Acta Math. Univ. Comenianae
Vol. LXXIV, 2(2005), pp. 149–161

COEFFICIENT ESTIMATES IN SUBCLASSES OF THE CARATHÉODORY CLASS RELATED TO CONICAL DOMAINS

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Abstract. We study some properties of subclasses of of the Carathéodory class of functions, related to conic sections, and denoted by $\mathcal{P}(p_k)$. Coefficients bounds, estimates of some functionals are given.

1. Introduction

We denote by $\mathcal{P}$ the class of Carathéodory functions analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$, e.g.

$$\mathcal{P} = \{p : p \text{ analytic in } \mathcal{U},\ p(0) = 1,\ \operatorname{Re} p(z) > 0\}. \tag{1.1}$$

Some special subclasses of $\mathcal{P}$ play an important role in geometric function theory because of their relations with subclasses of univalent functions. Many such classes have been introduced and studied; some became the well-known, for instance, the class of analytic functions p in the unit disk $\mathcal{U}$ such that $p(0) = 1$ and $p \prec (1+Az)/(1+Bz)$, that is the class of functions for which $p(\mathcal{U})$ is a subset of a disk, or a half-plane. The other choice is the class of all p such that $p \prec [(1+z)/(1-z)]^{\gamma}$. In this case $p(\mathcal{U})$ is a subset of a sector, contained in a right half-plane with a vertex at the origin and symmetric about the real axis. Here the symbol "$\prec$" denotes the subordinations (cf. e.g. [**7**]).

Let $k \in [0, \infty)$. For arbitrarily chosen k let Ω_k denote the following domain

$$\Omega_k = \left\{u + iv : u^2 > k^2(u-1)^2 + k^2v^2\right\}. \tag{1.2}$$

Note that Ω_k is convex and symmetric in the real axis and $1 \in \Omega_k$ for all k. Ω_0 is nothing but the right half-plane and when $0 < k < 1$, Ω_k is an unbounded domain enclosed by the right branch of the hyperbola

$$\left(\frac{(1-k^2)u + k^2}{k}\right)^2 - \left(\frac{(1-k^2)v}{\sqrt{1-k^2}}\right)^2 = 1 \tag{1.3}$$

Received March 3, 2003.
2000 *Mathematics Subject Classification.* Primary 34A25, 30C35,33E05, 30C45.
Key words and phrases. The Carathéodory class, extremal problems, Fekete-Szegö-Goluzin functional.

with foci at 1 and $-(1+k^2)/(1-k^2)$. When $k=1$, the domain Ω_1 is still unbounded domain enclosed by the parabola

$$2u = v^2 + 1$$

with the focus at 1. When $k > 1$, the domain Ω_k becomes bounded domain being the interior of the ellipse

$$\left(\frac{(k^2-1)u-k^2}{k}\right)^2 - \left(\frac{(k^2-1)v}{\sqrt{k^2-1}}\right)^2 = 1$$

with foci at 1 and $(k^2+1)/(k^2-1)$. It should be noted that, for no choice of parameter k, Ω_k reduce to a disk. $\{\Omega_k, k \in [0,\infty)\}$ forms the family of domains bounded by conic sections, convergent in the sense of the kernel convergence.

Let p_k denote the conformal mapping of $\mathcal{U}$ onto Ω_k determined by conditions $p_k(0)=1$, $p_k'(0)>0$. The concrete form of p_k was given in [**7**], [**8**], [**11**] and in [**5**].

Theorem 1.1. *Let $k \in [0,\infty)$. The conformal mapping of $\mathcal{U}$ onto Ω_k is of the form*

$$p_k(z) = \begin{cases} \frac{1+z}{1-z} & \text{for} \quad k=0, \\ 1+\frac{2}{1-k^2}\,\sinh^2\left(A(k)\operatorname{arctanh}\sqrt{z}\right) & \text{for} \quad k\in(0,1), \\ 1+\frac{8}{\pi^2}\left(\operatorname{arctanh}\sqrt{z}\right)^2 & \text{for} \quad k=1, \\ 1+\frac{2}{k^2-1}\sin^2\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right) & \text{for} \quad k>1, \end{cases} \tag{1.4}$$

where $A(k) = (2/\pi)\arccos k$, $\mathcal{F}(w,t)$ is the Jacobi elliptic integral of the first kind

$$\mathcal{F}(w,t) = \int_0^w \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}, \tag{1.5}$$

and $k = \cosh\mu(t) = \cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$, $t\in(0,1)$.

By $\mathcal{P}(p_k)$ we denote the subclass of the Carathéodory class $\mathcal{P}$, consisting of functions p, analytic in $\mathcal{U}$, $p(0)=1$, $\operatorname{Re}p(z)>0$ in $\mathcal{U}$, and such that $p \prec p_k$ in $\mathcal{U}$. Observe that when k varies, $\mathcal{P}(p_k)$ generate a number of subclasses of the class $\mathcal{P}$.

