + E\sum_{n=k}^{M-1}\left\langle (K_n + D_n^* R(M,n+1)D_n)z_n, z_n\right\rangle .
\end{aligned}$$

Since $R(M,M) = 0$ and $x_k = x$ we have:

$$\begin{aligned}
V(M,k,x,u_k^M) = {}& \left\langle R(M,k)x, x\right\rangle \\
& + E\sum_{n=k}^{M-1}\left\langle (K_n + D_n^* R(M,n+1)D_n)z_n, z_n\right\rangle
\end{aligned} \tag{21}$$

Let $\widetilde{x}_n$ be the solution of system

$$\left\{\begin{array}{rcl} x_{n+1} & = & (A_n + D_nF_n)x_n + \xi_n B_n x_n \\ x_k & = & x \in H \end{array}\right. , \tag{22}$$

where $F_n = -[K_n + D_n^* R(M, n+1)D_n]^{-1} D_n^* R(M, n+1)A_n$, $n \leq M-1$.

It is clear that $\widetilde{x}_n$ is also the solution of (2) with $\widetilde{u}_n = F_n\widetilde{x}_n$, $k \leq n \leq M-1$. From Lemma 1 it follows that $\widetilde{x}_n$ is $\mathcal{F}_n$-measurable, $\{\widetilde{x}_n, \xi_n\}$ are independent and $E\left\|\widetilde{x}_n\right\|^2 < \infty$, $k \leq n \leq M-1$. Therefore, we deduce that $\widetilde{u}_k^M = \{\widetilde{u}_n, k \leq n \leq M-1\} \in U_{k,M}$. If we denote

$$S_{k,M}(u_k^M) = E\sum_{n=k}^{M-1} \langle (K_n + D_n^* R(M, n+1)D_n)z_n, z_n\rangle ,$$

it is clear that $S_{k,M}(u_k^M) \geq 0$ and $S_{k,M}(\widetilde{u}_k^M) = 0$. Thus,

$$\begin{aligned} \min_{u_k^M \in U_{k,M}} V(M, k, x, u_k^M) &= \min_{u_k^M \in U_{k,M}} \{\langle R(M,k)x, x\rangle + S_{k,M}(u_k^M)\} \\ &= \langle R(M,k)x, x\rangle + \min_{u_k^M \in U_{k,M}} S_{k,M}(u_k^M). \end{aligned}$$

Since $\min\limits_{u_k^M \in U_{k,M}} S_{k,M}(u_k^M) = S_{k,M}(\widetilde{u}_k^M) = 0$ we deduce

$$\langle R(M,k)x, x\rangle = \min_{u_k^M \in U_{k,M}} V(M, k, x, u_k^M) \geq 0.$$

We obtain $R(M,k) \geq 0$ and the induction is complete. Thus the sequence $R(M,n)$ is well defined for all $0 < n \leq M$ and is called *solution of the Riccati equation* (17) *with the final condition* $R(M,M) = 0$.

Now, we obtain, for all $0 < k < M$

$$\min_{u_k^M \in U_{k,M}} V(M, k, x, u_k^M) = V(M, k, x, \widetilde{u}_k^M) = \langle R(M,k)x, x\rangle . \tag{23}$$

Let $\widetilde{u}_k^{M-1} = \{\widetilde{u}_k, \widetilde{u}_{k+1}, \ldots, \widetilde{u}_{M-2}\}$. It is clear that $\widetilde{u}_k^{M-1} \in U_{k,M-1}$ and

$$V(M-1, k, x, \widetilde{u}_k^{M-1}) \leq V(M, k, x, \widetilde{u}_k^M).$$

Using the above inequality and (23), we get $R(M,k) \geq R(M-1,k) \geq 0$ for all $M-1 \geq k > 0$. □

Proposition 25. *Suppose* (2) *is stabilizable. Then the Riccati equation* (17) *admits a nonnegative bounded on* $\mathbf{N}^*$ *solution.*

Proof. Since (2) is stabilizable it follows that there exists a bounded on $\mathbf{N}^*$ sequence $F = \{F_n\}_{n\in\mathbf{N}^*}$, $F_n \in L(H,U)$ such that $\{A + DF, B\}$ is uniformly exponentially stable. Then, there exist $a \in (0,1)$ and $\beta \geq 1$ such that $E\left\|x_n(k,x)\right\|^2 \leq \beta a^{n-k}\left\|x\right\|^2$, for all $n \geq k > 0$ and $x \in H$.

