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SUBDIRECTLY IRREDUCIBLE QUASI-MODAL ALGEBRAS

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Abstract. In this paper we give suitable notions of congruences and subdirectly irreducible algebras for the class of quasi-modal algebras introduced in [**1**]. We also prove some characterizations of subdirectly irreducible algebras following the similar results given by G. Sambin [**5**] for modal algebras.

1. Introduction

In [**1**] we introduce the class of quasi-modal algebras as a generalization of the class of the modal algebras. A quasi-modal algebra is a Boolean algebra $\boldsymbol{A}$ endowed with a map of A into the lattice $\mathrm{Id}\,(\boldsymbol{A})$ of ideals of $\boldsymbol{A}$ satisfying certains conditions. This type of maps, called quasi-modal operators, are not operations in the Boolean algebra, but have some properties similar to modal operators. Since the quasi-modal operator is not an operation, we have not the usual notions of congruences and simple and subdirectly irreducible algebras. The main of this paper is to introduce and study the natural analogous notions for the class of quasi-modal algebras.

In Section 2 we will recall some notions on Boolean duality and the duality for quasi-modal algebras given in [**1**]. In Section 3 we shall introduce the notion of q-congruence. We will prove that the set of q-congruences and the set of Δ-filters (introduced in [**1**]) are isomorphic lattices. In the study of any class of algebras the knowledge of the simple and subdirectly irreducible algebras is very important. In Section 4 we will define a notion of a simple and subdirectly irreducible quasi-modal algebra. Following the results given by G. Sambin in [**5**] on subdirectly irreducible modal algebras, we shall determinate the simple and subdirectly irreducible algebras of some classes of quasi-modal algebras introduced in [**1**].

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2. Preliminaries

We assume that the reader is familiar with basic concepts of modal algebras (see [**2**] or [**5**]). We shall recall some concepts of the topological duality for quasi-modal algebras. For more details see [**1**].

A *Boolean space* X is a topological space that is *compact and totally disconnected*, i.e., for given distinct points $x, y \in X$, there is a closed and open subset U of X such that $x \in U$ and $y \notin U$. If X is a Boolean space, then the set $\mathrm{Clop}\,(X)$ of the all closed and open subsets of X (= clopen) is a basis for X and it is a Boolean algebra under set-theoretical complement and intersection. We shall denote by $\mathcal{O}\,(X)$ $(\mathcal{C}\,(X))$ the set of all open subsets (closed subsets) of X.

If $\boldsymbol{A} = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra, by $\mathrm{Ul}\,(\boldsymbol{A})$ we shall denote the set of all ultrafilters (or proper maximal filters) of $\boldsymbol{A}$ while by $\mathrm{Id}(\boldsymbol{A})$ and $\mathrm{Fi}(\boldsymbol{A})$ we shall denote the families of all ideals and filters of $\boldsymbol{A}$, respectively.

Let X be a Boolean space. The map $\varepsilon : X \to \mathrm{Ul}\,(\mathrm{Clop}(X))$ given by $\varepsilon\,(x) = \{U \in \mathrm{Clop}\,(X) : x \in U\}$ is a bijective and continuous function. With each Boolean algebra $\boldsymbol{A}$ we can associate a Boolean space whose points are the elements of $\mathrm{Ul}\,(\boldsymbol{A})$ with the topology determined by the clopen basis $\beta\,(\boldsymbol{A}) = \{\beta\,(a) : a \in A\}$, where $\beta\,(a) = \{P \in \mathrm{Ul}\,(A) : a \in P\}$. By the above considerations we have that, if X is a Boolean space, then $X \cong \mathrm{Ul}\,(\mathrm{Clop}\,(X))$, and if $\boldsymbol{A}$ is a Boolean algebra, then $\boldsymbol{A} \cong \mathrm{Clop}\,(\mathrm{Ul}\,(\boldsymbol{A}))$.

If $\boldsymbol{A}$ is a Boolean algebra and $\mathrm{Ul}\,(\boldsymbol{A})$ is the associated Boolean space, then there exists a duality between ideals (filters) of $\boldsymbol{A}$ and open (closed) sets of $\mathrm{Ul}(\boldsymbol{A})$. More precisely, if $\beta : \boldsymbol{A} \to \mathcal{P}\,(\mathrm{Ul}\,(\boldsymbol{A}))$ is the map given by $\beta\,(a) = \{P \in \mathrm{Ul}\,(\boldsymbol{A}) : a \in P\}$, then for $I \in \mathrm{Id}\,(\boldsymbol{A})$ and $F \in \mathrm{Fi}\,(\boldsymbol{A})$, we have that

$$\beta\,(I) = \{P \in \mathrm{Ul}\,(\boldsymbol{A}) : I \cap P \neq \emptyset\} \in \mathcal{O}\,(\mathrm{Ul}\,(\boldsymbol{A}))\,,$$

defines an isomorphism between $\mathrm{Id}\,(\boldsymbol{A})$ and $\mathcal{O}\,(\mathrm{Ul}\,(\boldsymbol{A}))$, and

$$\beta\,(F) = \{P \in \mathrm{Ul}\,(\boldsymbol{A}) : F \subseteq P\} \in \mathcal{C}\,(\mathrm{Ul}\,(\boldsymbol{A}))\,,$$

defines a dual-isomorphism between $\mathrm{Fi}\,(\boldsymbol{A})$ and $\mathcal{C}\,(\mathrm{Ul}\,(\boldsymbol{A}))$ (see [**4**] and [**5**] for further information on Boolean duality).

