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ON P-EXTENDING MODULES

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Abstract. Let R be a ring. A right R-module M is called **quasi-principally injective** if it is M-principally injective.In this paper, we give some characterizations and properties of principally injective modules, which generalize results of Nicholson and Yousif. For a quasi-principally injective module M, we show: 1. For isomorphic submodules H, K of M, we have $SH = SK$, where S is the endomorphism ring of M. 2. M has (PC_2), and consequently has (PC_3). We characterize when a direct sum of P-extending modules is P-extending, and when a direct sum of a P-extending module and a semi-simple module is P-extending. We also characterize when a direct sum of FP-extending modules is FP-extending. Finally, we discuss when a direct sum of P-extending modules with relatively EC-injective is P-extending.

1. Introduction

In [**7**], Nicholson and Yousif have introduced and studied the structure of principally injective rings, and have given some characterizations of such rings in terms of the internal properties of these rings. In fact, they defined principally injective modules in the following sense: A right module M over a ring R is called *principally injective* (for short *P-injective*) if every R-homomorphism from a principal right ideal of R to M can be extended to R. In [**8**], Wongwai extended this notion to modules by making use of M-cyclic submodules of M.

Here, we adopt the extension of the concept of principally injective rings, which is given in [**7**], to modules. The fact that, cyclic and M-cyclic submodules of a module M are not the same (e.g., as $\mathbb{Z}$-modules, the integers $\mathbb{Z}$ is cyclic submodule, but not a $\mathbb{Q}$-cyclic submodule, of $\mathbb{Q}$, and $\mathbb{Q}$ is $\mathbb{Q}$-cyclic but not cyclic, of $\mathbb{Q}$), gives the independence of the concepts of N-principally injective by Wongwai and the one we are dealing with.

We also introduce the definitions of principally extending, (for short P-extending), and P-(quasi-)continuous modules as follows: For a right R-module M,

1. M is called a *P-extending* module if every cyclic submodule of M is essential in a direct summand of M or, equivalently, every EC-closed submodule of M is a summand. M is called an *FP-extending* module if every finite uniform dimension EC-closed submodule of M is a summand.

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2. M is called a *P-quasi-continuous* module if it is P-extending, and the following condition holds: (PC_3) For each $a, b \in M$, if aR and $bR \leq^{\oplus} M$ with $aR \cap bR = 0$, then $aR \oplus bR \leq^{\oplus} M$.
3. M is called a *P-continuous* module if it is P-extending, and the following condition holds: (PC_2) For each $a, b \in M$, if $aR \cong bR$ and $bR \leq^{\oplus} M$, then $aR \leq^{\oplus} M$.

It is known that in regular rings the condition (C_2) is satisfied, and so such rings are continous if and only if they are extending. Consequently, every regular ring is P-continuous as a module over itself. It is also clear that regular rings are P-injective rings. This allows us to find P-injective modules, which are not injective.

Direct sums of extending modules have been investigated in great detail, in a long series of papers, by Dung and Smith [**3**], and by Kamal and Muller [**4**], [**5**]. The present paper studies direct sums of P-extending modules, and we investigate when such direct sums are P-extending.

It is known that M is N-injective if and only if for every submodule A of $N \oplus M$ with $A \cap M = 0$, there exists a submodule B of $N \oplus M$ such that $A \leq B$, and $N \oplus M = B \oplus M$. In analogue, we introduce the concept of N-EC-injectivity, and give a characterization of such modules different from the diagram description. This helps us to build up blocks of P-extending modules, which are relatively EC-injective to obtain P-extending modules. We prove that, if $M = M_1 \oplus M_2$, then M_i is P-extending and is M_j-EC-injective $(i \neq j = 1, 2)$ if and only if $M = C \oplus M_i^{'} \oplus M_j$, where $M_i^{'} \leq M_i$, for every EC-closed submodule C of M with $C \cap M_j = 0$ $(i \neq j = 1, 2)$.

All modules here are right modules over a ring R. The right (respectively, left) annahilator of a subset X of a module is denoted by $r_R(X)$ (resp. $l_R(X)$). A submodule A of a module M is called *essential* in M or M is an *essential extension* of A (denoted by $A \leq^{e} M$), if every non-zero submodule of M has non-zero intersection with A. $X \leq^{\oplus} M$ signifies that X is a direct summand of M.

