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APPROXIMATION OF FUNCTIONS OF TWO VARIABLES BY SOME LINEAR POSITIVE OPERATORS

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Abstract. We introduce certain positive linear operators in weighted spaces of functions of two variables and we study approximation properties of these operators. We give theorems on the degree of approximation of functions from polynomial and exponential weighted spaces by introduced operators, using norms of these spaces.

I. Introduction

Approximation properties of Szasz-Mirakyan operators

$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1}$$

$x \in R_0 = [0, +\infty)$, $n \in N := \{1, 2, \dots\}$, in polynomial weighted spaces C_p were examined in [**1**]. The space C_p, $p \in N_0 := \{0, 1, 2, \dots\}$, is associated with the weighted function

$$w_0(x) := 1, \quad w_p(x) := (1 + x^p)^{-1}, \qquad \text{if} \qquad p \geq 1, \tag{2}$$

and consists of all real-valued functions f, continuous on R_0 and such that $w_p f$ is uniformly continuous and bounded on R_0. The norm on C_p is defined by the formula

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in R_0} w_p(x) \, |f(x)|. \tag{3}$$

In [**1**] there were proved theorems on the degree of approximation of $f \in C_p$ by the operators S_n defined by (1). From these theorems it was deduced that

$$\lim_{n \to \infty} S_n(f;x) = f(x), \tag{4}$$

for every $f \in C_p$, $p \in N_0$ and $x \in R_0$. Moreover the convergence (4) is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

The Szasz-Mirakyan operators are important in approximation theory. They have been studied intensively, in connection with different branches of analysis, such as numerical analysis. Recently in many papers various modifications of

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S_n were introduced [**4**]–[**8**], [**12**]–[**15**], [**19**], [**20**]. Approximation properties of modified Szasz-Mirakyan operators

$$L_n(f;r;x) := \frac{1}{g\left((nx+1)^2;r\right)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n(nx+1)}\right), \qquad x \in R_0, \quad n \in N, \tag{5}$$

where

$$g(t;r) := \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, \qquad t \in R_0, \tag{6}$$

i.e.

$$g(0;r) = \frac{1}{r!}, \qquad g(t,r) = \frac{1}{t^r}\left(e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!}\right) \qquad \text{if} \quad t > 0,$$

in polynomial weighted spaces were examined in [**13**].

In [**13**] it was proved that if $f \in C_p$, $p \in N_0$, then

$$\|L_n(f;r;\cdot) - f(\cdot)\|_p \le M_1 \omega_1\left(f; C_p; \frac{1}{n}\right), \qquad n, r \in N, \tag{7}$$

where

$$\omega_1(f;C_p;t) := \sup_{0 \le h \le t} \|\Delta_h f(\cdot)\|_p, \qquad t \in R_0, \tag{8}$$

where $\Delta_h f(x) := f(x+h) - f(x)$ for $x, h \in R_0$ and $M_1 = \text{const} > 0$.

In particular, if $f \in C_p^1$, $p \in N_0$, then

$$\|L_n(f;r;\cdot) - f(\cdot)\|_p \le \frac{M_2}{n}, \qquad n, r \in N, \tag{9}$$

where $M_2 = \text{const} > 0$. The above inequalities estimate the rate of uniform convergence of $\{L_n(f;r;\cdot)\}$

In [**14**] there were proved theorems on the degree of approximation of $f \in C_p$ by operators A_n defined by

$$A_n(f;r;\alpha;x) := \frac{1}{g\left((n^\alpha x+1)^2;r\right)} \sum_{k=0}^{\infty} \frac{(n^\alpha x+1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n^\alpha(n^\alpha x+1)}\right). \tag{10}$$

The degree of approximation is similar and in some cases better than for approximation by L_n.

Similar results in exponential weighted spaces can be found in [**15**], [**17**].

Thus the question arises, whether the operators introduced in [**18**] for function of two variables can be similarly modified. In connection with this question we introduce the operators (15).