Let $\boldsymbol{A}$ be a Boolean algebra. The filter (ideal) generated by a subset $Y \subseteq A$ is denoted by $F\,(Y)$ $(I\,(Y))$. The set complement of a subset $Y \subseteq A$ will be denoted by Y^c or $A - Y$.

Definition 1. Let $\boldsymbol{A}$ be a Boolean algebra. A *quasi-modal operator* defined on $\boldsymbol{A}$ is a function $\Delta : A \to \mathrm{Id}\,(\boldsymbol{A})$ such that it satisfies the following conditions for all $a, b \in A$:

(Q1) $\Delta\,(a \wedge b) = \Delta a \cap \Delta b$,
(Q2) $\Delta 1 = A$.

A quasi-modal algebra, or *qm*-algebra, is a structure $\boldsymbol{A} = \langle A, \vee, \wedge, \neg, \Delta, 0, 1\rangle$ where $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and Δ is a quasi-modal operator.

The class of qm-algebras is denoted by $\mathcal{QMA}$. Let $\boldsymbol{A} \in \mathcal{QMA}$. We define the dual operator $\nabla : A \to \mathrm{Fi}(\boldsymbol{A})$ by $\nabla a = \neg\Delta\neg a$, where $\neg\Delta x = \{\neg y : y \in \Delta x\}$. It is easy to see that the operator ∇ satisfies the following conditions:

(Q3) $\nabla(a \vee b) = \nabla a \cap \nabla b$,

(Q4) $\nabla 0 = A$.

Recall that a *modal algebra* is an algebra $\langle A, \vee, \wedge, \neg, \Box, 0, 1\rangle$, where $\langle A, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and $\Box$ is an operator defined on A satisfying the following conditions:

(M1) $\Box(a \wedge b) = \Box a \wedge \Box b$,

(M2) $\Box 1 = 1$.

Let $\boldsymbol{A} \in \mathcal{QMA}$ with the property that for each $a \in A$, Δa is a principal ideal. If we define the function $\Box : A \to A$ by $\Box a = x$ such that $\Delta a = I(x)$, then the structure $\langle A, \vee, \wedge, \neg, \Box, 0, 1\rangle$ is a modal algebra, called the associated modal algebra of $\boldsymbol{A}$. On the other hand, any modal algebra $\langle A, \vee, \wedge, \neg, \Box, 0, 1\rangle$ has an associated quasi-modal algebra $\boldsymbol{A} = \langle A, \vee, \wedge, \neg, \Delta_\Box, 0, 1\rangle$, where the operator $\Delta_\Box$ is defined by $\Delta_\Box a = I(\Box a)$, for each $a \in A$.

Let $\boldsymbol{A} \in \mathcal{QMA}$. For each $P \in \mathrm{Ul}(\boldsymbol{A})$ we define the set

$$\Delta^{-1}(P) = \{a \in A : \Delta a \cap P \neq \emptyset\}.$$

Dually, we can define the set $\nabla^{-1}(P) = \{a \in A : \nabla a \subseteq P\}$.

Lemma 2. [**1**] *Let $\boldsymbol{A} \in \mathcal{QMA}$.*

1. *For each $P \in \mathrm{Ul}(\boldsymbol{A})$, $\Delta^{-1}(P) \in \mathrm{Fi}(\boldsymbol{A})$,*
2. *For each $a \in A$, $a \in \Delta^{-1}(P) \Leftrightarrow \forall Q \in \mathrm{Ul}(\boldsymbol{A})$: if $\Delta^{-1}(P) \subseteq Q$ then $a \in Q$.*

Let $\boldsymbol{A} \in \mathcal{QMA}$. We define on $\mathrm{Ul}(\boldsymbol{A})$ a binary relation $R_{\boldsymbol{A}}$ by

$$(P,Q) \in R_{\boldsymbol{A}} \;\Leftrightarrow\; \forall a \in A: \text{ if } \Delta a \cap P \neq \emptyset \text{ then } a \in Q \;\Leftrightarrow\; \Delta^{-1}(P) \subseteq Q. \tag{1}$$

We note that the relation $R_{\boldsymbol{A}}$ can be defined using the operator ∇ as $(P,Q) \in R_{\boldsymbol{A}} \Leftrightarrow Q \subseteq \nabla^{-1}(P)$.

Definition 3. A descriptive quasi-modal space, or a q-descriptive space for short, is a structure $\mathcal{F} = \langle X, R, D\rangle$ such that:

1. X is a Boolean space and $D = \mathrm{Clop}(X)$,
2. R is a binary relation defined on X such that for each $x \in X$, $R(x) = \{y \in X : (x,y) \in R\}$ is a closed subset of X, and
3. $\Delta_R(O) = \{x \in X : R(x) \subseteq O\} \in \mathcal{O}(X)$, for any $O \in D$.

