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SOLUTIONS OF $f'' + A(z)f = 0$ WITH PRESCRIBED SEQUENCES OF ZEROS

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Dedicated to the memory of Prof. V. Šeda.

Abstract. The problem of when a given sequence (resp. two sequences) of complex points can be the zero-sequence(s) of a solution (resp. of two linearly independent solutions) of $f'' + A(z)f = 0$, where $A(z)$ is entire, has been studied by several authors during the last two decades. However, it is not well-known that problems of this type were first stated and studied by O. Borůvka and V. Šeda [**16**] almost fifty years ago. A historical review to these studies will be given below. We then offer some remarks and improvements on results due to S. Bank and A. Sauer found in [**1**] and [**15**]. Our reasoning towards these improvements is based on some growth estimates for Mittag-Leffler-type series in the complex plane. These estimates might be of independent interest.

1. Introduction

The differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is entire, has been actively investigated during the last two decades, starting from [**2**]. Some of these investigations have been directed to determine all such equations (1.1) with solutions having prescribed zero-sequences, see [**1**], [**6**], [**15**] and [**17**]. However, it seems to have remained unknown that such studies already started much earlier. Indeed, O. Borůvka [**16**] posed, in his Brno seminar in the 1950's, the following:

Problem 1. *Let $\{z_n\}$ be a given sequence of distinct points in the complex plane $\mathbb{C}$ with no finite limit points. Does there exist an entire function $A(z)$ such that the differential equation* (1.1) *possesses a solution f having zeros exactly at the points z_n?*

Problem 1 is completely solved by V. Šeda in [**16**]. In fact, it is shown in [**16**] that, for any entire function F, another entire function $A_F(z)$ exists such that the

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equation

$$f'' + A_F(z)f = 0 \tag{1.2}$$

possesses a solution f_F with zeros exactly at the prescribed points z_n. By using the subscript F we mean that the corresponding function depends on F. For two different entire functions F_1 and F_2 the method yields two different entire functions $A_{F_1}(z)$ and $A_{F_2}(z)$ both solving Problem 1.

Šeda also posed and solved the following problem in [**16**].

Problem 2. *Let $\{a_n\}$ and $\{b_n\}$ be two given sequences of distinct points in $\mathbb{C}$ with no points in common and with no finite limit points. Does there exist an entire function $A(z)$ such that* (1.1) *possesses two linearly independent solutions f_1 and f_2 with their zeros exactly at the points a_n and b_n, respectively?*

Just as in the case of Problem 1, we may expect infinitely many solutions to Problem 2. Indeed, for any entire function F, there exists an entire function $A_F(z)$ such that (1.2) has linearly independent solutions $f_{1,F}$ and $f_{2,F}$ with their zeros exactly at the prescribed points a_n and b_n, respectively. Moreover, for two entire functions F_1 and F_2 that do not differ by an integer multiple of $2\pi i$, the method in [**16**] yields two different entire functions $A_{F_1}(z)$ and $A_{F_2}(z)$ both solving Problem 2.

Apparently, this background history has been unknown to L.-C. Shen, who independently resolved Problem 2 in 1985, see [**17**]. However, his method of proof is different than the one offered in [**16**], relying on what are called *Bank-Laine functions* (BL-functions) today. An entire function E is a BL-function, see [**11**], if at all zeros ζ of E, it is true that either $E'(\zeta) = 1$ or $E'(\zeta) = -1$. BL-functions are closely related to (1.1) through the non-linear differential equation

$$-4A(z)E^2 = 1 - (E')^2 + 2EE'', \tag{1.3}$$

originally developed in [**2**], and the following lemma, see [**3**, Lemma C].

Lemma A. *If E is a BL-function, then $A(z)$ defined by* (1.3) *is entire and E is a product of two linearly independent solutions f_1 and f_2 of* (1.1), *normalized so that $W(f_1, f_2)$, the Wronskian of f_1 and f_2, satisfies $W(f_1, f_2) = 1$.*

Conversely, if E is a product of two linearly independent solutions of (1.1), *where $A(z)$ is defined by* (1.1), *then E is a BL-function.*

Methods to solve Problems 1 and 2 will be discussed in more detail later on.