II. Approximation in polynomial weighted spaces

1. Preliminaries

1.1. For given $p, q \in N_0$, we define the weighted function

$$w_{p,q}(x,y) := w_p(x)w_q(y), \qquad (x,y) \in R_0^2 := R_0 \times R_0, \tag{11}$$

and the weighted space $C_{p,q}$ of all real-valued functions f continuous on R_0^2 for which $w_{p,q}f$ is uniformly continuous and bounded on R_0^2. The norm on $C_{p,q}$ is defined by the formula

$$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup_{(x,y)\in R_0^2} w_{p,q}(x,y)\,|f(x,y)|. \tag{12}$$

The modulus of continuity of $f \in C_{p,q}$ we define as usual by the formula

$$\omega(f, C_{p,q}; t, s) := \sup_{0\le h\le t,\, 0\le\delta\le s} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{p,q}\,, \qquad t, s \ge 0, \tag{13}$$

where $\Delta_{h,\delta} f(x,y) := f(x+h, y+\delta) - f(x,y)$ and $(x+h, y+\delta) \in R_0^2$. Moreover let $C_{p,q}^1$ be the set of all functions $f \in C_{p,q}$ which first partial derivatives belong also to $C_{p,q}$.

From (13) it follows that

$$\lim_{t,s\to 0+} \omega(f, C_{p,q}; t, s) = 0 \tag{14}$$

for every $f \in C_{p,q}$, $p, q \in N_0$.

1.2. In this paper we introduce the following class of operators in $C_{p,q}$.

Definition 1. Fix $r, s \in N := \{1, 2, \cdots\}$ and $\alpha > 0$. Define a class of operators $A_{m,n}(f; r, s, \alpha)$ by the formula

$$\begin{aligned} A_{m,n}(f; r, s, \alpha; x, y) \equiv A_{m,n}(f; x, y) &:= \frac{1}{g\left((m^\alpha x+1)^2; r\right) g\left((n^\alpha y+1)^2; s\right)} \\ &\cdot \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(m^\alpha x+1)^{2j}}{(j+r)!} \frac{(n^\alpha y+1)^{2k}}{(k+s)!} f\left(\frac{j+r}{m^\alpha(m^\alpha x+1)}, \frac{k+s}{n^\alpha(n^\alpha y+1)}\right) \end{aligned} \tag{15}$$

for $(x,y) \in R_0^2$, $m, n \in N$.

The methods used to prove the Lemmas and the Theorems are similar to those used in construction of modified Szasz-Mirakyan operators [**16**], [**18**].

From (15), (10), (6) we deduce that $A_{m,n}(f; r, s)$ are well defined in every space $C_{p,q}$, $p, q \in N_0$. Moreover for fixed $r, s \in N$ and $\alpha > 0$ we have

$$A_{m,n}(1; r, s, \alpha; x, y) = 1 \quad \text{for} \quad (x,y) \in R_0^2, \quad m, n \in N, \tag{16}$$

and if $f \in C_{p,q}$ and $f(x,y) = f_1(x) f_2(y)$ for all $(x,y) \in R_0^2$, then

$$A_{m,n}(f; r, s, \alpha; x, y) = A_m(f_1; r, \alpha; x) A_n(f_2; s, \alpha; y) \tag{17}$$

for all $(x,y) \in R_0^2$ and $m, n \in N$.

In this paper by $M_k(\beta_1, \beta_2)$ we shall denote suitable positive constants depending only on indicated parameters β_1, β_2.

2. Lemmas and theorems

2.1. In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

From (10) and (6) we get for $x \in R_0$ and $n \in N$

$$\begin{aligned} &A_n(1; r, \alpha; x) = 1, \\ &A_n(t - x; r, \alpha; x) = \frac{1}{n^\alpha} + \frac{1}{n^\alpha(n^\alpha x + 1)(r-1)!g((n^\alpha x + 1)^2; r)} \end{aligned} \tag{18}$$

$$A_n((t-x)^2; r, \alpha; x) = \frac{2}{n^{2\alpha}} + \frac{r + (n^\alpha x + 1)^2 - 2n^\alpha x(n^\alpha x + 1)}{n^{2\alpha}(n^\alpha x + 1)^2(r-1)!g((n^\alpha x + 1)^2; r)}. \tag{19}$$

In the paper [**14**] was proved the following lemma for $A_n(f; r, \alpha)$ defined by (10).