Let $\mathcal{F} = \langle X, R, D\rangle$ be a q-descriptive space. Let us consider the structure

$$\mathcal{A}(\mathcal{F}) = \left\langle D, \cup, \cap, ^c, \overline{\Delta}, \emptyset, X\right\rangle,$$

where $\overline{\Delta} : D \to \mathrm{Id}(D)$ is defined by $\overline{\Delta}(O) = I(\Delta_R(O)) = \{U \in D : U \subseteq \Delta_R(O)\}$.

Theorem 4. [**1**] *Let $\mathcal{F} = \langle X, R, D\rangle$ be a q-descriptive space. Then $\mathcal{A}(\mathcal{F}) = \left\langle D, \cup, \cap, ^c, \overline{\Delta}, \emptyset, X\right\rangle$ is a quasi-modal algebra.*

Let $\mathcal{F} = \langle X, R, D \rangle$ be a q-descriptive space. Since $\mathcal{A}(\mathcal{F})$ is a quasi-modal algebra, we can define in the set $\mathrm{Ul}(D)$ the relation $R_D \subseteq \mathrm{Ul}(D)^2$ by $(P, Q) \in R_D \Leftrightarrow \Delta_R^{-1}(P) \subseteq Q$, where

$$\Delta_R^{-1}(P) = \{O \in D : I(\Delta_R O) \cap P \neq \emptyset\} = \{O \in D : \exists U \in P \quad U \subseteq \Delta_R(O)\}.$$

We note that R_D is the relation associated with the qm-algebra $\mathcal{A}(\mathcal{F})$ along the general description following Lemma 2. Since the space X is compact, for all $P \in \mathrm{Ul}(D)$ there exists $x \in X$ such that $\varepsilon(x) = P$. So,

$$\Delta_R^{-1}(\varepsilon(x)) = \{O \in D : \exists U \in \varepsilon(x) \quad U \subseteq \Delta_R(O)\} = \{O \in D : R(x) \subseteq O\},$$

and therefore we have that $(\varepsilon(x), \varepsilon(y)) \in R_D \Leftrightarrow \Delta_R^{-1}(\varepsilon(x)) \subseteq \varepsilon(y)$.

Definition 5. Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be two qm-algebras. A function $h : A_1 \to A_2$ is a q-homomorphism, if

1. h is a homomorphism of Boolean algebras, and
2. for any $a \in A_1$, $I(h(\Delta_1 a)) = \Delta_2(h(a))$.

A q-isomorphism is a Boolean isomorphism that is a q-homomorphism.

Theorem 6. [**1**] *Let $\boldsymbol{A} \in \mathcal{QMA}$. Then the structure $\mathcal{F}(\boldsymbol{A}) = \langle \mathrm{Ul}(\boldsymbol{A}), R_{\boldsymbol{A}}, \beta(\boldsymbol{A}) \rangle$ is a q-descriptive space such that $\mathcal{A}(\mathcal{F}(\boldsymbol{A})) \cong \boldsymbol{A}$.*

Theorem 7. [**1**] *Let $\mathcal{F} = \langle X, R, D \rangle$ be a q-descriptive space. Then the map $\varepsilon : X \to \mathrm{Ul}(D)$ is a bijective and continuous function such that $(\varepsilon(x), \varepsilon(y)) \in R_D$ iff $(x, y) \in R$, for any $x, y \in X$.*

3. Q-congruences

A filter F in a modal algebra $\langle A, \vee, \wedge, \neg, \Box, 0, 1 \rangle$ is said to be *open* if $\Box a \in F$ when $a \in F$. It is known that the lattice of open filters is isomorphic to the lattice of congruences (see [**5, 6**]). In [**1**] we introduce the following generalization of the notion of open filter.

Definition 8. Let $\boldsymbol{A}$ be a quasi-modal algebra. A filter F of $\boldsymbol{A}$ is called a Δ-filter, if $\Delta a \cap F \neq \emptyset$, provided $a \in F$.

Lemma 9. *Let $\boldsymbol{A} \in \mathcal{QMA}$. The set of all Δ-filters of $\boldsymbol{A}$ is a lattice.*

Proof. Let F_1, F_2 be Δ-filters. We prove that $F_1 \vee F_2$ is a Δ-filter. Let $a \in F_1 \vee F_2$. Then there exist $f_1 \in F_1$ and $f_2 \in F_2$ such that $f_1 \wedge f_2 \leq a$. So, $\Delta(f_1 \wedge f_2) = \Delta f_1 \cap \Delta f_2 \subseteq \Delta a$. Since $\Delta f_1 \cap F_1 \neq \emptyset$ and $\Delta f_2 \cap F_2 \neq \emptyset$, there exists $x_1 \in \Delta f_1 \cap F_1$ and there exists $x_2 \in \Delta f_2 \cap F_2$. As Δf_1, Δf_2 are ideals, $x_1 \wedge x_2 \in \Delta f_1 \cap \Delta f_2 \subseteq \Delta a$. It follows that $x_1 \wedge x_2 \in \Delta a \cap (F_1 \vee F_2)$. Therefore, $F_1 \vee F_2$ is a Δ-filter. The proof that $F_1 \wedge F_2 = F_1 \cap F_2$ is a Δ-filter is easy and left to the reader. $\Box$

The congruences in modal algebras are Boolean congruences compatible with the unary operator $\Box$ (see [**3**] and [**5**]). In the case of quasi-modal algebras we can define a notion of equivalence relation compatible with the boolean operations and *compatible,* in a certain sense, with the operator Δ.