The aim of this paper is to present some properties of the class $\mathcal{P}(p_k)$. In Section 2 we prove the continuity of functions "extremal" in $\mathcal{P}(p_k)$ as regards the parameter k. Some coefficients problems are treated in Section 3, in particular we obtain the sharp bound on the coefficient functional $|b_2-\mu b_1^2|$ $(-\infty<\mu<\infty)$.

2. General properties of the family $\mathcal{P}(p_k)$

We recall some notation and properties of Jacobi elliptic functions which will be used in next theorems (cf. e.g. [**1**], [**4**]).

The *elliptic integral* (or *normal elliptic integral*) *of the first kind* has been defined at (1.5). By $\mathcal{K}(t)$ we denote the *complete elliptic integral of the first kind*

$$\mathcal{K}(t) = \mathcal{F}(1,t), \quad \text{and let} \quad \mathcal{K}'(t) = \mathcal{K}(t'), \quad t' = \sqrt{1-t^2}, \quad t\in(0,1).$$

Let $\mathcal{E}(w,t)$ denote the *elliptic integral of the second kind*, e.g.

$$\mathcal{E}(w,t)=\int_0^w\sqrt{\frac{1-t^2x^2}{1-t^2}}dx, \tag{2.1}$$

and let $\mathcal{E}(t)=\mathcal{E}(1,t)$ be the *complete elliptic integral of the second kind*, $t\in(0,1)$. Also, set $\mathcal{E}'(t)=\mathcal{E}(t')$. Changing the variable by $x=\sin\theta$ integrals (1.5) and (2.1) reduce to the Legendre form

$$\mathcal{F}(\varphi,t)=\int_0^\varphi(1-t^2\sin^2\theta)^{-1/2}d\theta,$$
$$\mathcal{E}(\varphi,t)=\int_0^\varphi\sqrt{1-t^2\sin^2\theta}d\theta.$$

The equation $z=\mathcal{F}(\varphi,t)$, where z is assumed to be real, defines φ as a function of z which has been called by Jacobi the *amplitude* of z and denoted $\varphi=\mathrm{am}(z,t)$. Further Jacobi introduced sin(amz), and cos(amz) (*sinus* and *cosinus amplitudinus*) that have several applications in geometry and mechanics. Among numerous interesting properties of elliptic functions, the following will be used in the proof:

$$\lim_{t\to0^+}\mathcal{K}(t)=\lim_{t\to0^+}\mathcal{E}(t)=\lim_{t\to1^-}\mathcal{K}'(t)=\frac{\pi}{2}, \tag{2.2}$$

$$\lim_{t\to0^+}\mathcal{K}'(t)=\infty,\quad \lim_{t\to0^+}\mathcal{E}'(t)=1, \tag{2.3}$$

$$\begin{aligned}&\lim_{t\to1^-}\mathcal{K}(t)=\infty, && \lim_{t\to1^-}\mathcal{E}(t)=1,\\ &\lim_{t\to1^-}(1-t^2)\mathcal{K}(t)=0, && \lim_{t\to1^-}\frac{\mathcal{K}'(t)}{\mathcal{K}(t)}=0,\end{aligned} \tag{2.4}$$

$$\lim_{t\to0^+}\frac{\mathcal{K}'(t)}{\mathcal{K}(t)}=\infty,\quad\text{so that}\quad \lim_{t\to0^+}\sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)=\infty. \tag{2.5}$$

Functions $\mathcal{K},\mathcal{K}',\mathcal{E},\mathcal{E}'$ are continuous and differentiable on $(0,1)$, and

$$\begin{gathered}\frac{d\mathcal{K}(t)}{dt}=\frac{\mathcal{E}(t)-(1-t^2)\mathcal{K}(t)}{t(1-t^2)},\qquad \frac{d\mathcal{E}(t)}{dt}=\frac{\mathcal{E}(t)-\mathcal{K}(t)}{t},\\ \frac{d\mathcal{K}'(t)}{dt}=\frac{t^2\mathcal{K}'(t)-\mathcal{E}(t)}{t(1-t^2)},\end{gathered} \tag{2.6}$$

from which the Legendre identity can be derived (cf. [**1**, p. 112])

$$\mathcal{E}(t)\mathcal{K}'(t)+\mathcal{E}'(t)\mathcal{K}(t)-\mathcal{K}(t)\mathcal{K}'(t)=\frac{\pi}{2}.$$

Further, (2.6) and the above identity yields the result

$$\begin{aligned}\frac{d[\mathcal{K}'(t)/\mathcal{K}(t)]}{dt}&=\frac{\mathcal{K}(t)\mathcal{K}'(t)-\mathcal{E}(t)\mathcal{K}'(t)-\mathcal{E}'(t)\mathcal{K}(t)}{t(1-t^2)\mathcal{K}^2(t)}\\ &=-\frac{\pi}{2t(1-t^2)\mathcal{K}^2(t)}.\end{aligned} \tag{2.7}$$

Moreover

$$\frac{\pi}{1+\sqrt{1-t^2}} \le \mathcal{K}(t) \le \frac{\pi}{2\sqrt{1-t^2}}, \tag{2.8}$$

(c.f. [**2**], see also [**3**]).