A submodule A of M is called *M-cyclic* submodule of M if it is isomorphic to M/X, for some submodule X of M. The injective hull and the uniform dimension of a module M will be denoted by $E(M)$ and $U - \dim(M)$ respectively. The endomorphism ring of a module M is denoted by $\mathrm{End}\,(M)$. A submodule is *closed* in M if it has no proper essential extensions in M. The *graph* of a homomorphism $f : N \to M$ is the submodule $\langle f \rangle = \{n - f(n) : n \in N\}$ of $N \oplus M$.

A module M is *extending* (*n-extending*) if every closed submodule A (with $U - \dim(A) \leq n$) is a direct summand of M, or equivalently to the requirement that every submodule A (with $U - \dim(A) \leq n$) is essential in a direct summand of M.

A module M is called *quasi-continuous* if it is extending module, and the following condition holds: (C_3) For all X, and $Y \leq^{\oplus} M$, with $X \cap Y = 0$, one has $X \oplus Y \leq^{\oplus} M$. M is called *continuous* if it is extending module, and the following condition holds: (C_2) If a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M.

2. Principally Injective Modules

Let R be a ring and M, N be R-modules. M is called *N-principally injective* (for short *N-P-injective*) if every R-homomorphism from a cyclic submodule of N to M can be extended to N. Equivalently, for each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, there exists $f \in \operatorname{Hom}_R(N, M)$ such that $m = f(n)$.

Within the proof of [**2**, Proposition 1.1], it was observed that M is N-injective if and only if $N \oplus M = C \oplus M$, for every complement C of M in $N \oplus M$. The condition 3. in the next Proposition is analogous with such observation.

Proposition 2.1. *Let M and N be R-modules, and $S = \operatorname{End}(M)$. Then the following are equivalent:*

1. *M is N P-injective;*
2. *For each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, we have $Sm \subseteq \operatorname{Hom}_R(N, M)n$;*
3. *For each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, there is a complement C of M in $N \oplus M$ with $n - m \in C$ and $N \oplus M = C \oplus M$;*
4. *For each $n \in N$, $l_M r_R(n) = \operatorname{Hom}_R(N, M)n$;*
5. *For each $n \in N$ and $a \in R$, $l_M [aR \cap r_R(n)] = l_M(a) + \operatorname{Hom}_R(N, M)n$.*

Proof. 1. $\Rightarrow$ 2.: Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$. Since M is N-P-injective, then there exists a homomorphism $f : N \to M$ such that $m = f(n)$. Let $\phi \in S$, then $\phi(m) \in \operatorname{Hom}_R(N, M)n$. Therefore, $Sm \subseteq \operatorname{Hom}_R(N, M)n$.

2. $\Rightarrow$ 3.: Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, then by 2. there exists a homomorphism $f : N \to M$ such that $m = f(n)$. Hence $N \oplus M = \langle f \rangle \oplus M$, where $\langle f \rangle$ is the graph of a homomorphism $f : N \to M$. Therefore, $C = \langle f \rangle$ is a complement of M in $N \oplus M$ with $N \oplus M = C \oplus M$ and $n - m \in C$.

3. $\Rightarrow$ 4.: Let $n \in N$ and $x \in l_M r_R(n)$, then $r_R(n) \subseteq r_R(x)$. By 3. there is a complement C of M in $N \oplus M$ with $n - x \in C$ and $N \oplus M = C \oplus M$. So, there exists a homomorphism $f : N \to M$ such that $C = \langle f \rangle$. Since $n - x \in C$, then $n - x = n^{'} - f(n')$, for some $n' \in N$. So, $n = n'$ and $x = f(n') = f(n)$. Hence $x \in \operatorname{Hom}_R(N, M)n$, and $l_M r_R(n) \subseteq \operatorname{Hom}_R(N, M)n$. The other conclusion is obvious.