Definition 10. Let $\boldsymbol{A} \in \mathcal{QMA}$. Let θ be a Boolean congruence of $\boldsymbol{A}$. Let $a, b \in A$ such that $(a, b) \in \theta$. We shall say that $(\Delta a, \Delta b) \in \theta^{\Delta}$ iff

(Co1) For each $x \in \Delta a$ there exists $y \in \Delta b$ such that $(x, y) \in \theta$, and

(Co2) for each $y \in \Delta b$ there exists $x \in \Delta a$ such that $(x, y) \in \theta$.

We shall say that a Boolean congruence θ is a q-congruence if for all $a, b \in A$, $(a, b) \in \theta$ implies that $(\Delta a, \Delta b) \in \theta^{\Delta}$.

Lemma 11. *Let $\langle A, \vee, \wedge, \neg, \Box, 0, 1\rangle$ be a modal algebra and let $\langle A, \vee, \wedge, \neg, \Delta_{\Box}, 0, 1\rangle$ be the associated quasi-modal algebra. Then θ is a modal congruence iff θ is a q-congruence.*

Proof. Let θ be a congruence of the modal algebra $\langle A, \vee, \wedge, \neg, \Box, 0, 1\rangle$. Let $(a, b) \in \theta$. We prove that for each $x \leq \Box a$ there exists $y \leq \Box b$ such that $(x, y) \in \theta$. Let $x \leq \Box a$. Since $(\Box a, \Box b) \in \theta$ and $x = x \wedge \Box a$,

$$(x, x \wedge \Box b) = (x \wedge \Box a, x \wedge \Box b) \in \theta.$$

So, $y = x \wedge \Box b \leq \Box b$ is the necessary element.

Let us suppose that θ is a q-congruence. Let $(a, b) \in \theta$. Since $(I(\Box a), I(\Box b)) \in \theta^{\Delta}$, $\Box a \in I(\Box a)$, and $\Box b \in I(\Box b)$, there exist $x \leq \Box a$ and $y \leq \Box b$ such that $(\Box a, y) \in \theta$ and $(x, \Box b) \in \theta$. Let $|x|_{\theta} = \{y \in A : (x, y) \in \theta\}$ be the equivalence class of an element $x \in A$. Then

$$\begin{aligned} |\Box a|_{\theta} &= |\Box a \vee x|_{\theta} = |\Box a|_{\theta} \vee |x|_{\theta} \\ &= |y|_{\theta} \vee |\Box b|_{\theta} = |y \vee \Box b|_{\theta} \\ &= |\Box b|_{\theta}. \end{aligned}$$

Thus, $(\Box a, \Box b) \in \theta$ and consequently θ is a congruence of the modal algebra $\langle A, \vee, \wedge, \neg, \Box, 0, 1\rangle$. $\Box$

Theorem 12. Let $\boldsymbol{A} \in \mathcal{QMA}$. Let F be a Δ-filter and let θ be a q-congruence. Then

$$F(\theta) = \{a \in A : (a, 1) \in \theta\}$$

is a Δ-filter, and

$$\theta(F) = \left\{(a, b) \in A^2 : a \wedge f = b \wedge f \text{ for some } f \in F\right\}$$

is a q-congruence such that $\theta(F(\theta)) = \theta$ and $F = F(\theta(F))$. Therefore, there exists an order-isomorphism between of the lattice of Δ-filters and the lattice of the q-congruences of $\boldsymbol{A}$.

Proof. Let F be a Δ-filter. We prove that $\theta(F)$ is a q-congruence. Let $a, b \in A$ such that $a \wedge f = b \wedge f$ for some $f \in F$. Let $x \in \Delta a$. Since $\Delta f \cap F \neq \emptyset$, there exists $z \in \Delta f \cap F$, and since Δa and Δf are ideals,

$$x \wedge z \in \Delta a \cap \Delta f = \Delta(a \wedge f) = \Delta(b \wedge f) = \Delta b \cap \Delta f.$$

So, the element $y = x \wedge z$ belongs to Δb and satisfies $x \wedge z = y \wedge z$. Thus, $(x, y) \in \theta(F)$.

Let $y \in \Delta b$. Similarly, we can prove that there exists $w \in \Delta a$ such that $(y, w) \in \theta(F)$. Thus, $(\Delta a, \Delta b) \in \theta(F)^{\Delta}$.

Let θ be a q-congruence. Let $a \in F(\theta)$. Since $(a,1) \in \theta$, we get $(\Delta a, \Delta 1) = (\Delta a, A) \in \theta^{\Delta}$. As $1 \in A$, there exists $x \in \Delta a$ such that $(x,1) \in \theta$, i.e., $x \in F(\theta)$. Thus, $\Delta a \cap F(\theta) \neq \emptyset$, and this implies that $F(\theta)$ is a Δ-filter.

Since the transformation $F \to \theta(F)$ is isotone and, hence, is the required order-isomorphism, we conclude that $\theta(F(\theta)) = \theta$ and $F = F(\theta(F))$. □

Definition 13. Let $\mathcal{F} = \langle X, R, D\rangle$ be a q-descriptive space. A subset $Y \subseteq X$ is called an R-saturated, if $R(x) \subseteq Y$ for each $x \in Y$. A subset $Y \subseteq X$ is called an R-subset, if Y is closed and R-saturated.