Finally, by a simple computation we arrive at

$$\lim_{t\to 1^-} \mathcal{F}(\sqrt{z/t}, t) = \operatorname{arctanh}\sqrt{z}. \tag{2.9}$$

We now return to functions "extremal" in the class $\mathcal{P}(p_k)$.

Theorem 2.1. *Functions p_k is continuous as regards the parameter $k \in [0,\infty)$.*

Proof. First we observe that

$$\begin{aligned}
\lim_{k\to 0^+} p_k(z) &= \lim_{k\to 0^+} \frac{2}{1-k^2}\sinh^2\left(A(k)\operatorname{arctanh}\sqrt{z}\right)+1\\
&= 2\sinh^2\left(\operatorname{arcsinh}\frac{\sqrt{z}}{\sqrt{1-z}}\right)+1 = p_0(z),
\end{aligned}$$

since $\lim_{k\to 0^+} A(k) = \lim_{k\to 0^+}(2/\pi)\arccos k = 1$.

Simultaneously, by (1.4) and setting $t = 2\operatorname{arctanh}\sqrt{z}$ one gets

$$\begin{aligned}
\lim_{k\to 1^-} p_k(z) &= \lim_{k\to 1^-}\left(1+\frac{2}{1-k^2}\sinh^2\left(A(k)\operatorname{arctanh}\sqrt{z}\right)\right)\\
&= 1+2\lim_{k\to 1^-}\frac{\sinh^2\frac{A(k)t}{2}}{1-k^2}\\
&= 1+\frac{t^2}{2}\lim_{k\to 1^-}\left(\frac{A(k)}{\sqrt{1-k^2}}\right)^2\left(\frac{\sinh\frac{A(k)t}{2}}{\frac{A(k)t}{2}}\right)^2\\
&= 1+\frac{t^2}{2}\left(\lim_{k\to 1^-}\frac{\frac{-2}{\pi\sqrt{1-k^2}}}{\frac{-k}{\sqrt{1-k^2}}}\right)^2\left(\lim_{k\to 1^-}\frac{\sinh\frac{A(k)t}{2}}{\frac{A(k)t}{2}}\right)^2\\
&= 1+\frac{2}{\pi^2}t^2 = p_1(z).
\end{aligned}$$

Here, $A(k) = (2/\pi)\arccos k \to 0^+$ as $k \to 1^-$.

Finally, we will prove the right-hand continuity of p_k at $k = 1$ if we show that

$$\lim_{k\to 1^+}\frac{\sin\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)}{\sqrt{k^2-1}} = \frac{2}{\pi}\operatorname{arctanh}\sqrt{z}. \tag{2.10}$$

Note that $k = \cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$, so that $\sqrt{k^2-1} = \sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$ and if $k \to 1^+$ then $t \to 1^-$, thus (2.10) is equivalent to

$$\lim_{t\to 1^-}\frac{\sin\left(\frac{\pi}{4\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)}{\sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)} = \frac{2}{\pi}\operatorname{arctanh}\sqrt{z}. \tag{2.11}$$

Since by (2.4) and (2.9) both, the numerator and the denominator tend to 0 we need to prove, by the l'Hospital rule, that there exists the limit of the quotient of derivatives of (2.11), or equivalently

$$\lim_{t\to 1^-}\left(\cos\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)\cdot\right.$$
$$\left.\cdot\frac{\frac{\pi}{2}\left[\frac{\mathcal{E}(t)-(1-t^2)\mathcal{K}(t)}{t(1-t^2)\mathcal{K}^2(t)}\mathcal{F}(\sqrt{z/t},t)-\frac{1}{\mathcal{K}(t)}\frac{d[\mathcal{F}(\sqrt{z/t},t)]}{dt}\right]}{\cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)\frac{\pi^2}{4}\frac{1}{t(1-t^2)\mathcal{K}^2(t)}}\right). \tag{2.12}$$

Set

$$D(z,t)=\left(\mathcal{E}(t)-(1-t^2)\mathcal{K}(t)\right)\mathcal{F}(\sqrt{z/t},t)-t(1-t^2)\mathcal{K}(t)\frac{d[\mathcal{F}(\sqrt{z/t},t)]}{dt}.$$