Next, we relate the growth of the coefficient function $A(z)$ (or $A_F(z)$) with Problems 1 and 2.

Problem 3.

(a) *Estimate the growth of $A(z)$ in Problem 1 in terms of the exponent of convergence of the zero sequence.*
(b) *Estimate the growth of $A(z)$ in Problem 2 in terms of the exponents of convergence of the two zero sequences.*

This paper is organized as follows. In Section 2, we first review some basic concepts of canonical products, and we then introduce a new concept of q-separated sequences. In Section 3, a method to estimate the growth of the classical Mittag-Leffler series' is offered. These estimates may be of independent interest. In Section 4, we review two methods to solve Problem 1. In Section 5, we offer some remarks and improvements on earlier results due to S. Bank and A. Sauer, and discuss Problem 3(a). In Section 6, we review two methods to solve Problem 2. Section 7 includes a short discussion related to Problem3(b).

2. Canonical products and q-separated sequences

We first recall some of the basic concepts related to canonical products, since they play a fundamental role in the paper. Secondly, we introduce a new concept of q-separated sequences that turns out to be convenient in the subsequent sections.

To begin with, let $\{z_n\}$ be a sequence of complex points, not necessarily distinct. The exponent of convergence of $\{z_n\}$ is a number $\lambda \geq 0$ such that

$$\sum_n \frac{1}{|z_n|^{\lambda-\varepsilon}} = \infty \quad \text{and} \quad \sum_n \frac{1}{|z_n|^{\lambda+\varepsilon}} < \infty$$

for any $\varepsilon > 0$. The genus of $\{z_n\}$ is the unique integer $p \geq 0$ such that

$$\sum_n \frac{1}{|z_n|^{p}} = \infty \quad \text{and} \quad \sum_n \frac{1}{|z_n|^{p+1}} < \infty.$$

It follows that $p \leq \lambda$. For $k = 0$ and $k \in \mathbb{N}$, the well-known Weierstrass convergence factors are given, respectively, by the formulae

$$e_0(z) = 1 \quad \text{and} \quad e_k(z) = \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\right).$$

If $\{z_n\}$ has a finite genus p, then the canonical product

$$G(z) = \prod_n \left(1 - \frac{z}{z_n}\right) e_p\left(\frac{z}{z_n}\right) \tag{2.1}$$

represents an entire function having zeros exactly at the points z_n. As soon as the genus of a sequence $\{z_n\}$ is finite, the canonical product (2.1) is uniquely determined.

Definition 2.1. Let $G(z)$ be the canonical product associated with a sequence $\{z_n\}$ of finite genus. If

$$\inf_k \left\{ |z_k| e^{|z_k|^q} |G'(z_k)| \right\} > 0 \tag{2.2}$$

holds for some $q \geq 0$, then $\{z_n\}$ is called q-separated.

Lemma 2.2. *A sequence $\{z_n\}$ of finite genus p is q-separated, if and only if,*

$$\inf_k \left\{ e^{|z_k|^q} \prod_{n \neq k} \left|1 - \frac{z_k}{z_n}\right| \left|e_p\left(\frac{z_k}{z_n}\right)\right| \right\} > 0.$$

Proof. An elementary computation yields

$$G'(z_k) = -\frac{e_p(1)}{z_k}\prod_{n\neq k}\left(1-\frac{z_k}{z_n}\right)e_p\left(\frac{z_k}{z_n}\right), \tag{2.3}$$

from which the assertion follows. □

Remarks.

1. Note that the points in a q-separated sequence must be simple, hence distinct.
2. By Lemma 2.2, 0-separated sequences of genus 0 in the plane (see Example 2.3 below) resemble uniformly separated sequences in the unit disc (see Chapter 9 in [**5**]).