Let $\mathcal{F} = \langle X, R, D\rangle$ be a q-descriptive space. The family of subsets of X that are R-saturated will be denoted by $\mathcal{S}_R(X)$. It is easy to check that the intersection and union of any subfamily of $\mathcal{S}_R(X)$ is an R-saturated set. Moreover, since X and $\emptyset$ are R-saturated sets, the family $\mathcal{S}_R(X)$ is a complete sublattice of $\mathcal{P}(X)$. Also, we can see that the R-subsets, ordered by inclusion, form a sublattice of $\mathcal{C}(X)$ closed under arbitrary intersections. The family of subsets of X that are R-subsets will be denoted by $\mathcal{C}_R(X)$.

Theorem 14. [**1**] *Let $\boldsymbol{A}$ be a quasi-modal algebra. Then the lattice of Δ-filters of $\boldsymbol{A}$ is anti-isomorphic to the lattice of $R_{\boldsymbol{A}}$-subsets of $\mathcal{F}(\boldsymbol{A}) = \langle \mathrm{Ul}(\boldsymbol{A}), R_{\boldsymbol{A}}, \beta(\boldsymbol{A})\rangle$.*

Corollary 15. *Let $\boldsymbol{A} \in \mathcal{QMA}$. There exists an anti-isomorphism between the lattice of the q-congruences of $\boldsymbol{A}$ and the lattice of $R_{\boldsymbol{A}}$-subsets of $\mathcal{F}(\boldsymbol{A})$.*

Let $\boldsymbol{A} \in \mathcal{QMA}$. Let θ be a q-congruence of $\boldsymbol{A}$. We shall define a quotient structure A/θ as follows:

$$A/\theta = \{|x|_\theta : x \in A\},$$

where $|x|_\theta = \{y \in A : (x,y) \in \theta\}$ is the equivalence class of x. Since θ is a Boolean congruence, A/θ is a Boolean algebra. We define on A/θ a structure of quasi-modal algebra taking a quasi-modal operator Δ_θ as follows:

$$\Delta_\theta |a| = I(\{|x| : x \in \Delta a\}),$$

i.e., $\Delta_\theta |a|$ is the ideal in A/θ generated by the set $\{|x| : x \in \Delta a\}$.

Theorem 16. *Let $\boldsymbol{A} \in \mathcal{QMA}$. Let θ be a q-congruence of $\boldsymbol{A}$. Then the structure*

$$\boldsymbol{A}/\theta = \langle A/\theta, \vee, \wedge, \Delta_\theta, \neg, |0|_\theta, |1|_\theta\rangle$$

is a quasi-modal algebra.

Proof. It is easy and left to the reader. □

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{QMA}$. Let us recall that a q-homomorphism is a Boolean homomorphism $h : A \to B$ such that $I(h(\Delta a)) = \Delta h(a)$, for all $a \in A$. The *kernel* of h is the relation

$$\mathrm{Ker}\, h = \left\{(a,b) \in A^2 : h(a) = h(b)\right\}.$$

It is known that $\mathrm{Ker}\, h$ is a Boolean congruence.

Theorem 17. *The kernel of a q-homomorphism is a q-congruence.*

Proof. Let $a, b \in A$ such that $h(a) = h(b)$. Let $x \in \Delta a$. Since $h(x) \in \Delta h(a) = \Delta h(b)$, there exists $y \in \Delta b$ such that $h(x) = h(y)$, i.e., $(x, y) \in \mathrm{Ker}h$. Thus, $\mathrm{Ker}h$ is a q-congruence. □

Finally, we can give a Homomorphism Theorem, whose proof is easy and left to the reader.

Theorem 18. *Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{QMA}$ and let $h : A \to B$ be a q-homomorphism. Then, $\boldsymbol{A}/\,\mathrm{Ker}h \cong h(\boldsymbol{A})$.*

4. Simple and subdirectly irreducible algebras

In the above section we introduced a satisfactory notion of congruence for quasi-modal algebras, in the sense that we can define an appropriate notion of quotient structure. Then we may expect that it is possible also to give an adequate notion of simple and subdirectly irreducible quasi-modal algebra. In this section we shall define these notions and we give characterizations of simple and subdirectly irreducible algebras of some classes of quasi-modal algebras.

Definition 19. Let $\boldsymbol{A} \in \mathcal{QMA}$. We shall say that $\boldsymbol{A}$ *is subdirectly irreducible* iff there exists a minimal non trivial q-congruence θ **in** $\boldsymbol{A}$. Similarly, we shall say that $\boldsymbol{A}$ is simple if $\boldsymbol{A}$ has only two q-congruences.

By Theorem 14 and Corollary 15, we can affirm that a quasi-modal algebra $\boldsymbol{A}$ is simple iff it has only two Δ-filters iff there are only two $R_{\boldsymbol{A}}$-subsets on $\mathrm{Ul}(\boldsymbol{A})$, and $\boldsymbol{A}$ is subdirectly irreducible iff there is a minimal non-trivial Δ-filter in $\boldsymbol{A}$ iff there is a maximal $R_{\boldsymbol{A}}$-subset of $\mathrm{Ul}(\boldsymbol{A})$ distinct from $\mathrm{Ul}(\boldsymbol{A})$ and $\emptyset$.