Then (2.12) reduces to

$$\frac{2}{\pi}\lim_{t\to 1^-}\frac{\cos\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)}{\cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)}D(z,t).$$

Differentiating with respect to t we obtain from (1.5)

$$\frac{d[\mathcal{F}(\sqrt{z/t},t)]}{dt}=\int_0^{\sqrt{z/t}}\frac{tx^2}{\sqrt{(1-x^2)(1-t^2x^2)^3}}dx-\frac{\sqrt{z}}{2t\sqrt{t-z}\sqrt{1-tz}}$$

so that

$$\lim_{t\to 1^-}\frac{d[\mathcal{F}(\sqrt{z/t},t)]}{dt}=\frac{1}{4}\log\frac{1-\sqrt{z}}{1+\sqrt{z}}-\frac{1}{2(1-\sqrt{z})}.$$

Since, by (2.4)

$$\lim_{t\to 1^-}t(1-t^2)\mathcal{K}(t)=0,\quad\text{and}\quad\lim_{t\to 1^-}\mathcal{E}(t)=1$$

then using (2.9) we obtain

$$\lim_{t\to 1^-}D(z,t)=\operatorname{arctanh}\sqrt{z}.$$

Therefore, the above and first and fourth relation from (2.4) finally yield

$$\frac{2}{\pi}\lim_{t\to 1^-}\frac{\cos\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(\sqrt{z/t},t)\right)}{\cosh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)}D(z,t)=\frac{2}{\pi}\operatorname{arctanh}\sqrt{z},$$

that is equivalent to (2.11). It completes the proof. □

Theorem 2.2. *Let $k\in[0,\infty)$ be fixed. The function $p_k(z)$ has the positive Taylor coefficients around the origin.*

The proof of Theorem 2.2 appeared complicated for the case $k \in (1,\infty)$ and has been proved in [**11**] using the theory of continued fractions. We quote it here for the sake of completeness. Applying Theorem 2.2 estimates of the modulus and the real part of $p \in \mathcal{P}(p_k)$ were derived [**11**].

3. Coefficient bounds

Now, we find some bounds in the family $\mathcal{P}(p_k)$. The first problem, we discus, is the Fekete-Szegö-Goluzin's problem in the class $\mathcal{P}(p_k)$. We begin by proving the theorem that is itself interesting, since it improves the Livingston result in $\mathcal{P}(p_k)$ [**12**]. For fixed k, set

$$p_k(z) = 1 + P_1(k)z + P_2(k)z^2 + \cdots, \quad z \in \mathcal{U}.$$

Theorem 3.1. *Let* $0 \le k < \infty$ *be fixed. Then*

$$|P_1^2(k) - P_2(k)| \le P_1(k). \tag{3.1}$$

Proof. The inequality (3.1) is obvious for $k = 0$, by Livingston result [**12**], therefore we assume $k > 0$. We consider separately cases:

1. $k \in (0,1)$,
2. $k = 1$,
3. $k > 1$.

Case 1. By virtue of (1.4) a precise form of coefficients of p_k were derived (cf. [**5**], [**9**])

$$P_1(k) = \frac{2A^2(k)}{1-k^2}, \qquad P_2(k) = \frac{2A^2(k)(A^2(k)+2)}{3(1-k^2)} = P_1(k)\frac{A^2(k)+2}{3},$$

$$A(k) = \frac{2}{\pi}\arccos k.$$

Since $P_1(k)$ is positive for $k \in (0,1)$, the inequality (3.1) reduces to proving $|P_1(k) - (A^2(k)+2)/3| \le 1$. Note that

$$P_1(k) - \frac{A^2(k)+2}{3} = \frac{A^2(k)(5+k^2)+1-k^2}{3(1-k^2)} > 0$$

for $k \in (0,1)$, then it suffices to show the inequality $P_1(k) - (A^2(k)+2)/3 \le 1$, or equivalently

$$\frac{5(1-k^2)}{k^2+5} - \frac{4}{\pi^2}\arccos^2 k \ge 0.$$

Set

$$h(k) := \sqrt{\frac{5(1-k^2)}{k^2+5}} - \frac{2}{\pi}\arccos k.$$

Then functions h is well defined on a closed interval $[0,1]$ and

$$h'(k) = \frac{2}{\pi\sqrt{1-k^2}}\left[1 - \frac{3\sqrt{5}\pi k}{(k^2+5)^{3/2}}\right].$$

Note that $h'(0) > 0, h'(1^-) < 0$. Since $g(k) = \frac{3\sqrt{5}\pi k}{(k^2+5)^{3/2}}$ is monotone then there exists the only point $k_0 \in (0,1)$ such that $h'(k_0) = 0$. Then $h'(k) > 0$ in $0 < k < k_0$ and $h'(k) < 0$ in $0 < k_0 < k < 1$. Therefore $h(k) \geq \min\{h(0), h(1)\} = 0$ in $0 \leq k < 1$, so that the proof of the case 1. is complete.