Let us consider $\overline{u} = \{\overline{u}_n = F_n x_n, n \geq k\}$, where x_n is the solution of $\{A + DF, B\}$ with the initial condition $x_k = x$. Since F_n is bounded on $\mathbf{N}^*$, it is not difficult to see that $\overline{u} \in \widetilde{U}_{k,x}$. If $\overline{u}_k^M = \{\overline{u}_n, M-1 \geq n \geq k\}$, we have

$$\begin{aligned} V(M,k,x,\overline{u}_k^M) &= E\sum_{n=k}^{M-1}[\|C_nx_n\|^2 + \langle K_n\overline{u}_n, \overline{u}_n\rangle] \\ &= E\sum_{n=k}^{M-1}[\|C_nx_n\|^2 + \langle K_nF_nx_n, F_nx_n\rangle] \\ &\leq \sum_{n=k}^{M-1}(\widetilde{C}^2 + \widetilde{K}\widetilde{F}^2)E\|x_n\|^2 \leq \sum_{n=k}^{\infty}\eta E\|x_n\|^2 \end{aligned}$$

for all $M > k$, where $\eta = \widetilde{C}^2 + \widetilde{K}\widetilde{F}^2$ and $\widetilde{L} = \sup_{n\in\mathbf{N}^*}\|L_n\|$, $L = C, K, F$. For all $M > k > 0$, we get

$$V(M,k,x,\overline{u}_k^M) \leq \eta\sum_{n=k}^{\infty}\beta a^{n-k}\|x\|^2 = \lambda\|x\|^2.$$

Let $R(M,n)$ be the solution of the Riccati equation (17) with the final condition $R(M,M) = 0$. Using (23), we deduce that, for all $M > k > 0$

$$\lambda\|x\|^2 \geq V(M,k,x,\overline{u}_k^M) \geq \langle R(M,k)x, x\rangle.$$

From Lemma 24 and the last inequality, it follows

$$0 \leq R(M-1,k) \leq R(M,k) \leq \lambda I$$

for all $M-1 \geq k > 0$.

Thus, there exists $R(k) \in L(H)$ such that $0 \leq R(M,k) \leq R(k) \leq \lambda I$ for $M \in \mathbf{N}^*$, $M \geq k > 0$ and $R(M,k) \underset{M\to\infty}{\to} R(k)$, in the strong operator topology.

We denote $L = \lim_{M\to\infty}(< \mathcal{G}_n(R(M,n+1))x, x > - < \mathcal{G}_n(R(n+1))x, x >)$, $P_{M,n} = K_n + D_n^*R(M,n+1)D_n$ and $P_n = K_n + D_n^*R(n+1)D_n$. From the definition of $\mathcal{G}_n$ we get

$$\begin{aligned} L = &\lim_{M\to\infty}\Big(\Big\langle P_{M,n}^{-1}D_n^*R(M,n+1)A_nx, D_n^*R(M,n+1)A_nx\Big\rangle \\ &- \left\langle P_n^{-1}D_n^*R(n+1)A_nx, D_n^*R(n+1)A_nx\right\rangle\Big) \\ = &\lim_{M\to\infty}\Big(\Big\langle\left(P_{M,n}^{-1} - P_n^{-1}\right)D_n^*R(n+1)A_nx, D_n^*R(n+1)A_nx\Big\rangle \\ &+ \Big\langle P_{M,n}^{-1}D_n^*\left[R(M,n+1) - R(n+1)\right]A_nx, y_{M,n}\Big\rangle\Big), \end{aligned}$$

where we denote $y_{M,n} = D_n^*\left[R(M,n+1) + R(n+1)\right]A_nx$. Thus,

$$\begin{aligned} |L| \leq &\lim_{M\to\infty}\Big(\left\|P_{M,n}^{-1}\right\|\left\|D_n^*\left[R(M,n+1) - R(n+1)\right]A_nx\right\| \\ &\cdot\left\|D_n^*\left[R(M,n+1) + R(n+1)\right]A_nx\right\| \\ &+ \Big\langle\left(P_{M,n}^{-1} - P_n^{-1}\right)D_n^*R(n+1)A_nx, D_n^*R(n+1)A_nx\Big\rangle\Big). \end{aligned}$$