Lemma 1. *For every fixed $p \in N_0$, $r \in N$ and $\alpha > 0$ there exist positive constants $M_i \equiv M_i(p, r)$, $i = 3, 4$, such that for all $x \in R_0$, $n \in N$*

$$w_p(x)\, A_n\left(1/w_p(t); r, \alpha; x\right) \le M_1, \tag{20}$$

$$w_p(x)\, A_n\left((t-x)^2/w_p(t); r, \alpha, x\right) \le \frac{M_2}{n^{2\alpha}}. \tag{21}$$

Applying Lemma 1 we shall prove the main lemma on $A_{m,n}$ defined by (15).

Lemma 2. *Fix $p, q \in N_0$, $r, s \in N$ and $\alpha > 0$. Then there exists a positive constant $M_5 \equiv M_5(p, q, r, s)$ such that*

$$\left\| A_{m,n}\left(1/w_{p,q}(t, z); r, s, \alpha; \cdot, \cdot\right) \right\|_{p,q} \le M_5 \quad \textit{for} \quad m, n \in N. \tag{22}$$

Moreover for every $f \in C_{p,q}$ we have

$$\left\| A_{m,n}\left(f; r, s, \alpha; \cdot, \cdot\right) \right\|_{p,q} \le M_5 \left\| f \right\|_{p,q} \quad \textit{for} \quad m, n \in N, \ r, s \in N. \tag{23}$$

The formulas (15), (5) and the inequality (23) show that $A_{m,n}$, $m, n \in N$, defined by (15) are linear positive operators from the space $C_{p,q}$ into $C_{p,q}$.

Proof. The inequality (22) follows immediately from (11), (17) and (20).

From (15) and (12) we get for $f \in C_{p,q}$ and $r, s \in N$

$$\left\| A_{m,n}(f; r, s, \alpha) \right\|_{p,q} \le \left\| f \right\|_{p,q} \left\| A_{m,n}(1/w_{p,q}; r, s, \alpha) \right\|_{p,q}, \quad m, n \in N,$$

which by (22) implies (23). This completes the proof of Lemma 2. □

2.2. Now we shall give two theorems on the degree of approximation of functions by $A_{m,n}$.

Theorem 1. *Suppose that* $f \in C^1_{p,q}$ *with fixed* $p, q \in N_0$. *Then there exists a positive constant* $M_6 = M_6(p,q,r,s)$ *such that for all* $m, n \in N$ *and* $r, s \in N$

$$\|A_{m,n}(f; r, s, \alpha; \cdot, \cdot) - f(\cdot, \cdot)\|_{p,q} \le M_4 \left\{ \frac{1}{m^\alpha} \|f'_x\|_{p,q} + \frac{1}{n^\alpha} \|f'_y\|_{p,q} \right\}. \tag{24}$$

Proof. Let $(x,y) \in R^2_0$ be a fixed point. Then for $f \in C^1_{p,q}$ we have

$$f(t,z) - f(x,y) = \int_x^t f'_u(u,z)du + \int_y^z f'_v(x,v)dv, \qquad (t,z) \in R^2_0.$$

From this and by (16) we get

$$\begin{aligned} A_{m,n}(f(t,z); r, s, \alpha; x, y) - f(x,y) &= A_{m,n}\left(\int_x^t f'_u(u,z)du; r, s, \alpha; x, y \right) \\ &\quad + A_{m,n}\left(\int_y^z f'_v(x,v)dv; r, s, \alpha; x, y \right). \end{aligned} \tag{25}$$

By (2), (11), (12) we have

$$\begin{aligned} \left| \int_x^t f'_u(u,z)du \right| &\le \|f'_x\|_{p,q} \left| \int_x^t \frac{du}{w_{p,q}(u,z)} \right| \\ &\le \|f'_x\|_{p,q} \left(\frac{1}{w_{p,q}(t,z)} + \frac{1}{w_{p,q}(x,z)} \right) |t-x|, \end{aligned}$$

which by (2), (10) (11), (15) and (16)–(18) implies that

$$\begin{aligned} & w_{p,q}(x,y) \left| A_{m,n}\left(\int_x^t f'_u(u,z)du; r, s, \alpha; x, y \right) \right| \\ &\quad \le w_{p,q}(x,y) A_{m,n}\left(\left| \int_x^t f'_u(u,z)du \right|; r, s, \alpha; x, y \right) \\ &\quad \le \|f'_x\|_{p,q}\, w_{p,q}(x,y) \left\{ A_{m,n}\left(\frac{|t-x|}{w_{p,q}(t,z)}; r, s, \alpha; x, y \right) \right. \\ &\qquad \left. + A_{m,n}\left(\frac{|t-x|}{w_{p,q}(x,z)}; r, s, \alpha; x, y \right) \right\} \\ &\quad \le \|f'_x\|_{p,q}\, w_q(y) A_n\left(\frac{1}{w_q(z)}; s; \alpha y \right) \\ &\qquad \cdot \left\{ w_p(x) A_m\left(\frac{|t-x|}{w_p(t)}; r, \alpha; x \right) + A_m\left(|t-x|; r; x \right) \right\}. \end{aligned}$$