Let $\mathcal{F} = \langle X, R, D\rangle$ be a q-descriptive space. We define $R_0 = \{(x, x) : x \in X\}$ and $R^{n+1} = R \circ R^n$. The reflexive and transitive closure of R is denoted as R^∞. We note that $R^\infty(x) = \bigcup_{n\geq 0} R^n(x)$, for $x \in X$. The domain of the relation R is the set $\mathrm{dom}R = \{x \in X : (x, y) \in R \text{ for some } y \in X\}$.

As the family $\mathcal{C}_R(X) = \mathcal{C}(X) \cap \mathcal{S}_R(X)$ is closed under arbitrary intersections, we can define the set

$$Y_x = \bigcap \{Y : x \in Y \text{ and } Y \in \mathcal{C}_R(X)\},$$

for each $x \in X$. It is clear that $Y_x \in \mathcal{C}_R(X)$.

Let us define the set

$$\mathcal{I}_{\mathcal{F}} = \{x \in X : Y_x = X\},$$

Let $\mathcal{H}_{\mathcal{F}} = X - \mathcal{I}_{\mathcal{F}}$. We note that the set $\mathcal{I}_{\mathcal{F}}$ is not exactly the same set as the set considered in [**5**].

Theorem 20. *Let $\boldsymbol{A} \in \mathcal{QMA}$. Then:*

1. *$\boldsymbol{A}$ is simple iff $Y_P = \mathrm{Ul}(\boldsymbol{A})$, for each $P \in \mathrm{Ul}(\boldsymbol{A})$.*
2. *$\boldsymbol{A}$ is subdirectly irreducible but non-simple iff the set $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})} \in \mathcal{C}_{R_{\boldsymbol{A}}}(\mathrm{Ul}(\boldsymbol{A})) - \{\emptyset, \mathrm{Ul}(\boldsymbol{A})\}$.*

Proof. 1. ($\Rightarrow$) Let $P \in \mathrm{Ul}(\boldsymbol{A})$.
Since $P \in Y_P$ and $Y_P \in \mathcal{C}_{R_{\boldsymbol{A}}}(\mathrm{Ul}(\boldsymbol{A})) = \{\emptyset, \mathrm{Ul}(\boldsymbol{A})\}$, $Y_P = \mathrm{Ul}(\boldsymbol{A})$.

($\Leftarrow$) Let $Y \in \mathcal{C}_{R_{\boldsymbol{A}}}(\mathrm{Ul}(\boldsymbol{A})) - \{\emptyset\}$.
Then there exists $P \in X(\boldsymbol{A})$ such that $P \in Y$. It follows that $Y_P \subseteq Y$. Thus, $Y = \mathrm{Ul}(\boldsymbol{A})$, and consequently $\boldsymbol{A}$ is simple.

2. ($\Rightarrow$) Assume that $\boldsymbol{A}$ is subdirectly irreducible but non-simple.
So there exists a minimal non-trivial Δ-filter F of $\boldsymbol{A}$. Let us consider the set

$$\beta(F) = \{P \in \mathrm{Ul}(\boldsymbol{A}) : F \subseteq P\}.$$

It is clear that $\beta(F)$ is an $R_{\boldsymbol{A}}$-subset. It is enough to prove that $\beta(F) = \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$. If $F \subseteq P$, then $Y_P \subseteq \beta(F)$, because $\beta(F)$ is a $R_{\boldsymbol{A}}$-subset of $\mathrm{Ul}(\boldsymbol{A})$. Since $\beta(F) \neq \mathrm{Ul}(\boldsymbol{A})$, we get $P \in \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$.

Let $P \in \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$. Then $Y_P \neq \mathrm{Ul}(\boldsymbol{A})$. Then $\boldsymbol{A}$ is subdirectly irreducible and $\beta(F)$ is the maximal $R_{\boldsymbol{A}}$-subset of $\mathrm{Ul}(\boldsymbol{A})$ distinct from $\mathrm{Ul}(\boldsymbol{A})$ and $\emptyset$, $P \in Y_P \subseteq \beta(F)$. It follows that $F \subseteq P$, i.e., $P \in \beta(F)$. Thus, $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is an $R_{\boldsymbol{A}}$-subset of $\mathrm{Ul}(\boldsymbol{A})$.

($\Leftarrow$) Assume that $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})} \in \mathcal{C}_{R_A}(\mathrm{Ul}(\boldsymbol{A})) - \{\emptyset, \mathrm{Ul}(\boldsymbol{A})\}$.
We prove that $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is the maximal $R_{\boldsymbol{A}}$-subset of $\mathrm{Ul}(\boldsymbol{A})$. Let $P \in \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ and $Q \in R_{\boldsymbol{A}}(P)$. Then $Q \in Y_P$, and consequently $Y_Q \subseteq Y_P \neq \mathrm{Ul}(\boldsymbol{A})$. It follows that $Q \in \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$. Thus, $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is an $R_{\boldsymbol{A}}$-subset.

Let Z be an $R_{\boldsymbol{A}}$-subset of $\mathrm{Ul}(\boldsymbol{A})$ non-void and different from $\mathrm{Ul}(\boldsymbol{A})$. Let $P \in Z$. So, $Y_P \subseteq Z \neq \mathrm{Ul}(\boldsymbol{A})$, and thus $P \in \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$. So, $Z \subseteq \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$. Therefore, $\boldsymbol{A}$ is subdirectly irreducible. $\square$

Now we shall determine the simple and subdirectly irreducible algebra in some classes of quasi-modal algebras.