4. $\Rightarrow$ 5.: Let $n \in N$, $a \in R$, and $x \in l_M [aR \cap r_R(n)]$, then $x(aR \cap r_R(n)) = 0$ and so $r_R(na) \subseteq r_R(xa)$. Hence $l_M r_R(xa) \subseteq l_M r_R(na) = \operatorname{Hom}_R(N, M)na$, by 4. Therefore, $xa = f(na) = f(n)a$, for some $f \in \operatorname{Hom}_R(N, M)$. So $(x - f(n))a = 0$ and $x - f(n) \in l_M(a)$. Thus $x \in l_M(a) + \operatorname{Hom}_R(N, M)n$, and so $l_M [aR \cap r_R(n)] \subseteq l_M(a) + \operatorname{Hom}_R(N, M)n$. On the other hand, let $x \in l_M(a) + \operatorname{Hom}_R(N, M)n$, then $x = m + f(n)$ for some $m \in l_M(a)$ and $f \in \operatorname{Hom}_R(N, M)$. So $xa = ma + f(n)a = f(na)$. Let $ar \in aR \cap r_R(n)$, then $x(ar) = f(na)r = f(nar) = 0$, and so $x \in l_M [aR \cap r_R(n)]$. Thus $l_M(a) + \operatorname{Hom}_R(N, M)n \subseteq l_M [aR \cap r_R(n)]$.

5. $\Rightarrow$ 1.: Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, then $l_M r_R(m) \subseteq l_M r_R(n)$. By 5. we get $l_M r_R(n) = \operatorname{Hom}_R(N, M)n$, and so there is a homomorphism $f : N \to M$ such that $f(n) = m$. Thus M is N-P-injective. $\square$

Proposition 2.2. *Let M be N-P-injective, then M is X-P-injective, for every submodule X of N. If, in addition, X is a direct summand of N, then M is N/X-P-injective.*

Proof. It is clear. □

Lemma 2.3. *Let M be N-P-injective and $K \leq^{\oplus} M$, then K is N-P-injective.*

Proof. It is obvious. □

Lemma 2.4. *Let $\{M_i\}_{i\in I}$ be a family of modules. Then the direct product $\prod\limits_{i\in I} M_i$ is N-P-injective if and only if M_i is N-P-injective, for every $i \in I$.*

Proof. It is clear. □

Proposition 2.5. *If M is a quasi-principally injective module, and $S = \mathrm{End}(M)$, then $SH = SK$, for any isomorphic R-submodules H, K of M.*

Proof. Since $H \cong K$, then there is a right R-isomorphism $\sigma : H \to K$. For each $k \in K$, $k = \sigma(h)$ for some $h \in H$ and $r_R(h) = r_R(k)$. Since M is quasi-principally injective, then $Sh = Sk$ by Proposition 2.1, and so $Sk \subseteq SH$, for each $k \in K$. Then $SK \subseteq SH$. Similarly, we get $SH \subseteq SK$, and so the result. □

Corollary 2.6. *Let R be a P-injective ring and H, K be two-sided ideals of R. If $H \cong K$, as right ideals of R, then $H = K$.*

Remark. In Corollary 2.6, the condition P-injective for the ring R is not avoided. In fact, there are rings which do not satisfy the result in 2.6, for example, the ring $\mathbb{Z}$ of integers.

Theorem 2.7. *Let M be a quasi-principally injective module, then M has (PC_2).*

Proof. Let $a, b \in M$ with $aR \cong bR$ and $bR \leq^{\oplus} M$. Then $bR = eM$ for some idempotent $e \in \mathrm{End}(M)$. Since $aR \cong bR$, then there is an isomorphism $\sigma : bR \to aR$. Let $\sigma e = h$, then $aR = hM$ and $\sigma^{-1}h = e$. Since $bR \leq^{\oplus} M$, then by Lemma 2.3, bR is M-P-injective, and so there exists a homomorphism $\phi : M \to bR$ such that $\phi(a) = \sigma^{-1}(a)$. Then ϕ is an epimorphism, $\phi h = e$, and so $f = h\phi$ is an idempotent endomorphism of M. Hence $fM = h\phi M = h(bR) = heM = hM$, and so $aR \leq^{\oplus} M$. □

Remark. It is known that every summand right ideal of a ring R is generated by an idempotent element in R. Then every summand right ideal of R is cyclic and so, R has (PC_i) if and only if R has (C_i), $i = 2, 3$. Therefore by [**6**, Proposition 2.2.], if R has (PC_2), then R has (C_3).