Case 2. In this instance $P_1(k) = 8/\pi^2$ and $P_2(k) = 16/(3\pi^2) = 2P_1(k)/3$ (cf. [**14**], [**15**]). The inequality (3.1) now follows immediately by means of the relation $-1/3 < P_1(k) < 5/3$.

Case 3. To this purpose we use the following form of p_k for $k > 1$

$$p_k(z) = \frac{1}{k^2-1} \sin\left(\frac{\pi}{2\mathcal{K}(t)}\mathcal{F}(u(z)/\sqrt{t}, t\right) + \frac{k^2}{k^2-1}, \tag{3.2}$$

(cf. [**5**, p. 20]), where $u(z) = (z-\sqrt{t})/(1-\sqrt{t}z)$ and $k = \cosh(\mu(t)/2)$ for $t \in (0,1)$. In view of [**5**]

$$P_1(k) = \frac{\pi^2}{4(k^2-1)\mathcal{K}^2(t)\sqrt{t}(1+t)},$$

$$P_2(k) = P_1(k)\frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{24\mathcal{K}^2(t)\sqrt{t}(1+t)} =: P_1(k)D(k),$$

see also [**9**]. Since $P_1(k)$ is positive for all $t \in (0,1)$, the inequality (3.1) will hold if $|P_1(k) - D(k)| \leq 1$, equivalently $P_1(k) \leq D(k)+1$ and $D(k) \leq P_1(k)+1$. Now, we will show that the inequality $D(k) \leq P_1(k) + 1$ holds. We rewrite the inequality $D(k) \leq P_1(k) + 1$ into the form

$$\frac{4\mathcal{K}^2(t)(t^2+6t+1) - \pi^2}{4\mathcal{K}^2(t)} \leq \frac{3\pi^2}{2(k^2-1)\mathcal{K}^2(t)} + 6\sqrt{t}(1+t). \tag{3.3}$$

Observing that $k^2 - 1 = \sinh^2(\pi\mathcal{K}'(t)/(4\mathcal{K}(t)))$, the relation (3.3) becomes

$$t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - \frac{\pi^2}{4\mathcal{K}^2(t)} \leq \frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}. \tag{3.4}$$

Observe next, that if $k \to 1^+$ then $t \to 1^-$ and the case $k \to \infty$ corresponds to the case $t \to 0^+$. Thus, we may study the inequality (3.4) as regards $t \in (0,1)$. The left-hand side function satisfies

$$t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - \frac{\pi^2}{4\mathcal{K}^2(t)} \leq t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - (1-t^2). \tag{3.5}$$

by means of the right-hand estimation in (2.8). Set

$$w(t) := t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - (1-t^2) = 2t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t}.$$

The function $w(t)$ is defined on the closed interval $[0,1]$ and it decreases continuously from $w(0) = 0$ to $w(1) = -4$. Indeed $w'(t) = 4t - 9\sqrt{t} + 6 - 3/\sqrt{t} = (t-3) + 3(\sqrt{t}-1)^3/\sqrt{t} < 0$ on $(0,1)$. Therefore $w(t) < 0$ on $(0,1)$.

Now we show that the right-hand function of (3.4) is positive in $(0,1)$ and increases from 0 to $96/\pi^2$. Let

$$W(t) := \frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

Making use the first relation in (2.2) and the last relation in (2.5) we find that

$$\lim_{t\to 0^+} W(t) = 0.$$

Also, after necessary transformations, we obtain

$$\lim_{t\to 1^-} W(t) = \lim_{t\to 1^-} \frac{24\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2}{(\mathcal{K}')^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} = \frac{96}{\pi^2},$$

because of the last formula in (2.4) and the fact that $\lim\limits_{x\to 0}\sinh x/x = 1$. Now, we will show that $W(t)$ is increasing. Differentiating $W(t)$ and using (2.6) and (2.7) one gets

$$W'(t) = -\frac{3\pi^2\left[(\mathcal{E}(t)-(1-t^2)\mathcal{K}(t))\sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2}{8\mathcal{K}(t)}\cosh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)\right]}{t(1-t^2)\mathcal{K}^3(t)\sinh^3\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

In order to show that $W'(t) > 0$ it suffices to prove that the expression in the square brackets of $W'(t)$ is negative for $t \in (0,1)$. Such relation may be rewritten in the form

$$(\mathcal{E}(t)-(1-t^2)\mathcal{K}(t))\sinh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) < \frac{\pi^2}{8\mathcal{K}(t)}\cosh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)$$

or

$$\frac{8}{\pi^2}\left[\frac{\mathcal{E}(t)-(1-t^2)\mathcal{K}(t)}{\mathcal{K}(t)}\right] < \coth\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right). \tag{3.6}$$