Theorem 21. [**1**] *Let $\boldsymbol{A} \in \mathcal{QMA}$. Then:*

1. $\Delta a \subseteq I(a)$ *for all* $a \in A \Leftrightarrow R_{\boldsymbol{A}}$ *is reflexive.*
2. $\Delta a \subseteq \Delta^2 a$ *for all* $a \in A \Leftrightarrow R_{\boldsymbol{A}}^2 \subseteq R_{\boldsymbol{A}}$, *i.e.,* $R_{\boldsymbol{A}}$ *is transitive.*
3. $I(a) \subseteq \Delta\nabla a = \bigcap_{x \in \nabla a} \Delta x$ *for all* $a \in A \Leftrightarrow R_{\boldsymbol{A}}$ *is symmetric.*

Definition 22. Let $\boldsymbol{A} \in \mathcal{QMA}$. We shall say that $\boldsymbol{A}$ is a *quasi-topological algebra* if it satisfies the conditions $\Delta a \subseteq I(a)$ and $\Delta a \subseteq \Delta^2 a$, for every $a \in A$.
A *quasi-monadic algebra* is a quasi-topological algebra $\boldsymbol{A}$ such that $I(a) \subseteq \Delta\nabla a$, for every $a \in A$.

We note that from Theorem 21 it follows that a quasi-modal algebra $\boldsymbol{A}$ is a quasi-topological algebra iff $R_{\boldsymbol{A}}$ is reflexive and transitive, and $\boldsymbol{A}$ is a quasi-monadic algebra iff the relation $R_{\boldsymbol{A}}$ is an equivalence.

Theorem 23. *Let $\boldsymbol{A}$ be a quasi-topological algebra.*

1. *$\boldsymbol{A}$ is simple if and only if for each $P \in \mathrm{Ul}(\boldsymbol{A})$, $R_{\boldsymbol{A}}(P) = \mathrm{Ul}(\boldsymbol{A})$.*
2. *$\boldsymbol{A}$ is subdirectly irreducible but non-simple if and only if there exists $a \in A$, $a \neq 1$ such that $I(a) = \Delta a$ and for every $x \neq 1$, $\Delta x \subseteq I(a)$.*

Proof. First, we note that since $R_{\boldsymbol{A}}$ is reflexive and transitive, $R_{\boldsymbol{A}}^{\infty}(P) = R_{\boldsymbol{A}}(P) \neq \emptyset$, for every $P \in \mathrm{Ul}(\boldsymbol{A})$. So, $R_{\boldsymbol{A}}(P)$ is a closed and R-saturated set, for each $P \in \mathrm{Ul}(\boldsymbol{A})$. Thus, $Y_P = R_{\boldsymbol{A}}(P)$, for each $P \in \mathrm{Ul}(\boldsymbol{A})$.

The assertion 1 follows from Theorem 20.

2. ($\Rightarrow$) It follows from Theorem 20.2 that $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is a proper and nonvoid closed $R_{\boldsymbol{A}}$-subset of $\mathrm{Ul}(\boldsymbol{A})$. We prove that $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is open. Let $P \in \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$. Then, $Y_P = R_{\boldsymbol{A}}(P) \neq \mathrm{Ul}(\boldsymbol{A})$. So, there exists $Q \in \mathrm{Ul}(\boldsymbol{A})$ such that $Q \notin R_{\boldsymbol{A}}(P)$. It follows that there exists $a \in A$ such that $R_{\boldsymbol{A}}(P) \subseteq \beta(a)$ and $a \notin Q$. Then, $P \in \beta(\Delta a) = \Delta_R \beta(a)$. Thus,

$$\mathcal{H}_{\mathcal{F}(\boldsymbol{A})} \subseteq \bigcup \{\Delta_R \beta(a) : a \in A - \{1\}\}.$$

Let $P \in \bigcup \{\Delta_R \beta(a) : a \in A - \{1\}\}$. Then $P \in \Delta_R \beta(a)$ for some $a \in A - \{1\}$. Thus, $R_{\boldsymbol{A}}(P) \neq \mathrm{Ul}(\boldsymbol{A})$, i.e., $P \in \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$. Therefore, $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is the union of open subsets, and consequently is it open. Since $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is closed, it follows that $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is a clopen subset of $\mathrm{Ul}(\boldsymbol{A})$. So, there exists $a \in A$ such that $\beta(a) = \mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$. Therefore $\mathcal{H}_{\mathcal{F}(\boldsymbol{A})}$ is an $R_{\boldsymbol{A}}$-subset,

$$\beta(a) = \beta(I(a)) = \Delta_R(\beta(a)) = \beta(\Delta a),$$

i.e., $I(a) = \Delta a$.

Let $x \neq 1$. Let $P \in \Delta_R(\beta(x)) = \beta(\Delta x)$. Then, $R_{\boldsymbol{A}}(P) \subseteq \beta(x)$, and hence $x \neq 1, R_A(P) \neq \mathrm{Ul}(\boldsymbol{A})$, i.e., $P \in \mathcal{H}_{\mathcal{F}(\boldsymbol{A})} = \beta(a)$. Thus,

$$\Delta_R(\beta(x)) = \beta(\Delta x) \subseteq \beta(a) = \beta(I(a)).$$

We conclude that $\Delta x \subseteq I(a)$.