Corollary 2.8 ([**7**], Theorem 2.3.)**.** *If R is a P-injective ring, then R has (C_2).*

Lemma 2.9. *Let M be an R-module. If M has (PC_2), then M has (PC_3).*

Proof. Let $aR \leq^{\oplus} M$ and $bR \leq^{\oplus} M$ with $aR \cap bR = 0$, then $aR = eM = \operatorname{Im} e$, for some $e^2 = e \in \operatorname{End}(M)$, and so $aR \oplus bR = eM \oplus (1-e)bR$. Since $(1-e)bR \cong bR \leq^{\oplus} M$ and M has (PC_2), then $(1-e)bR = fM$ for some $f^2 = f \in \operatorname{End}(M)$. Then $ef = 0$, and $h = e + f - fe$ is an idempotent in $\operatorname{End}(M)$. Therefore, $aR \oplus bR = eM \oplus fM = (e + f - fe)M = hM \leq^{\oplus} M$. □

Corollary 2.10. *If M is a quasi-principally injective module, then M has (PC_3).*

Definition 2.1. By an *EC-(closed) submodule* C of a module M, we mean a (closed) submodule C which contains essentially a cyclic submodule; i.e. there exists $c \in C$ such that $cR \leq^e C$.

Lemma 2.11. *Every summand of an EC- submodule of M is EC-submodule.*

Proof. Let $cR \leq^e C$ be an EC-submodule of M, and $C_1 \leq^{\oplus} C$, then $C = C_1 \oplus C_2$, for some submodule C_2 in C. Let $c = c_1 + c_2$, where $c_1 \in C_1$ and $c_2 \in C_2$. It is easy to see that $c_1R \leq^e C_1$. Therefore, C_1 is an EC-submodule of M. □

Corollary 2.12. *Every summand of an EC-closed submodule of M is EC-closed.*

Lemma 2.13. *Every summand of a P-(quasi-)continuous module is P-(quasi-) continuous.*

Proof. It is obvious by Corollary 2.12 □

Lemma 2.14. *For an indecomposable module M, the following are equivalent:*

1. *M is extending;*
2. *M is P-extending;*
3. *M is uniform.*

Lemma 2.15. *A module M over a right noetherian ring R, is 1-extending if and only if it is P-extending.*

Proof. Let M be a 1-extending module, and $cR \leq^e C$ be an EC-closed submodule of M. Since R is a noetherian ring, then C has a finite uniform dimension. Since M is 1-extending, then by Proposition (4) in [**4**], M is n-extending. Hence C is a summand, and so M is P-extending. For the converse, it is obvious. □

Corollary 2.16. *Let M be a module with finite uniform dimension, then the following are equivalent:*

1. *M is extending;*
2. *M is 1-extending;*
3. *M is P-extending.*

Proposition 2.17. *Let $M = M_1 \oplus M_2$, and let $C \cap M_1$ be an EC-submodule of M, for every EC-closed submodule C of M. Then M is P-extending if and only if every EC-closed submodule C, with $C \cap M_1 = 0$, or $C \cap M_2 = 0$, is a summand.*

Proof. The necessary condition is obvious. For the sufficient condition, let $cR \leq^e C$ be an EC-closed submodule of M. If $C \cap M_1 = 0$, then we are done. Otherwise, $C \cap M_1$ is an EC-submodule of M, by assumption. Let C_1 be a maximal essential extension of $C \cap M_1$ in C, then C_1 is an EC-closed submodule of M, with $C_1 \cap M_2 = 0$. Hence by the assumption, C_1 is a summand of M. Write $M = C_1 \oplus C_2$, by the modular law, $C = C_1 \oplus (C \cap C_2)$. By Corollary 2.12, $C \cap C_2$ is an EC-closed submodule of M with $(C \cap C_2) \cap M_1 = 0$, and therefore, $C \cap C_2$ is a summand of M. Thus C is a summand of M, and therefore, M is P-extending. □

Proposition 2.18. *Let $M = M_1 \oplus M_2$, where M_1 is of finite uniform dimension. Then M is P-extending if and only if every EC-closed submodule C of M, with $C \cap M_1 = 0$, or C is of finite uniform dimension, is a summand.*