The purpose of the following two examples is to show that 1. there exists q-separated sequences, and that 2. not every sequence of finite genus is q-separated for any $q \geq 0$.

Example 2.3. Let $z_n = n^2$ for $n = 1, 2, \ldots$ Then $\lambda = \frac{1}{2}$, $p = 0$, and so the associated canonical product is simply

$$G(z) = \prod_n \left(1 - \frac{z}{n^2}\right).$$

By (2.3), we have

$$G'(z_k) = G'(k^2) = -\frac{1}{k^2}\prod_{n\neq k}\left(1-\frac{k^2}{n^2}\right),$$

so that

$$|z_1||G'(z_1)| = \prod_{n\geq 2}\left(1-\frac{1}{n^2}\right) > 0,$$

and, for $k \geq 2$, we have

$$\begin{aligned}|z_k||G'(z_k)| &= \prod_{n\leq k-1}\left(\frac{k^2}{n^2}-1\right)\prod_{n=k+1}^{k^2-1}\left(1-\frac{k^2}{n^2}\right)\prod_{n\geq k^2}\left(1-\frac{k^2}{n^2}\right)\\ &=: P_1(k)P_2(k)P_3(k).\end{aligned}$$

By the series test, the product $\prod_{n\neq k}\left|1-\frac{k^2}{n^2}\right|$ converges for every finite $k \in \mathbb{N}$. Clearly, the products $P_1(k)$, $P_2(k)$ and $P_3(k)$ converge for every finite $k \in \mathbb{N}\setminus\{1\}$ as well. We proceed to show that the products $P_1(k)$, $P_2(k)$ and $P_3(k)$ remain bounded away from zero as k tends to infinity. This will show, by the definition, that the sequence $\{z_n\}$ is q-separated for every index $q \geq 0$.

Suppose that $k \geq 2$. We obtain the immediate estimates

$$P_1(k) \geq (k-1)\left(\frac{k^2}{(k-1)^2}-1\right) = \frac{2k-1}{k-1} \geq 2$$

and

$$\begin{aligned} P_2(k) &\geq (k^2 - k - 1)\left(1 - \frac{k^2}{(k+1)^2}\right) \\ &= (k^2 - k - 1)\left(\frac{k}{(k+1)^2} + \frac{1}{k+1}\right) \geq \frac{5}{9}. \end{aligned}$$

To estimate $P_3(k)$, we use the fact that $1 + x \leq e^x$ for $x \geq 0$. It follows that

$$\begin{aligned} P_3(k)^{-1} &= \prod_{n \geq k^2} \frac{n^2}{n^2 - k^2} = \prod_{n \geq k^2} \left(1 + \frac{k^2}{n^2 - k^2}\right) \\ &\leq \exp\left(\sum_{n \geq k^2} \frac{k^2}{n^2 - k^2}\right) \leq \exp\left(k^2 \int_{k^2 - 1}^{\infty} \frac{dx}{x^2 - k^2}\right) \\ &= \exp\left(\frac{k}{2} \log \frac{k^2 + k - 1}{k^2 - k - 1}\right) \leq \exp\left(\frac{k}{2} \log\left(\frac{k+1}{k-1}\right)^2\right) \\ &= \left(\frac{k+1}{k-1}\right)^k = \left(\left(1 + \frac{2}{k-1}\right)^{k-1}\right)^{\frac{k}{k-1}} \leq e^4. \end{aligned}$$

Hence,

$$P_3(k) \geq e^{-4},$$

and we are done.

Example 2.4. We now make use of the sequence constructed in [**1**, Corollary 1]. Namely, let $\{\varepsilon_n\}$ be an infinite sequence of real numbers satisfying

$$0 < \varepsilon_n < \exp\left(-\exp\left(2^n\right)\right), \quad n = 1, 2, \ldots,$$

and let $\{z_n\}$ be the increasing sequence of real numbers defined by setting $z_k = 2^{n+1}$ if $k = 2n + 1$, and $z_k = 2^n + \varepsilon_n$ if $k = 2n$.