($\Leftarrow$) Suppose that there exists $a \in A$, $a \neq 1$ such that $I(a) = \Delta a$ and for every $x \neq 1$, $\Delta x \subseteq I(a)$. Let us consider the filter $F(a)$. We prove that $F(a)$ is a minimal Δ-filter. Let $a \leq b \neq 1$. Then, $\Delta a \subseteq \Delta b$. By the hypothesis, $\Delta b \subseteq I(a) = \Delta a$. Thus, $\Delta a = \Delta b = I(a)$, and this implies that $\Delta b \cap F(a) \neq \emptyset$. So, $F(a)$ is a Δ-filter.

Let F be a Δ-filter. Let $b \in F$ with $b \neq 1$. Then $\Delta b \subseteq I(a)$, and hence $\Delta b \cap F \neq \emptyset$, $I(a) \cap F \neq \emptyset$, i.e., $a \in F$. Therefore, $F(a) \subseteq F$. Now, from Theorem 12, we deduce that $\boldsymbol{A}$ is subdirectly irreducible. □

Theorem 24. *Let $\boldsymbol{A}$ be a quasi-monadic algebra. Then the following conditions are equivalent:*

1. *$\boldsymbol{A}$ is simple*
2. *$\boldsymbol{A}$ is subdirectly irreducible,*
3. *For every $x \in A - \{1\}$, $\Delta x = \{0\}$,*
4. *For every $P \in \mathrm{Ul}(\boldsymbol{A})$, $R_{\boldsymbol{A}}(P) = \mathrm{Ul}(\boldsymbol{A})$.*

Proof. ($1 \Rightarrow 2$) is immediate.

($2 \Rightarrow 3$) Suppose that there exists $x \in A - \{1\}$, $\Delta x \neq \{0\}$. Then, $\Delta_{R_{\boldsymbol{A}}}(\beta(x)) \neq \emptyset$. So,

$$(\Delta_{R_{\boldsymbol{A}}}(\beta(x)))^c = \nabla_{R_{\boldsymbol{A}}}(\beta(\neg x)) \neq \mathrm{Ul}(A).$$

By Theorem 23, there exists $U = \beta(a) \neq \mathrm{Ul}(\boldsymbol{A})$ such that $\Delta_{R_A}(\beta(x)) \subseteq U$ and $\Delta_{R_{\boldsymbol{A}}}(\nabla_{R_{\boldsymbol{A}}}(\beta(\neg x))) \subseteq U$. Since $R_{\boldsymbol{A}}$ is an equivalence, it is easy to see that

$$\Delta_{R_{\boldsymbol{A}}}(\beta(x)) \cup \Delta_{R_{\boldsymbol{A}}}(\nabla_{R_{\boldsymbol{A}}}(\beta(\neg x))) = \mathrm{Ul}(\boldsymbol{A}).$$

Thus, $U = \mathrm{Ul}(\boldsymbol{A})$, which is a contradiction. Therefore, $\Delta_{R_{\boldsymbol{A}}}(\beta(x)) = \emptyset$, i.e. $\Delta x = \{0\}$.

(3 $\Rightarrow$ 4) Suppose that there exists $P \in \mathrm{Ul}(\boldsymbol{A})$ such that $R_{\boldsymbol{A}}(P) \neq \mathrm{Ul}(\boldsymbol{A})$. Then there exists $Q \in \mathrm{Ul}(\boldsymbol{A})$ and $a \in A$ such that $R_{\boldsymbol{A}}(P) \subseteq \beta(a)$ and $Q \notin \beta(a)$. It follows that $a \neq 1$, but by the hypothesis we have that $\Delta_{R_{\boldsymbol{A}}}(\beta(a)) = \emptyset$, which is a contradiction. Thus, $R_{\boldsymbol{A}}(P) = \mathrm{Ul}(\boldsymbol{A})$, for every $P \in \mathrm{Ul}(\boldsymbol{A})$.

(4 $\Rightarrow$ 1) Since $R_{\boldsymbol{A}}(P) = \mathrm{Ul}(\boldsymbol{A})$ for every $P \in \mathrm{Ul}(\boldsymbol{A})$, we conclude by assertion 1 of Theorem 23 that $\boldsymbol{A}$ is simple. □

References

1. Celani S. A., *Quasi-modal algebras*, Mathematica Bohemica **126**(4) (2001), 721–736.

2. Goldblatt R., *Mathematics of Modality,* CSLI Lecture Notes **43**, 1993.

3. Kracht M., *Tools and Techniques in Modal Logic*, Studies in Logic, **142**, Elsevier.

4. Koppelberg S., *Topological duality*, in Handbook of Boolean algebras 1, (J. D. Monk and R. Bonnet, eds.), North-Holland, Amsterdan-New York-Oxford-Tokyo, 1989.

5. Sambin G., *Subdirectly irreducible modal algebras and initial frames*, Studia Logica **62** (1999), 269–282.

6. Sambin G. and Vaccaro V., *Topology and duality in modal logic*, Annals of Pure and Applied Logic **37** (1988), 249–296.

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