Proof. The necessary condition is obvious. For the sufficient condition, let $mR \leq^e C$ be an EC-closed submodule of M. If $C \cap M_1 = 0$, then we are done. Now let $0 \neq c \in C \cap M_1$, and C_1 be a maximal essential extension of cR in C. Since M_1 is of finite uniform dimension, so is C_1. By the given assumption, C_1 is a summand of M. Write $M = C_1 \oplus K$. Hence $C = C_1 \oplus C^*$, where $C^* := K \cap C$ is closed in M. Let $m = c_1 + c^*$, where $c_1 \in C_1$ and $c^* \in C^*$. Since C^* is a summand of an EC-closed submodule C, then by Corollary 2.12, C^* is EC-closed. If $C^* \cap M_1 = 0$, then by assumption C^* is a summand, and hence C is a summand of M. On the other hand, if $C^* \cap M_1 \neq 0$, then by repeating the previous steps, we have $C^* = C_2 \oplus C_3$, where C_2 is a summand and has a nonzero intersection with M_1. Continuing in this manner, we should stop after a finite steps (due to M_1 a finite uniform dimensional module) and end with $C = C_1 \oplus C_2 \oplus \ldots \oplus C_n$, where C_i is a summand of M $(i = 1, 2, \ldots, n-1)$, and C_n contains an essential cyclic submodule with $C_n \cap M_1 = 0$. Hence C_n is a summand of M, by assumption, and therefore C is a summand of M. □

Corollary 2.19. *Let $M = M_1 \oplus M_2$, where M_1 is of finite uniform dimension. Then M is P-extending if and only if every EC-closed submodule C of M, with $C \cap M_1 = 0$, or $C \cap M_2 = 0$, is a summand.*

Proposition 2.20. *Let $M = M_1 \oplus M_2$. Then M is FP-extending if and only if every EC-closed submodule C of M with finite uniform dimensional such that $C \cap M_1 = 0$, or $C \cap M_2 = 0$, is a summand.*

Proof. Is similar to the proof of Proposition 2.18 □

Proposition 2.21. *Let $M = M_1 \oplus M_2$, where M_1 is a semisimple module. Then M is P-extending if and only if every EC-closed submodule C of M with $C \cap M_1 = 0$, is a summand.*

Proof. The necessary condition is obvious. For the sufficient condition, let C be an EC-closed submodule of M. If $C \cap M_1 = 0$, then we are done. On the other hand, since M_1 is a semisimple, we get $C \cap M_1 \leq^{\oplus} M_1$ and so $C = C \cap M_1 \oplus C^*$. Since C^* is an EC-closed submodule of M and $C^* \cap M_1 = 0$, then C^* is a summand of M. Therefore, C is a summand of M. □

Proposition 2.22. *Let $M = M_1 \oplus M_2$, where M_1 is P-extending and M_2 is M_1-P-injective. If M_2 is nonsingular, then every EC-closed submodule C of M, with $C \cap M_2 = 0$, is a summand of M.*

Proof. Let $cR \leq^e C$ be an EC-closed submodule of M with $C \cap M_2 = 0$, and write $c = c_1 + c_2$, where $c_1 \in M_1$ and $c_2 \in M_2$. Since M_2 is M_1-P-injective, then by Lemma 5 in [**4**],

$$cR = (c_1R)^* = \{c_1r + \phi(c_1)r : r \in R\} \subseteq (M_1)^* := \{m_1 + \phi(m_1) : m_1 \in M_1\} \cong M_1$$

and that $M = (M_1)^* \oplus M_2$, where $\phi \in \mathrm{Hom}_R(M_1, M_2)$. Let $x \in C$ and write $x = y + m_2$, where $y \in (M_1)^*$ and $m_2 \in M_2$. Since $cR \leq^e C$, then there exists an essential right ideal I of R such that $m_2I = 0$. Since M_2 is nonsingular, then $m_2 = 0$. It follows that $C \subseteq (M_1)^*$. Since $(M_1)^*$ is P-extending, we have $C \leq^\oplus (M_1)^* \leq^\oplus M$. □

Definition 2.2. Let $M = M_1 \oplus M_2$ be a module. The module M_2 is called *M_1-EC-injective*, if for every EC-(closed) submodule N of M_1, and every homomorphism from N to M_2 can be extended to M_1.

This is equivalent to for every EC-(closed) submodule N of M such that $N \cap M_2 = 0$, there exists $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$.

Observe that every module over a regular ring R is R-EC-injective.

Lemma 2.23. *Let $M = M_1 \oplus M_2$ and M_2 be M_1-EC-injective. Then:*

1. *M_2 is K-EC-injective, for all $K \leq M_1$.*
2. *H is M_1-EC-injective, for all $H \leq^\oplus M_2$.*
3. *H is K-EC-injective, for all $K \leq^\oplus M_1$, and $H \leq^\oplus M_2$.*

Proof. Let K be a submodule of M_1, and N be an EC-submodule of $K \oplus M_2$ with $N \cap M_2 = 0$. Then N is an EC-submodule of M. Since M_2 is M_1-EC-injective, then there is $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$. Then $K \oplus M_2 = (K \oplus M_2) \cap (N' \oplus M_2) = (N' \cap (K \oplus M_2)) \oplus M_2$ and $N \leq N' \cap (K \oplus M_2)$. Hence M_2 is K-EC-injective.