We put

$$L_1 = \lim_{M\to\infty} \left\| P_{M,n}^{-1} \right\| \left\| D_n^* \left[R(M,n+1) - R(n+1) \right] A_n x \right\| \\ \cdot \left\| D_n^* \left[R(M,n+1) + R(n+1) \right] A_n x \right\|$$

and

$$L_2 = \lim_{M\to\infty} \left\langle \left(P_{M,n}^{-1} - P_n^{-1} \right) D_n^* R(n+1) A_n x, D_n^* R(n+1) A_n x \right\rangle .$$

Since $P_{M,n} \geq K_n \geq \delta I$, $\delta > 0$ we deduce that $\left\| P_{M,n}^{-1} \right\| \leq \frac{1}{\delta}$ for all $M \geq n+1 \geq k$. From the strong convergence of $\{R(M,n+1)\}_{M\in\mathbf{N}^*,\, M\geq n+1}$ to R_{n+1} it follows $L_1 = 0$.

We see that $\left\| P_{M,n}^{-1} x - P_n^{-1} x \right\| \leq \left\| P_{M,n}^{-1} \right\| \left\| P_{M,n} y - P_n y \right\|$, where $y = P_n^{-1} x$. Since $\lim_{M\to\infty} \left\| P_{M,n} y - P_n y \right\| = 0$ we get $\lim_{M\to\infty} \left\| P_{M,n}^{-1} x - P_n^{-1} x \right\| = 0$.

Now, it is clear that $L_2 = 0$. Hence $L = 0$ and

$$\lim_{M\to\infty} \left\langle \mathcal{G}_n(R(M,n+1))x, x \right\rangle = \left\langle \mathcal{G}_n(R(n+1))x, x \right\rangle .$$

From the definition of $R(M,n)$ and the above result, we deduce that $R(n)$ is a solution of (17). The proof is complete. □

The next result is the infinite dimensional version of Proposition 9 in [**7**], where we replace the Markov perturbations with independent random perturbations.

Proposition 26. *If the system* $\{A, B, C\}$ *is detectable, then every nonnegative bounded solution of* (17) *is stabilizing.*

Proof. Let us consider the bounded on $\mathbf{N}^*$ sequence $P = \{P_n\}_{n\in\mathbf{N}^*}, P_n \in L(V,H)$ such that the stochastic system $\{A + PC, B\}$ is uniformly exponentially stable. Let R_n be a nonnegative, bounded on $\mathbf{N}^*$ solution of (17).

We will prove, according Definition 22, that the stochastic system $\{A+DF, B\}$, where F is given by (18), is uniformly exponentially stable. If the control sequence $\overline{u} \in \widetilde{U}_{k,x}$ is defined as in the previous proposition, we get

$$E \left\langle R_n x_n, x_n \right\rangle = \left\langle R_k x, x \right\rangle - V(n,k,x,\overline{u}_k^n)$$

and $V(n,k,x,\overline{u}_k^n) \leq \left\langle R_k x, x \right\rangle \leq M \left\| x \right\|^2$, where $M > 0$ is the positive constant such as $R_n \leq MI$ for all $n \in \mathbf{N}^*$. Consequently $\lim_{n\to\infty} V(n,k,x,\overline{u}_k^n) \leq M \left\| x \right\|^2$. The system $\{A + DF, B\}$ can be written

$$\begin{cases} x_{n+1} &= (A_n + P_n C_n) x_n + \xi_n B_n x_n + \nu_n \\ x_k &= x \in H \end{cases}$$

with $\nu_n = (D_n F_n - P_n C_n) x_n$. Using (1), and H_1 we obtain successively

$$E \left\| \nu_n \right\|^2 \leq \frac{2\widetilde{D}^2}{\delta} \delta E \left\| \overline{u}_n \right\|^2 + 2\widetilde{P}^2 E \left\| C_n x_n \right\|^2$$

and

$$E \left\| \nu_n \right\|^2 \leq \kappa (E \left\langle K_n \overline{u}_n, \overline{u}_n \right\rangle + E \left\| C_n x_n \right\|^2),$$

where $\kappa = \max\{\frac{2\widetilde{D}^2}{\delta}, 2\widetilde{P}^2\}$ and $\widetilde{L} = \sup\limits_{n\in\mathbf{N}^*} \|L_n\|$ for $L = D, P$. Hence

$$\sum_{n=k}^{\infty} E\|\nu_n\|^2 \le \kappa \lim_{n\to\infty} V(n,k,x,\overline{u}_k^n) \le \widetilde{M}\|x\|^2, \qquad \widetilde{M} = \kappa M.$$