Applying the Hölder inequality and (18)–(21), we get

$$\begin{aligned} A_m\left(|t-x|; r, \alpha; x \right) &\le \left\{ A_m((t-x)^2; r, \alpha; x) A_m(1; r, \alpha; x) \right\}^{\frac{1}{2}} \\ &\le \frac{M_7(p,r)}{m^\alpha}, \end{aligned}$$

$$\begin{aligned}
&w_p(x)A_m\left(\frac{|t-x|}{w_p(t)};r,\alpha;x\right)\\
&\qquad\le\left\{w_p(x)A_m\left(\frac{(t-x)^2}{w_p(t)};r,\alpha;x\right)\right\}^{\frac{1}{2}}\left\{w_p(x)A_m\left(\frac{1}{w_p(t)};r,\alpha;x\right)\right\}^{\frac{1}{2}}\\
&\qquad\le\frac{M_8(p,r)}{m^\alpha}
\end{aligned}$$

for $x\in R_0$ and $m\in N$. This implies that

$$w_{p,q}(x,y)\left|A_{m,n}\left(\int_x^t f'_u(u,z)du;r,s,,\alpha;x,y\right)\right|\le\frac{M_9(p,q,r,s)}{m^\alpha}\|f'_x\|_{p,q},\quad m\in N.$$

Analogously we obtain

$$w_{p,q}(x,y)\left|A_{m,n}\left(\int_y^z f'_v(x,v)dv;r,s,\alpha;x,y\right)\right|\le\frac{M_{10}(p,q,r,s)}{n^\alpha}\|f'_y\|_{p,q},\quad n\in N.$$

Combining these estimations, we derive from (25)

$$w_{p,q}(x,y)\left|A_{m,n}(f;r,s;x,y)-f(x,y)\right|\le M_{11}\left\{\frac{1}{m^\alpha}\|f'_x\|_{p,q}+\frac{1}{n^\alpha}\|f'_y\|_{p,q}\right\},$$

for all $m,n\in N$, where $M_{11}=M_{11}(p,q,r,s)=\text{const}>0$. This ends the proof of (24). □

Theorem 2. *Suppose that $f\in C_{p,q}$, $p,q\in N_0$. Then there exists a positive constant $M_{11}\equiv M_{11}(p,q,r,s)$ such that*

$$\|A_{m,n}(f;r,s,\alpha;\cdot,\cdot)-f(\cdot,\cdot)\|_{p,q}\le M_{11}\,\omega\left(f,C_{p,q};\frac{1}{m^\alpha},\frac{1}{n^\alpha}\right)\tag{26}$$

for all $m,n\in N$, $r,s\in N$ and $\alpha>0$.

Proof. We apply the Steklov function $f_{h,\delta}$ for $f\in C_{p,q}$

$$f_{h,\delta}(x,y):=\frac{1}{h\delta}\int_0^h du\int_0^\delta f(x+u,y+v)dv,\quad (x,y)\in R_0^2,h,\delta>0.\tag{27}$$

From (27) it follows that

$$\begin{aligned}
f_{h,\delta}(x,y)-f(x,y)&=\frac{1}{h\delta}\int_0^h du\int_0^\delta\Delta_{u,v}f(x,y)dv,\\
(f_{h,\delta})'_x(x,y)&=\frac{1}{h\delta}\int_0^\delta\left(\Delta_{h,v}f(x,y)-\Delta_{0,v}f(x,y)\right)dv,\\
(f_{h,\delta})'_y(x,y)&=\frac{1}{h\delta}\int_0^h\left(\Delta_{u,\delta}f(x,y)-\Delta_{u,0}f(x,y)\right)du.
\end{aligned}$$