Now, $\lambda = 0 = p$, so that the associated canonical product is simply

$$G(z) = \prod_n \left(1 - \frac{z}{z_n}\right).$$

Define the numbers

$$\lambda_k := \sum_{n \neq k} \frac{1}{z_n - z_k}, \quad k = 1, 2, \ldots$$

Then, a routine computation (see (11) and (12) in [**1**]) shows that

$$\lambda_k = -\frac{G''(z_k)}{2G'(z_k)}, \quad k = 1, 2, \ldots$$

Now, by the proof of [**1**, Corollary 1], we learn that

$$|\lambda_{2n+1}| \geq \exp\left(\exp\left(z_{2n+1}\right)\right) - M, \quad n = 1, 2, \ldots,$$

where $M = \frac{6\sqrt{2}-2}{\sqrt{2}-1}$. We obtain, for $n = 1, 2, \ldots$, that

$$|G'(z_{2n+1})| = \frac{|G''(z_{2n+1})|}{2|\lambda_{2n+1}|} \le \frac{|G''(z_{2n+1})|}{2\left(\exp\left(\exp\left(2^{n+1}\right)\right) - M\right)}. \tag{2.4}$$

Now, G is of order of growth zero, so is G''. Therefore, by comparing (2.2) and (2.4), we see that $\{z_n\}$ is not q-separated for any $q \ge 0$.

3. Growth estimates for Mittag-Leffler series'

In studying solutions of (1.1) with prescribed zeros, some kind of interpolation reasoning seems to be unavoidable. This situation leads, in a natural way, to apply the classical Mittag-Leffler theorem, see [**9**, Theorem 7.9.8], which ensures that a meromorphic function in a domain $\Omega \subset \mathbb{C}$ with prescribed principal parts can be constructed. If $\Omega = \mathbb{C}$, the solution of this problem may be given in the form of a Mittag-Leffler series

$$\sum_n \left(P_n\left(\frac{1}{z - z_n}\right) - Q_n\left(\frac{1}{z - z_n}\right)\right),$$

where P_n's are polynomials containing the desired principal parts, and Q_n's are polynomials used to guarantee the convergence of the series, see [**9**, Theorem 7.1.2].

In its original form, the Mittag-Leffler theorem does not say anything about the growth of the resulting function. In the 1930's, J. M. Whittaker gave growth estimates for meromorphic functions with prescribed principal parts, see [**18**]. The main idea in [**18**] is to divide the given pole sequence into pole clusters, that Whittaker called "nebulae".[1] Each nebula contains finitely many poles, but, as we approach to infinity, the number of poles in the nebulae may increase. The construction in [**18**] is rather complicated, but the results therein are rather general, and the growth estimates seem to be sharp.

In the next theorem, we offer a simple method to estimate the growth of a Mittag-Leffler series. Moreover, this result may be applied to interpolation problems for q-separated sequences, related to equation (1.1). Corollary 3.3 below is the first step in this direction.

Theorem 3.1. *Suppose the following assumptions hold:*

(a) $\{z_n\}$ *is an infinite sequence of distinct non-zero complex points ordered according to increasing moduli and having no finite limit points,*
(b) $\{z_n\}$ *has finite genus* p *and finite exponent of convergence* λ*,*
(c) $\{c_n\}$ *is an infinite sequence of non-zero complex points, not necessarily distinct,*
(d) $g : [0, \infty) \longrightarrow [0, \infty)$ *is a continuous and eventually nondecreasing function such that* $\frac{\log|c_n|}{\log|z_n|} \le g(|z_n|)$ *for all* $n \in \mathbb{N}$*,*

[1] It might be worth noting that a similar idea is behind the proof of the classical Cartan's lemma, see [**13**, pp. 19–21]. This fact will come up later in the present section.