We denote by $\overline{X}(n,k)$, respectively $\widehat{X}(n,k)$, the random evolution operators associated with the systems $\{A+DF, B\}$ and $\{A+PC, B\}$. It is not difficult to see that $\overline{X}(n,k)x = \widehat{X}(n,k)x + \sum\limits_{p=k}^{n-1} \widehat{X}(n,p+1)\nu_p, n \ge k+1$.

Let $\widehat{T}(n,k)$ be the operator defined by the Theorem 5, for the uniformly exponentially stable system $\{A+PC, B\}$, and let $\beta \ge 1$ and $a \in (0,1)$ be such that $E\left\|\widehat{X}(n,k)x\right\|^2 \le \beta a^{n-k}$ for all $n \ge k > 0$. Taking expectations we have

$$\begin{aligned}
&E\left\|\overline{X}(n,k)x\right\|^2 \\
&\quad\le 2E\left\|\widehat{X}(n,k)x\right\|^2 + 2E\left(\sum_{p=k}^{n-1} a^{\frac{n-p-1}{4}} a^{\frac{p-n+1}{4}} \left\|\widehat{X}(n,p+1)\nu_p\right\|\right)^2 \\
&\quad\le 2E\left\|\widehat{X}(n,k)x\right\|^2 + 2\sum_{p=k}^{n-1} a^{\frac{n-p-1}{2}} \sum_{p=k}^{n-1} a^{\frac{p-n+1}{2}} E\left\langle \widehat{T}(n,p+1)(I)\nu_p, \nu_p\right\rangle \\
&\quad\le 2[\beta a^{n-k}\|x\|^2 + \frac{a^{\frac{n-k}{2}}-1}{a^{\frac{1}{2}}-1} \sum_{p=k}^{n-1} a^{\frac{p-n+1}{2}} \left\|\widehat{T}(n,p+1)\right\| E\|\nu_p\|^2] \\
&\quad\le 2[\beta a^{n-k}\|x\|^2 + \frac{\beta}{1-a^{\frac{1}{2}}} \sum_{p=k}^{n-1} a^{\frac{n-p-1}{2}} E\|\nu_p\|^2].
\end{aligned}$$

Now it is not difficult to deduce that

$$\sum_{n=k+1}^{\infty} \left\langle \overline{T}(n,k)(I)x, x\right\rangle = \sum_{n=k+1}^{\infty} E\left\|\overline{X}(n,k)x\right\|^2 \le 2\beta\left(\frac{1}{1-a} + \frac{\widetilde{M}}{(1-a^{\frac{1}{2}})^2}\right)\|x\|^2.$$

By the proof of Theorem 13 in [**8**], it follows the uniform exponential stability of the system $\{A+DF, B\}$. The proof is complete. □

Theorem 27. *Assume that the system* $\{A, B; C\}$ *is uniformly observable. If* R_n *is a nonnegative, bounded on* $\mathbf{N}^*$ *solution of* (17)*, then:*

- a) *there exists* $m > 0$ *such that* $R_n \ge mI$*, for all* $n \in \mathbf{N}^*$ *(*R_n *is uniformly positive on* $\mathbf{N}^*$*).*
- b) R_n *is stabilizing for* (2).

Proof. The main idea is the one of [**6**]. Let R_n be a nonnegative, bounded on $\mathbf{N}^*$ solution of (17) and let $\widetilde{X}(n,k)$ be the random evolution operator associated to the stochastic system $\{A+DF, B\}$ with F_n given by (18). Let n_0 and ρ be the numbers introduced by the Definition 11. We consider the operator $T_n \in \mathcal{H}$ given by

$$\langle T_n x, x\rangle = \sum_{j=n}^{n+n_0} \left(E\left\| C_j \widetilde{X}(j,n)x \right\|^2 + E\left\langle K_j F_j \widetilde{X}(j,n)x, F_j \widetilde{X}(j,n)x \right\rangle \right).$$

We will prove that $\inf\{\langle T_n x, x\rangle, n \in \mathbf{N}^*, \|x\| = 1\} > 0$.