This implies that $f_{h,\delta}\in C^1_{p,q}$ for $f\in C_{p,q}$ and $h,\delta>0$. Moreover

$$\|f_{h,\delta}-f\|_{p,q}\le\omega\left(f,C_{p,q};h,\delta\right),\tag{28}$$

$$\left\|(f_{h,\delta})'_x\right\|_{p,q}\le 2h^{-1}\omega\left(f,C_{p,q};h,\delta\right),\tag{29}$$

$$\left\|(f_{h,\delta})'_y\right\|_{p,q}\le 2\delta^{-1}\omega(f,C_{p,q};h,\delta),\tag{30}$$

for all $h, \delta > 0$. Observe that

$$\begin{aligned} & w_{p,q}(x,y)\,|A_{m,n}(f;r,s,\alpha;x,y)-f(x,y)| \\ &\qquad \le w_{p,q}(x,y)\,\{|A_{m,n}\,(f(t,z)f_{h,\delta}(t,z);r,s,\alpha;x,y)| \\ &\qquad\quad + \ |A_{m,n}\,(f_{h,\delta}(t,z);r,s,\alpha;x,y)-f_{h,\delta}(x,y)| \\ &\qquad\quad + |f_{h,\delta}(x,y)-f(x,y)|\} := T_1+T_2+T_3. \end{aligned}$$

By (12), (23) and (28) we obtain

$$T_1 \le \ \|A_{m,n}(f-f_{h,\delta};r,s,\alpha;\cdot,\cdot)\|_{p,q} \le M_5\,\|f-f_{h,\delta}\|_{p,q} \le M_5\,\omega(f,C_{p,q};h,\delta),$$

$$T_3 \le \omega\left(f,C_{p,q};h,\delta\right).$$

Applying Theorem 1 and (29) and (30), we get

$$\begin{aligned} T_2 \le &\ \ M_6\left\{\frac{1}{m^\alpha}\,\|(f_{h,\delta})'_x\|_{p,q}+\frac{1}{n^\alpha}\,\left\|(f_{h,\delta})'_y\right\|_{p,q}\right\} \\ \le &\ \ 2M_6\omega(f,C_{p,q};h,\delta)\left\{h^{-1}\frac{1}{m^\alpha}+\delta^{-1}\frac{1}{n^\alpha}\right\}. \end{aligned}$$

From the above we deduce that there exists a positive constant $M_{13} \equiv M_{13}(p,q,r,s)$ such that

$$\begin{aligned} &\|A_{m,n}(f;r,s,\alpha;\cdot,\cdot)-f(\cdot,\cdot)\|_{p,q} \\ &\qquad\qquad \le M_{13}\omega\left(f,C_{p,q};h,\delta\right)\left\{1+h^{-1}\frac{1}{m^\alpha}+\delta^{-1}\frac{1}{n^\alpha}\right\}, \end{aligned} \tag{31}$$

for $m, n \in N$ and $h, \delta > 0$. Now, for $m, n \in N$ setting $h = \frac{1}{m^\alpha}$ and $\delta = \frac{1}{n^\alpha}$ to (31), we obtain (26). □

From Theorem 2 and the property (14) it follows that

Corollary. *Let $f \in C_{p,q}$, $p, q \in N_0$. Then for $r, s \in N$ and $\alpha > 0$ we have*

$$\lim_{m,n\to\infty}\|A_{m,n}(f;r,s,\alpha;\cdot,\cdot)-f(\cdot,\cdot)\|_{p,q}=0. \tag{32}$$

III. Approximation in exponential weighted spaces

3. Preliminaries

3.1. Let as in [**15**], for a fixed $p, q > 0$,

$$v_{2p}(x) := \exp\left(-2px\right), \qquad x \in R_0, \tag{33}$$

and

$$v_{2p,2q}(x,y) := v_{2p}(x)v_{2q}(y), \qquad (x,y) \in R_0^2. \tag{34}$$

Denote by $C_{2p,2q}$ the set of all real-valued functions f continuous on R_0^2 for which $v_{2p,2q}f$ is uniformly continuous and bounded on R_0^2 The norm on $C_{2p,2q}$ is defined by

$$\|f\|_{2p,2q} \equiv \|f\left(\cdot,\cdot\right)\|_{2p,2q} := \sup_{(x,y)\in R_0^2} v_{2p,2q}(x,y)\,|f(x,y)|\,. \tag{35}$$