Set

$$\phi(t) = \mathcal{E}(t) - (1-t^2)\mathcal{K}(t).$$

Then, in view of (2.6) we have

$$\frac{d\phi(t)}{dt} = \frac{\mathcal{E}(t)-\mathcal{K}(t)}{t} - \frac{-2t^2\mathcal{K}(t)+\mathcal{E}(t)-\mathcal{K}(t)+t^2\mathcal{K}(t)}{t} = t\mathcal{K}(t) > 0$$

in $(0,1)$. Moreover $\phi(0^+) = 0$ and $\phi(1^-) = 1$ by the second and third relation in (2.4). Thus $0 < \phi(t) < 1$ in $(0,1)$. Note also that $\mathcal{K}(t)$ is increasing from $\pi/2$ to ∞, when $t \in (0,1)$. Therefore

$$\frac{8}{\pi^2}\left[\frac{\mathcal{E}(t)-(1-t^2)\mathcal{K}(t)}{\mathcal{K}(t)}\right] < \frac{8}{\pi^2}\frac{1}{\pi/2} = \frac{16}{\pi^3} < 1,$$

whereas the right-hand side of (3.6) is greater than 1 since $\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) > 0$. Then the inequality (3.6) holds, equivalently $W'(t) > 0$ on $(0,1)$ so that W increases on $(0,1)$. Thus, having in view properties of $w(t)$ and $W(t)$ we conclude that $w(t) \leq W(t)$ for all $t \in (0,1)$, so that (3.4) is satisfied.

Next, we will show that $P_1(k) \leq D(k) + 1$ holds for $t \in (0,1)$, or equivalently

$$\frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} \leq t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t} + 1 - \frac{\pi^2}{2\mathcal{K}^2(t)},$$

by reversing the inequality in (3.4). Since $\mathcal{K}(t) > \frac{\pi}{2}$ we have $-\frac{\pi^2}{4\mathcal{K}^2(t)} > -1$ so that it suffices to show that

$$\frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)} \leq t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t}. \tag{3.7}$$

Set

$$r(t) := t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t}.$$

Then $r'(t) = 2t + 9\sqrt{t} + 6 + 3/\sqrt{t} > 0$ on $(0,1)$ and $r(0) = 0$, $r(1) = 19$. Moreover $r(1/\sqrt{2}) \approx 14.158379 > \frac{96}{\pi^2}$ whereas the value $\frac{96}{\pi^2}$ is the supremum of the left hand side of (3.7) as was shown in the first part of the proof of that case. Thus, it suffices to show (3.7) for $t \in (0, 1/\sqrt{2})$. Since $\mathcal{K}(t) > \frac{\pi}{2}$ then

$$\frac{3\pi^2}{2\mathcal{K}^2(t)\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} < \frac{6}{\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)}.$$

Now, we will show that

$$\frac{6}{\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)} \leq 6(t + \sqrt{t}) \leq r(t)$$

for $t \in (0, 1/\sqrt{2}]$, which concludes the desired result. The last inequality is obvious therefore it suffices to show that

$$(t + \sqrt{t})\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - 1 \geq 0. \tag{3.8}$$

Let

$$s(t) := (t + \sqrt{t})\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - 1.$$

Now we prove that $s(t)$ decreases in $(0, 1/\sqrt{2})$ to $s(1/\sqrt{2}) > 0$. Differentiating, we obtain

$$s'(t) = \frac{1}{2}\sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)\left[\left(1 + \frac{1}{2\sqrt{t}}\right)\tanh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2(\sqrt{t}+1)}{4\sqrt{t}(1-t^2)\mathcal{K}^2(t)}\right].$$

Since $\sinh\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)$ is positive for $t \in (0, 1/\sqrt{2}]$ then $s'(t) < 0$ if and only if the expression in square brackets of $s'(t)$ is negative, or equivalently

$$\left(1 + \frac{1}{2\sqrt{t}}\right)\tanh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right) - \frac{\pi^2(\sqrt{t}+1)}{4\sqrt{t}(1-t^2)\mathcal{K}^2(t)} < 0.$$