Assume, by contradiction, that $inf\{\langle T_n x, x\rangle, n \in \mathbf{N}^*, \|x\| = 1\} = 0$. Then for every $\varepsilon > 0$ there exist $n_\varepsilon \in \mathbf{N}^*, x_\varepsilon \in H, \|x_\varepsilon\| = 1$ such that $\langle T_{n_\varepsilon} x_\varepsilon, x_\varepsilon\rangle < \varepsilon$. Let us denote $\widehat{x}_j^\varepsilon = \widetilde{X}(j, n_\varepsilon)x_\varepsilon$, $u_j^\varepsilon = F_j \widehat{x}_j^\varepsilon$. From (1) it follows

$$\varepsilon > \langle T_{n_\varepsilon} x_\varepsilon, x_\varepsilon\rangle \geq \delta E \sum_{j=n_\varepsilon}^{n_\varepsilon+n_0} \left\| u_j^\varepsilon \right\|^2. \tag{24}$$

On the other hand, $\varepsilon > \langle T_{n_\varepsilon} x_\varepsilon, x_\varepsilon\rangle \geq \sum_{j=n_\varepsilon}^{n_\varepsilon+n_0} E \left\| C_j \widetilde{X}(j, n_\varepsilon)x_\varepsilon \right\|^2$.

From Lemma 7, we deduce that for all $j \geq n_\varepsilon + 1$ we have

$$\widetilde{X}(j, n_\varepsilon)x_\varepsilon = X(j, n_\varepsilon)x_\varepsilon + \sum_{i=n_\varepsilon}^{j-1} X(j, i+1)D_i u_i^\varepsilon.$$

So, we obtain

$$\begin{aligned}
\varepsilon &> \sum_{j=n_\varepsilon+1}^{n_\varepsilon+n_0} E \left\| C_j \widetilde{X}(j, n_\varepsilon)x_\varepsilon \right\|^2 + E\left\| C_{n_\varepsilon} x_\varepsilon \right\|^2 \\
&\geq \sum_{j=n_\varepsilon+1}^{n_\varepsilon+n_0} E \left\| C_j X(j, n_\varepsilon)x_\varepsilon + C_j \sum_{i=n_\varepsilon}^{j-1} X(j, i+1)D_i u_i^\varepsilon \right\|^2 + \frac{1}{2} E\left\| C_{n_\varepsilon} x_\varepsilon \right\|^2 \\
&\geq \frac{1}{2}\left[\left(\sum_{j=n_\varepsilon+1}^{n_\varepsilon+n_0} E\left\| C_j X(j, n_\varepsilon)x_\varepsilon \right\|^2 + E\left\| C_{n_\varepsilon} x_\varepsilon \right\|^2 \right) \right. \\
&\qquad \left. - 2\widetilde{C}^2 \sum_{j=n_\varepsilon+1}^{n_\varepsilon+n_0} E \left\| \sum_{i=n_\varepsilon}^{j-1} X(j, i+1)D_i u_i^\varepsilon \right\|^2 \right].
\end{aligned}$$

From H_1 and Corollary 2, it follows that

$$E\left\| X(n,k)\eta \right\|^2 \leq \max\{1, (\widetilde{A}^2 + \widetilde{b}\widetilde{B}^2)^{n-k}\} E\left\| \eta \right\|^2$$

for all $n \geq k$ and $\eta \in L_k^2(H)$. Since D_i is bounded on $\mathbf{N}^*$ we can use the above results and (24) to deduce

$$E\left\| \sum_{i=n_\varepsilon}^{j-1} X(j, i+1)D_i u_i^\varepsilon \right\|^2 \leq n_0 \widetilde{D}^2 \mu_{n_0} \sum_{i=n_\varepsilon}^{n_\varepsilon+n_0} E\left\| u_i^\varepsilon \right\|^2 \leq c\varepsilon,$$

where $\mu_{n_0} = \max\{1, (\widetilde{A}^2 + \widetilde{b}\,\widetilde{B}^2)^{n_0}\}$, $\widetilde{D} = \sup\limits_{n \in \mathbf{N}^*} \|D_n\|$ and $c = \frac{n_0 \widetilde{D}^2 \mu_{n_0}}{\delta}$.