The modulus of continuity of function $f \in C_{2p,2q}$ we define as in section 1.1. by formula

$$\omega(f, C_{2p,2q}; t, z) := \sup_{0\le h\le t,\, 0\le\delta\le z} \|\Delta_{h,\delta} f(\cdot,\cdot)\|_{2p,2q}, \qquad t, z \ge 0,$$

and we have

$$\lim_{t,z\to 0+} \omega(f, C_{2p,2q}; t, z) = 0 \qquad \text{for } f \in C_{2p,2q}. \tag{36}$$

Analogously as in section 1.1, for fixed $p, q > 0$, we denote by $C^1_{2p,2q}$ the set of all functions $f \in C_{2p,2q}$ which first partial derivatives belong also to $C_{2p,2q}$.

3.2. Similarly as in Section II we introduce

Definition 2. Fix $r, s \in N$ and $\alpha > 0$. For functions $f \in C_{2p,2q}$, $p, q > 0$, we define the operators

(37)

$$B_{m,n}(f; p, q, r, s, \alpha; x, y) \equiv B_{m,n}(f; x, y) := \frac{1}{g\left((m^\alpha x+1)^2; r\right) g\left((n^\alpha y+1)^2; s\right)}$$
$$\cdot \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(m^\alpha x+1)^{2j}}{(j+r)!}\frac{(n^\alpha y+1)^{2k}}{(k+s)!} f\left(\frac{j+r}{m^\alpha(m^\alpha x+1)+2p}, \frac{k+s}{n^\alpha(n^\alpha y+1)+2q}\right)$$

for $(x, y) \in R_0^2$, $m, n \in N$.

In [**15**] there were examined the operators

$$B_n(f; x) \equiv B_n(f; q, r, \alpha; x)$$
$$:= \frac{1}{g\left((n^\alpha x+1)^2; r\right)} \sum_{k=0}^{\infty} \frac{(n^\alpha x+1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n^\alpha(n^\alpha x+1)+2q}\right) \tag{38}$$

for functions f of one variable, belonging to exponential weighted spaces.

In this paper we shall give similar results for operators $B_{m,n}(f)$.

4. Lemmas and theorems

4.1. In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems. From (37) and (6) we deduce that $B_{m,n}(f)$ is well-defined in every space $C_{2p,2q}$, $p, q > 0$,$r, s \in N$. In particular

$$B_{m,n}(1; x, y) = 1, \qquad (x, y) \in R_0^2, \quad m, n \in N, \tag{39}$$

and if $f \in C_{2p,2q}$ and $f(x, y) = f_1(x)f_2(y)$ for all $(x, y) \in R_0^2$, then

$$B_{m,n}(f; p, q, r, s, \alpha; x, y) = B_m(f_1; p, r, \alpha; x)B_n(f_2; q, s, \alpha; y) \tag{40}$$

for all $(x, y) \in R_0^2$ and $m, n \in N$. Moreover from (38) and (6) we get

$$B_n(1; q, r; x) = 1 \qquad x \in R_0, \quad n \in N. \tag{41}$$

In the paper [**15**] the following two lemmas for $B_n(f; q, r; \cdot)$ defined by (38) were proved.

Lemma 3. *Let $q, \alpha > 0$, $r \in N$ be fixed numbers. Then for all $n \in N$ and $x \in R_0$, we have*

$$\begin{aligned} B_n\left(t-x;q,r,\alpha;x\right) &= \frac{(n^\alpha x+1)^2}{n^\alpha(n^\alpha x+1)+2q} - x + \\ &+ \frac{1}{(n(nx+1)+2q)(r-1)!g((nx+1)^2;r)}, \\ B_n\left((t-x)^2;q,r,\alpha;x\right) &= \left(\frac{(n^\alpha x+1)^2}{n^\alpha(n^\alpha x+1)+2q} - x\right)^2 + \left(\frac{n^\alpha x+1}{n^\alpha(n^\alpha x+1)+2q}\right)^2 \\ &+ \frac{r+(n^\alpha x+1)^2-2x(n^\alpha(n^\alpha x+1)+2q)}{(n^\alpha(n^\alpha x+1)+2q)^2(r-1)!g((n^\alpha x+1)^2;r)}, \\ B_n\left(\mathrm{e}^{2qt};q,r,\alpha;x\right) &= \frac{g\left((n^\alpha x+1)^2\mathrm{e}^{2q/(n^\alpha(n^\alpha x+1)+2q)};r\right)}{g((n^\alpha x+1)^2;r)}\mathrm{e}^{2qr/(n^\alpha(n^\alpha x+1)+2q)}, \end{aligned}$$