2. Let H be a summand of M_2, and N be an EC-submodule of $M_1 \oplus H$ with $N \cap H = 0$. Then N is an EC-submodule of M and $N \cap M_2 = 0$. Since M_2 is M_1-EC-injective, then there is $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$. Since $H \leq^\oplus M_2$, then $M_2 = H \oplus H'$, and so $M_1 \oplus H = (M_1 \oplus H) \cap (N' \oplus H \oplus H') = H \oplus (M_1 \oplus H) \cap (N' \oplus H')$. Since $N \leq N'$, then $N \leq (M_1 \oplus H) \cap (N' \oplus H')$. Therefore, H is M_1-EC-injective.

3. Follows from 1. and 2. □

Proposition 2.24. *Let $M = M_1 \oplus M_2$, where M_1 is P-extending and M_2 is M_1-EC-injective. Then $M = C \oplus M_1' \oplus M_2$; where $M_1' \leq M_1$, for every EC-closed submodule C of M, with $C \cap M_2 = 0$.*

Proof. Let $cR \leq^e C$ be an EC-closed submodule of M with $C \cap M_2 = 0$. Define $X := M_1 \cap (C \oplus M_2)$. Then $c_1R \leq^e X$, where $c = c_1 + c_2$, where $c_1 \in M_1$ and $c_2 \in M_2$. Let N_1 be a maximal essential extension of X in M_1. Then N_1 is an EC-closed submodule of M_1. Since M_1 is P-extending, we have $N_1 \leq^\oplus M_1$.

Write $M_1 = N_1 \oplus M_1^{'}$, where $M_1^{'} \leq M_1$. Now $C \oplus M_2 = X \oplus M_2 \leq^e N_1 \oplus M_2$; i.e. $C \leq N_1 \oplus M_2$, and $C \leq^c N_1 \oplus M_2$. Then C is a complement of M_2 in $N_1 \oplus M_2$. Since M_2 is M_1-EC-injective, and N_1 is a summand of M_1, then by Lemma 2.23 1., M_2 is N_1-EC-injective, and so there exists $N^{'} \leq N_1 \oplus M_2$ such that $C \leq N^{'}$, and $N_1 \oplus M_2 = N^{'} \oplus M_2$. Hence $N^{'}$ is a complement of M_2 in $N_1 \oplus M_2$, but C is a complement of M_2 in $N_1 \oplus M_2$. Therefore, $N^{'} = C$ and $M = M_1 \oplus M_2 = N_1 \oplus M_1^{'} \oplus M_2 = C \oplus M_1^{'} \oplus M_2$. □

Corollary 2.25. *Let $M = M_1 \oplus M_2$, where M_i is P-extending and is M_j-EC-injective $(i \neq j = 1, 2)$ if and only if $M = C \oplus M_i^{'} \oplus M_j$; where $M_i^{'} \leq M_i$, for every EC-closed submodule C of M, with $C \cap M_j = 0$ $(i \neq j = 1, 2)$.*

Proposition 2.26. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are relatively EC-injective, and either M_1 or M_2 is of finite uniform dimension. Then M is P-extending if and only if M_1 and M_2 are P-extending.*

Proof. It is follows by Corollaries 2.25, and 2.19. □

Proposition 2.27. *Let $M = \bigoplus_{i \in I} M_i$ be an R-module, where $M(F)$ is P-extending and $M(I \setminus F)$ is $M(F)$-EC-injective, for all finite subset F of I. Then M is P-extending.*

Proof. Let Let $c \in M$ and C be a maximal essential extension of cR in M. Then $cR \leq M(F)$ and $cR \cap M(I \setminus F) = 0$, for a finite subset F of I. Since $cR \leq^e C$, then $C \cap M(I \setminus F) = 0$. Since $M(I \setminus F)$ is $M(F)$-EC-injective and C is EC-closed submodule of M, then by Proposition 2.24, C is a summand of M. Hence M is P-extending. □

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