Now we have $\varepsilon > \frac{1}{2}(\sum\limits_{j=n_\varepsilon}^{n_\varepsilon+n_0} E\left\|C_{n_\varepsilon}X(j,n_\varepsilon)x_\varepsilon\right\|^2 - 2\widetilde{C}^2 n_0 c\varepsilon)$ and, from Definition 11, we obtain $\varepsilon > \frac{1}{2}(\rho - 2C^2 n_0 c\varepsilon)$.

We get a contradiction because $\varepsilon > 0$ is arbitrary. Hence there exists $m > 0$ such that

$$\langle T_n x, x\rangle \geq m\left\|x\right\|^2 \tag{25}$$

for all $n \in \mathbf{N}^*$ and $x \in H$. We have (see the proof of Lemma 24)

$$E\left\langle R_{n_0+n+1}\widetilde{X}(n_0+n+1,n)x, \widetilde{X}(n_0+n+1,n)x\right\rangle - \langle R_n x, x\rangle = -\langle T_n x, x\rangle\,.$$

From the last equality and from the hypothesis ($R_n$is bounded on $\mathbf{N}^*$, that is there exists $M > m$ such that $R_n \leq MI$), we have

$$m\left\|x\right\|^2 \leq \langle T_n x, x\rangle \leq \langle R_n x, x\rangle \leq M\left\|x\right\|^2. \tag{26}$$

Now, we obtain from (25)

$$\begin{aligned} E\left\langle R_{n_0+n+1}\widetilde{X}(n_0+n+1,n)x, \widetilde{X}(n_0+n+1,n)x\right\rangle - \langle R_n x, x\rangle &\leq -m\left\|x\right\|^2 \\ &\leq -m/M\,\langle R_n x, x\rangle\,. \end{aligned}$$

Thus $E\left\langle R_{n_0+n+1}\widetilde{X}(n_0+n+1,n)x, \widetilde{X}(n_0+n+1,n)x\right\rangle \leq q\,\langle R_n x, x\rangle$ for all $n \in \mathbf{N}^*$ and $x \in H$, where $q = 1 - m/M, q \in (0,1)$.

Let $\widetilde{T}(n,k)$ be the operator introduced by Theorem 5 for the random evolution operator $\widetilde{X}(n,k)$. The previous inequality is equivalent with

$$\widetilde{T}(n_0+n+1,n)\left(R_{n_0+n+1}\right) \leq qR_n.$$

Since $\widetilde{T}(n,k)$ is monotone (that is $\widetilde{T}(n,k)(P) \leq \widetilde{T}(n,k)(R)$ for $P \leq R$, $n \geq k$) we deduce

$$\widetilde{T}(n,k)\left(\widetilde{T}(n_0+n+1,n)\left(R_{n_0+n+1}\right)\right) \leq q\widetilde{T}(n,k)(R_n)$$

and

$$\widetilde{T}(n_0+n+1,k)R_{n_0+n+1} \leq q\widetilde{T}(n,k)(R_n)$$

for all $n \geq k$.

Let $n \geq k$ be arbitrary. Then there exists $c, r \in \mathbf{N}$ such that $n-k = (n_0+1)c+r$ and $0 \leq r \leq n_0$. We obtain by induction:

$$\widetilde{T}(n,k)(R_n) \leq q^c\widetilde{T}(r+k,k)(R_r).$$

From (26) and Theorem 5 we get $m\widetilde{T}(n,k)(I) \leq Mq^c\left\|\widetilde{X}(r+k,k)\right\|^2 I$.

Using Corollary 2, we put $G = M\max\limits_{0\leq r\leq n_0}\{(\widetilde{A}^2 + \widetilde{b}\widetilde{B}^2)^r\}$ and we get

$$m\widetilde{T}(n,k)(I) \leq q^c G I.$$

We take $a = q^{1/(n_0+1)}$, $\beta = q^{-n_0/(n_0+1)}(G/m) \geq 1$ and it follows

$$\widetilde{T}(n,k)(I) \leq \beta a^{n-k} I.$$

From Theorem 5 we deduce

$$E\left\|\widetilde{X}(n,k)x\right\|^2 \leq \beta a^{n-k}\left\|x\right\|^2$$

for all $x \in H$ and $0 < k \leq n$, $k, n \in \mathbf{N}^*$. Therefore R_n is stabilizing for (2). The proof is complete. □

Now, we can state the main result of this section.