$$\begin{aligned} B_n\left((t-x)^2\mathrm{e}^{2qt};q,r,\alpha;x\right) = \Bigg[\left(\frac{(n^\alpha x+1)^2}{n^\alpha(n^\alpha x+1)+2q}\mathrm{e}^{2q/(n^\alpha(n^\alpha x+1)+2q)} - x\right)^2 \\ + \left(\frac{n^\alpha x+1}{n^\alpha(n^\alpha x+1)+2q}\right)^2 \mathrm{e}^{2q/(n^\alpha(n^\alpha x+1)+2q)}\Bigg] B_n\left(\mathrm{e}^{2qt};q,r,\alpha;x\right) \\ + \frac{r+(n^\alpha x+1)^2\mathrm{e}^{2q/(n^\alpha(n^\alpha x+1)+2q)}-2x(n^\alpha(n^\alpha x+1)+2q)}{(n^\alpha(n^\alpha x+1)+2q)^2(r-1)!g((n^\alpha x+1)^2;r)}\mathrm{e}^{2qr/(n^\alpha(n^\alpha x+1)+2q)}. \end{aligned}$$

Lemma 4. *For every fixed $q, \alpha > 0$ and $r \in N$ there exist positive constants $M_i \equiv M_i(p,r)$, $i = 14, 15$, such that for all $x \in R_0$, $n \in N$*

$$\begin{aligned} v_{2q}(x)\, B_n\left(1/v_{2q}(t);q,r,\alpha;x\right) &\le M_{14}, \\ v_{2q}(x)\, B_n\left((t-x)^2/v_{2q}(t);q,r,\alpha;x\right) &\le \frac{M_{15}}{n^{2\alpha}}. \end{aligned}$$

Applying (33) – (35) and (39) – (41) and Lemma 4 and arguing as in the proof of Lemma 2, we can prove the basic property of $B_{m,n}(f)$.

Lemma 5. *For fixed $p, q, \alpha > 0$ and $r, s \in N$ there exists a positive constant $M_{16} \equiv M_{16}(p,q,r,s)$ such that*

$$\left\|B_{m,n}\left(1/v_{2p,2q}(t,z);p,q,r,s,\alpha;\cdot,\cdot\right)\right\|_{2p,2q} \le M_{16} \quad \textit{for} \quad m,n \in N. \tag{42}$$

Moreover for every $f \in C_{2p,2q}$ we have

$$\left\|B_{m,n}\left(f;p,q,r,s;\cdot,\cdot\right)\right\|_{2p,2q} \le M_{16}\left\|f\right\|_{2p,2q} \quad \textit{for} \quad m,n \in N,\ r,s \in N. \tag{43}$$

The formula (37) and the inequality (43) show that $B_{m,n}$, $m,n \in N$, are linear positive operators from the space $C_{2p,2q}$ into $C_{2p,2q}$.

4.2. Applying Lemma 3 – Lemma 5 and (33)–(35) and (39)–(41) and reasoning as in the proof of Theorem 1, we can prove the following

Theorem 3. *Suppose that* $f \in C^1_{2p,2q}$ *with given* $p, q > 0$ *and* $r, s \in N$. *Then there exists a positive constant* $M_{17} = M_{17}(p,q,r,s)$ *such that for all* $m, n \in N$ *and* $\alpha > 0$

$$\|B_{m,n}(f;p,q,r,s,\alpha;\cdot,\cdot) - f(\cdot,\cdot)\|_{2p,2q} \le M_{17}\left\{\frac{1}{m^\alpha}\|f'_x\|_{2p,2q} + \frac{1}{n^\alpha}\|f'_y\|_{2p,2q}\right\}.$$

Theorem 4. *Suppose that* $f \in C_{2p,2q}$, $p, q, \alpha > 0$, $r, s \in N$. *Then there exists a positive constant* $M_{18} \equiv M_{18}(p,q,r,s)$ *such that*

$$\|B_{m,n}(f;p,q,r,s;\cdot,\cdot) - f(\cdot,\cdot)\|_{2p,2q} \le M_{18}\,\omega\left(f, C_{2p,2q}; \frac{1}{m^\alpha}, \frac{1}{n^\alpha}\right), \tag{44}$$

for all $m, n \in N$.