(e) *given* $\alpha > 1$, $\{q_n\}$ *is a sequence such that each* q_n *is the smallest positive integer satisfying*

$$q_n \geq \max\left\{\alpha\left(\frac{\log|c_n|}{\log|z_n|}+p\right), \frac{\log\frac{|c_n|}{n}}{\log|z_n|}+p+1\right\}.$$

Then

$$H(z) := \sum_{n=1}^{\infty} \frac{c_n}{z-z_n}\left(\frac{z}{z_n}\right)^{q_n} \tag{3.1}$$

is meromorphic in $\mathbb{C}$ *with simple poles exactly at the points* z_n *with residue* c_n*. Moreover, we have the growth estimates*

$$\lambda \leq \rho(H) \leq \max\left\{\lambda, \limsup_{r\to\infty} \frac{\log g(\alpha r)}{\log r}\right\}. \tag{3.2}$$

Proof. 1. Denote $\beta = \sqrt[3]{\alpha}$ (> 1). Suppose $|z| = r \leq R < \infty$ and write

$$\begin{aligned} H(z) &= \sum_{|z_n|\leq\beta R} \frac{c_n}{z-z_n}\left(\frac{z}{z_n}\right)^{q_n} + \sum_{|z_n|>\beta R} \frac{c_n}{z-z_n}\left(\frac{z}{z_n}\right)^{q_n} \\ &=: S_1(z) + S_2(z). \end{aligned} \tag{3.3}$$

The expression $S_1(z)$ in (3.3) is a finite sum and therefore represents a rational meromorphic function in $\mathbb{C}$. Hence, in order to prove that $H(z)$ is meromorphic in $\mathbb{C}$, it suffices to show that $S_2(z)$ converges uniformly. But

$$\begin{aligned} |S_2(z)| &\leq \sum_{|z_n|>\beta R} \frac{|c_n|}{|z-z_n|}\left|\frac{z}{z_n}\right|^{q_n} = \sum_{|z_n|>\beta R} \frac{|c_n|}{|z_n|\left|1-\frac{z}{z_n}\right|}\left|\frac{z}{z_n}\right|^{q_n} \\ &\leq \frac{\beta}{\beta-1}\sum_{|z_n|>\beta R} \frac{|c_n|R^{q_n}}{|z_n|^{q_n+1}}. \end{aligned}$$

Therefore, by assumption (b), the convergence follows, if we are able to show that

$$\frac{|c_n|R^{q_n}}{|z_n|^{q_n+1}} \leq \frac{1}{|z_n|^{p+1}} \tag{3.4}$$

holds for all n large enough. To prove (3.4), we define $\delta := \frac{\alpha-1}{\alpha} \in (0,1)$. Then, by assumption (e),

$$\frac{\log(|c_n||z_n|^p)}{\log|z_n| - \log R} \leq \frac{\log(|c_n||z_n|^p)}{(1-\delta)\log|z_n|} = \alpha\frac{\log(|c_n||z_n|^p)}{\log|z_n|} \leq q_n$$

holds for all n large enough. Hence

$$\log\left(|c_n||z_n|^{p+1}\right) \leq \log\frac{|z_n|^{q_n+1}}{R^{q_n}}$$

for all n large enough, and (3.4) follows.

2. Obviously, all poles of $H(z)$ are simple and of residue c_n.

3. The inequality $\lambda \leq \rho(H)$ being clear, we proceed to prove the second inequality in (3.2). Consider first the finite sum $S_1(z)$ in (3.3). We denote by $n(t)$

the number of points z_n in the disc $|z| \leq t$. Applying inequalities (7.9) in [**7**] (with different notation, the reasoning being based on Cartan's lemma) together with assumptions (d) and (e), we get

$$\begin{aligned}
|S_1(z)| &\leq \sum_{|z_n|\leq \beta R} \frac{|c_n|}{|z-z_n|}\left|\frac{z}{z_n}\right|^{q_n} \\
&\leq \beta\frac{n\left(\beta^2 R\right)}{R}(\log R)^\beta \sum_{|z_n|\leq \beta R} \frac{|c_n|}{n}\left|\frac{z}{z_n}\right|^{q_n} \\
&\leq \beta n\left(\beta^2 R\right)(\log R)^\beta R^{\alpha(g(\beta R)+p+1)} \sum_{|z_n|\leq \beta R} \frac{|c_n|}{n|z_n|^{q_n}},
\end{aligned}$$