Theorem 28. *Assume that:*

1) *System* (2) *is stabilizable;*
2) *System* $\{A, B; C\}$ *is either uniformly observable or detectable;*

Then the Riccati equation (17) *admits a unique nonnegative, bounded on* $\mathbf{N}^*$ *and stabilizing solution.*

Proof. From the Proposition 25 and the assumption 1) we deduce that (17) admits a nonnegative, bounded on $\mathbf{N}^*$ solution. Now, we use the above theorem or Proposition 26 and 2) to deduce that this solution is stabilizing. A stabilizing and bounded on $\mathbf{N}^*$ solution of the Riccati equation is unique, by Proposition 23. The proof is complete. □

The above theorem is proved in [**6**] for the discrete time stochastic systems in finite dimensional spaces. The continuous case for stochastic systems on infinite dimensional spaces is treated in [**9**].

6. QUADRATIC CONTROL

The theorem below gives a solution to the control problem (2), (3).

Theorem 29. *Assume* H_1 *holds. If the hypotheses of the Theorem 28 hold, then*

$$\min_{u \in U_{k,x}} I_k(x,u) = I_k(x,\widetilde{u}) = \langle R_k x, x\rangle,$$

where $R_n, n \in \mathbf{N}^*$ *is the solution of* (17), $\widetilde{u} = \{\widetilde{u}_n = F_n\widetilde{x}_n,\ n \geq k > 0\}$, F_n *is given by* (18) *and* $\widetilde{x}_n$ *is the solution of the stochastic system* $\{A + DF, B\}$, $F = \{F_n\}_{\mathbf{n}\in\mathbf{N}^*}$.

Proof. Let x_n be the solution of system (2) and R_n be the unique solution of (17). Arguing as in the proof of Lemma 24 we have

$$\begin{aligned} E\langle R_n x_n, x_n\rangle = {} & \langle R_k x, x\rangle - E\sum_{i=k}^{n-1}[\|C_i x_i\|^2 + \langle K_i u_i, u_i\rangle] \\ & + E\sum_{i=k}^{n-1}\langle (K_i + D_i^* R_{i+1} D_i)(u_i - F_i x_i), (u_i - F_i x_i)\rangle \end{aligned} \tag{27}$$

Let $\widetilde{u} = \{\widetilde{u}_n = F_n\widetilde{x}_n, n \geq k > 0\}$, where F_n is given by (18) and $\widetilde{x}_n$ is the solution of the stochastic system $\{A + DF; B\}$. It is easy to see that $\widetilde{u} \in \widetilde{U}_{k,x}$.

Then (27) becomes

$$E\left\langle R_n\widetilde{x}_n,\widetilde{x}_n\right\rangle = \left\langle R_kx,x\right\rangle - E\sum_{i=k}^{n-1}[\|C_i\widetilde{x}_i\|^2 + \langle K_i\widetilde{u}_i,\widetilde{u}_i\rangle]. \tag{28}$$

Since R_n is bounded on $\mathbf{N}^*$ (there exists $M > 0$ such that $\langle R_nx,x\rangle \leq M\|x\|^2$ for all $x \in H$ and $n \in \mathbf{N}^*$) and stabilizing, we get

$$E\left\langle R_n\widetilde{x}_n,\widetilde{x}_n\right\rangle \leq ME\left\|\widetilde{x}_n\right\| \underset{n\to\infty}{\rightarrow} 0.$$

As $n \to \infty$ in (28) and by using the Monotone Convergence Theorem, it follows $\langle R_kx,x\rangle = I_k(x,\widetilde{u})$ and consequently $\widetilde{u} \in U_{k,x}$. Thus

$$\min_{u\in U_{i,x}} I_k(x,u) \leq I_k(x,\widetilde{u}) = \left\langle R_kx,x\right\rangle.$$

Let $R(M,k)$ be the solution of (17) with $R(M,M) = 0$. If $u \in U_{k,x}$ it is clear that the sequence $\overline{u}_k^M = \{u_k,u_{k+1},\ldots,u_{M-1}\}$ belongs to $U_{k,M}$. From Lemma 24 we have:

$$\langle R(M,k)x,x\rangle \leq V(M,k,x,\overline{u}_k^M) \leq I_k(x,u),$$

for all $M > k$. As $M \to \infty$, it follows

$$\langle R_kx,x\rangle \leq I_k(x,u)$$

for all $u \in U_{k,x}$. The conclusion follows. □

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