The above will be fulfilled if

$$\frac{2\sqrt{t}+1}{2(\sqrt{t}+1)}\tanh\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)-\frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)}<0,$$

or, by means of the relation $\tanh x<1$, when the inequality

$$\frac{2\sqrt{t}+1}{2(\sqrt{t}+1)}-\frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)}<0$$

holds. It is easy to see that $b(t)=\frac{2\sqrt{t}+1}{2(1+\sqrt{t})}$ is increasing on $(0,1/\sqrt{2})$ with the maximal value $b(1/\sqrt{2})\approx 0.73$. Let

$$c(t):=\frac{\pi^2}{4(1-t^2)\mathcal{K}^2(t)}.$$

Since, by (2.6),

$$c'(t)=\frac{\pi^2}{2}\frac{\mathcal{K}(t)-\mathcal{E}(t)}{t(1-t^2)^2\mathcal{K}^3(t)}$$

and $\mathcal{K}(t)>\mathcal{E}(t)$ on $(0,1)$, then $c'(t)>0$ on $(0,1)$ so does on $(0,1/\sqrt{2})$ and therefore $c(t)>c(0^+)=1$ for all $t\in(0,1/\sqrt{2})$. Thus $b(t)-c(t)<0$ for all $t\in(0,1/\sqrt{2})$ and hence $s(t)$ decreases on $(0,1/\sqrt{2})$.

Next, we show that $s(0^+)=\infty$. Note that

$$\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)=\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2\left[1+\frac{1}{3!}\left(\frac{\pi\mathcal{K}'(t)}{4\mathcal{K}(t)}\right)^2+\cdots\right]^2.$$

Then

$$\begin{aligned}&\lim_{t\to 0^+}(t+\sqrt{t})\sinh^2\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)\\ =\quad&\lim_{t\to 0^+}\frac{\pi^2(1+\sqrt{t})}{4\mathcal{K}^2(t)}\sqrt{t}\,(\mathcal{K}'(t))^2\left[1+\frac{1}{3!}\left(\frac{\pi\mathcal{K}'(t)}{2\mathcal{K}(t)}\right)^2+\cdots\right]^2.\end{aligned}$$

By properties of $\mathcal{K}(t)$ at 0^+ (the relation (2.2)) we have

$$\lim_{t\to 0^+}\frac{\pi^2(1+\sqrt{t})}{4\mathcal{K}^2(t)}=1,$$

so that we need to calculate the limit

$$\lim_{t\to 0^+}\sqrt{t}\,(\mathcal{K}'(t))^2.$$

Applying (2.8) to the value $\sqrt{1-t^2}$ we obtain

$$\frac{\pi}{1+t}\le\mathcal{K}(\sqrt{1-t^2})=\mathcal{K}'(t)\le\frac{\pi}{2t},\tag{3.9}$$

and since $\sqrt{t}/(1+t)^2$ and $\sqrt{t}/t^2$ tend to ∞, as $t\to 0^+$, we conclude that

$$\lim_{t\to 0^+}\sqrt{t}\,(\mathcal{K}'(t))^2=\infty.\tag{3.10}$$

Thus also $\lim_{t\to 0^+} s(t) = \infty$. Moreover $s(1/\sqrt{2}) \approx 0.9614 > 0$ so that we obtain the desired result. Hence the proof of the case 3. is complete. $\square$

Theorem 3.2. *Let* $0 \le k < \infty$ *be fixed, and let a function* $p \in \mathcal{P}(p_k)$ *be such that* $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$. *Then*

$$|b_1^2 - b_2| \le P_1(k). \tag{3.11}$$

The equality holds if $p(z)$ *is* $p_k(z^2)$ *or one of its rotation.*

Proof. Since $p \prec p_k$ then, in view of a definition of the subordination, there exists a function $\omega(z) = \alpha_1 z + \alpha_2 z^2 + \cdots$, $|\omega(z)| < 1$ such that $p(z) = p_k(\omega(z))$, therefore

$$1 + b_1 z + b_2 z^2 + \cdots = 1 + P_1(k)\alpha_1 z + z^2(P_1(k)\alpha_2 + P_2(k)\alpha_1^2) + \cdots .$$

Comparing the coefficients of z and z^2 we have $b_1 = P_1(k)\alpha_1$ and $b_2 = P_1(k)\alpha_2 + P_2(k)\alpha_1^2$, thus

$$\begin{aligned} |b_1^2 - b_2| &= |P_1^2(k)\alpha_1^2 - P_1(k)\alpha_2 - P_2(k)\alpha_1^2| \\ &= |\alpha_1^2(P_1^2(k) - P_2(k)) - P_1(k)\alpha_2| \\ &\le |\alpha_1|^2|P_1^2(k) - P_2(k)| + |P_1(k)||\alpha_2|. \end{aligned}$$

For the Schwarz' function ω the classical inequality $|\alpha_2| \le 1 - |\alpha_1|^2$ holds then, on account (3.1), we conclude

$$\begin{aligned} |b_1^2 - b_2| &\le |\alpha_1|^2|P_1^2(k) - P_2(k)| + |P_1(k)|(1 - |\alpha_1|^2) \\ &= |\alpha_1|^2\left[|P_1^2(k) - P_2(k)| - P_1(k)\right] + P_1(k) \\ &\le P_1(k), \end{aligned}$$

and the proof of the inequality of (3.11) is complete.