Proof. Similarly as in the proof of Theorem 2 we shall apply the Steklov function $f_{h,\delta}$ for $f \in C_{2p,2q}$, defined by (27). Analogously as in (28)–(30) we get

$$\|f_{h,\delta} - f\|_{2p,2q} \le \omega(f, C_{2p,2q}; h, \delta), \tag{45}$$

$$\left\|(f_{h,\delta})'_x\right\|_{2p,2q} \le 2h^{-1}\omega(f, C_{2p,2q}; h, \delta), \tag{46}$$

$$\left\|(f_{h,\delta})'_y\right\|_{2p,2q} \le 2\delta^{-1}\omega(f, C_{2p,2q}; h, \delta) \tag{47}$$

for all $h, \delta > 0$, which show that $f_{h,\delta} \in C^1_{2p,2q}$ if $f \in C_{2p,2q}$ and $h, \delta > 0$.

Now, for $B_{m,n}$, we can write

$$\begin{aligned} v_{2p,2q}(x,y)\,&|B_{m,n}(f;p,q,r,s,\alpha;x,y) - f(x,y)| \\ &\le v_{2p,2q}(x,y)\,\{|B_{m,n}(f(t,z) - f_{h,\delta}(t,z);p,q,r,s,\alpha;x,y)| \\ &\quad + |B_{m,n}(f_{h,\delta}(t,z);p,q,r,s,\alpha;x,y) - f_{h,\delta}(x,y)| \\ &\quad + |f_{h,\delta}(x,y) - f(x,y)|\} := T_1 + T_2 + T_3. \end{aligned}$$

By (35), (43) and (45), we get

$$\begin{aligned} T_1 &\le \|B_{m,n}(f - f_{h,\delta};p,q,r,s,\alpha;\cdot,\cdot)\|_{2p,2q} \\ &\le M_{16}\,\|f - f_{h,\delta}\|_{2p,2q} \le M_{14}\,\omega(f, C_{2p,2q}; h, \delta), \\ T_3 &\le \omega(f, C_{2p,2q}; h, \delta). \end{aligned}$$

Applying Theorem 3 and (46) and (47), we get

$$\begin{aligned} T_2 &\le M_{17}\left\{\frac{1}{m^\alpha}\left\|(f_{h,\delta})'_x\right\|_{2p,2q} + \frac{1}{n^\alpha}\left\|(f_{h,\delta})'_y\right\|_{2p,2q}\right\} \\ &\le 2M_{17}\omega(f, C_{2p,2q}; h, \delta)\left\{h^{-1}\frac{1}{m^\alpha} + \delta^{-1}\frac{1}{n^\alpha}\right\}. \end{aligned}$$

From the above we deduce that there exists a positive constant $M_{19} \equiv M_{19}(p,q,r,s)$ such that

$$\begin{aligned} &\|B_{m,n}(f;p,q,r,s,\alpha;\cdot,\cdot)-f(\cdot,\cdot)\|_{2p,2q} \\ &\qquad \le M_{19}\omega\left(f,C_{2p,2q};h,\delta\right)\left\{1+h^{-1}\frac{1}{m}+\delta^{-1}\frac{1}{n}\right\}, \end{aligned} \tag{48}$$

for $m,n \in N$ and $h,\delta > 0$. Now, for $m,n \in N$ setting $h=\frac{1}{m^\alpha}$ and $\delta=\frac{1}{n^\alpha}$ to (48), we obtain (44). □

Theorem 4 and (36) imply

Corollary. *Let $f \in C_{2p,2q}$, $p,q,\alpha > 0$, $r,s \in N$. Then*

$$\lim_{m,n\to\infty}\|B_{m,n}(f;p,q,r,s,\alpha;\cdot,\cdot)-f(\cdot,\cdot)\|_{p,q}=0.$$

Remark. Theorems and Corollaries in our paper show that $A_{m,n}$ and $B_{m,n}$, $m,n \in N$, give for $\alpha > 1/2$ a better degree of approximation of functions belonging to weighted spaces of functions of two variables than classical Szasz-Mirakyan operator $S_{m,n}$, examined for continuous and bounded functions in [**11**].

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