The equality in (3.11) holds if $|b_1| = 0$ and $|b_2| = P_1(k)$, or equivalently, $p(z)$ is $p_k(z^2)$ or one of its rotations. $\square$

Remark. Observe that the bound like (3.11) can be used in those subclasses of Carathéodory class for which the inequality similar to (3.1) holds. Let

$$q(z) = 1 + c_1 z + c_2 z^2 + \cdots , \tag{3.12}$$

be such that

$$|c_1^2 - c_2| \le c_1, \quad \text{with} \quad c_1 \ge 0, \tag{3.13}$$

it means q satisfies the bounds similar to (3.1). Then, reasoning along the same line as in Theorem 3.2 we may prove that for $p \in \mathcal{P}(q)$, $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$ the inequality

$$|b_1^2 - b_2| \le c_1, \tag{3.14}$$

is satisfied.

For instance, if $0 < \gamma \le 1$ then the function $\varphi(z) = [(1+z)/(1-z)]^\gamma$ maps the unit disk onto an angle, symmetric with respect to real axis, of width $\gamma\pi$ and contained in the right half-plane. Moreover, $\varphi(z) = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots$. Then, $|c_1^2 - c_2| = 2\gamma^2 \le 2\gamma = c_1$ therefore (3.13) is satisfied, so that (3.14) applies.

Concluding, if $p \in \mathcal{P}(\varphi) = \{q : q \prec \varphi\}$ and $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$, then $|b_1^2 - b_2| \le 2\gamma$. This inequality remarkable improves the result $|b_1^2 - b_2| \le 2$, due to Ma and Minda [**13**]. Similarly, the family $\mathcal{P}((1 + (1 - 2\beta)z)/(1 - z))$ can be treated. In this instance we obtain the inequality $|b_1^2 - b_2| \le 2(1 - \beta)$ for $p \in \mathcal{P}((1 + (1 - 2\beta)z)/(1 - z))$.

Theorem 3.3. *Let* $0 \le k < \infty$ *be fixed, and let a function* $p \in \mathcal{P}(p_k)$ *be of the form* $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$. *Then*

$$|b_2 - \mu b_1^2| \le \begin{cases} P_1(k) - \mu P_1^2(k) & \mu \le 0, \\ P_1(k) & \mu \in (0, 1], \\ P_1(k) + (\mu - 1)P_1^2(k) & \mu \ge 1. \end{cases} \tag{3.15}$$

When $\mu < 0$ *or* $\mu > 1$, *the equality holds if* $p(z)$ *is* $p_k(z)$ *or one of its rotations. If* $0 < \mu < 1$ *then the equality holds if* $p(z) = p_k(z^2)$ *or one of its rotation.*

Proof. Since $p \prec p_k$ then by Rogosinski Subordination Theorem we have $|b_n| \le P_1(k)$ for $n \ge 1$ and each fixed $k \in [0, \infty)$ First assume that $\mu \ge 1$. In view of Theorem 3.2, we have $|b_2 - b_1^2| \le P_1(k)$ therefore we obtain

$$|b_2 - \mu b_1^2| \le |b_1^2 - b_2| + (\mu - 1)|b_1|^2 \le P_1(k) + (\mu - 1)P_1^2(k).$$

Next, suppose that $\mu \le 0$. Then

$$|b_2 - \mu b_1^2| \le |b_2| + (-\mu)|b_1|^2 \le P_1(k) - \mu P_1^2(k).$$

Finally, if $0 < \mu \le 1$ then $\mu = 1/t$ with $t \ge 1$. Hence one gets

$$\begin{aligned} |b_2 - \mu b_1^2| &= |b_2 - \frac{1}{t} b_1^2| = \frac{1}{t}|t b_2 - b_1^2| = \frac{1}{t}|(t - 1)b_2 + b_2 - b_1^2| \\ &\le \frac{1}{t}[(t - 1)|b_2| + |b_2 - b_1^2|] \\ &\le \frac{1}{t}[(t - 1)P_1(k) + P_1(k)] = P_1(k), \end{aligned}$$

and the proof of all cases of (3.15) is complete.

When $\mu < 0$ or $\mu > 1$, the equality holds if and only if $|b_1| = P_1(k)$, that is, $p(z) = p_k(z)$ or one of its rotation. If $0 < \mu < 1$ then the equality holds if $|b_1| = 0$ and $|b_2| = P_1(k)$, or equivalently, $p(z)$ is $p_k(z^2)$ or one of its rotations. $\square$

Remark. In the paper [**13**] Ma and Minda proved similar bounds in the class $\mathcal{P}$. For instance, when $\mu \le 0$ authors obtained the estimate $|b_2 - \mu b_1^2| \le 2 - 4\mu$. Observe that in the case of $\mathcal{P}(p_k)$ the result is far better; the same it holds in the remaining range of the constant μ.

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