provided z is outside a certain sequence of closed Euclidean discs. As R tends to infinity, these discs form an exceptional set whose projection E on the positive real axis is of finite logarithmic measure, see [**7**]. Using the assumptions (e) and (b), we obtain

$$\sum_{|z_n|\leq \beta R} \frac{|c_n|}{n|z_n|^{q_n}} \leq \sum_{|z_n|\leq \beta R} \frac{1}{|z_n|^{p+1}} \leq \sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} < \infty.$$

Hence there exists a $C > 0$ such that

$$|S_1(z)| \leq Cn\left(\beta^2 R\right)(\log R)^\beta R^{\alpha(g(\beta R)+p+1)},$$

provided $|z| \leq R$ and $|z| \notin E$.

Let G be the canonical product associated with the sequence $\{z_n\}$, hence being of order of growth λ. Then GH is entire and

$$|G(z)H(z)| \leq C|G(z)|\left(n\left(\beta^2 R\right)(\log R)^\beta R^{\alpha(g(\beta R)+p+1)} + 1\right),$$

provided $|z| \leq R$ and $|z| \notin E$. Taking now $R = \beta r$ and applying [**8**, Lemma 5], we see that there exists an $r_0 = r_0(\beta) > 0$ such that

$$M(r, GH) \leq CM(\beta r, G)\left(n(\beta^4 r)\left(\log\left(\beta^2 r\right)\right)^\beta \left(\beta^2 r\right)^{\alpha(g(\alpha r)+p+1)} + 1\right)$$

for all $r \geq r_0$. Therefore,

$$\rho(GH) \leq \max\left\{\lambda, \limsup_{r\to\infty} \frac{\log g(\alpha r)}{\log r}\right\},$$

and the second inequality in (3.2) is now proved. □

Remarks.

1. Using Theorem 3.1 with $g \equiv 1$, we see that the Mittag-Leffler series in (3.1) represents a function meromorphic in $\mathbb{C}$ of order of growth equal to λ. In particular, if $c_n = \pm 1$, we may choose $g \equiv 1$. As a consequence, we have obtained the growth of the Mittag-Leffler series discussed in [**9**, Remark, p. 210].

2. The upper bound in (3.2) agrees with the corresponding bound offered in [**18**], although the methods of proof are different.

The following variant of Theorem 3.1 may easily be deduced by a similar proof.

Theorem 3.2. *Let $k \in \mathbb{N}$ and suppose that the assumptions (a)–(d) of Theorem 3.1 are valid. Suppose further that*

(e′) *given $\alpha > 1$, $\{q_n\}$ is a sequence such that each q_n is the smallest positive integer satisfying*

$$q_n \geq \max\left\{\alpha\left(\frac{\log|c_n|}{\log|z_n|} + p + 1 - k\right), \frac{\log\frac{|c_n|}{n^k}}{\log|z_n|} + p + 1\right\}.$$

Then

$$H(z) := \sum_{n=1}^{\infty} \frac{c_n}{(z - z_n)^k}\left(\frac{z}{z_n}\right)^{q_n}$$

is meromorphic in $\mathbb{C}$ with poles of multiplicity k exactly at the points z_n. Moreover, we have the growth estimates

$$\lambda \leq \rho(H) \leq \max\left\{\lambda, \limsup_{r\to\infty} \frac{\log g(\alpha r)}{\log r}\right\}.$$

Next, we relate a general interpolation problem with q-separated sequences, so that the growth of the resulting function could be estimated.

Corollary 3.3. *Let $\{z_n\}$ be a sequence satisfying the assumptions (a) and (b) of Theorem 3.1. Suppose further that $\{z_n\}$ is q-separable for some $q \geq 0$. Let $\{\sigma_n\}$ be an infinite sequence of non-zero complex points, not necessarily distinct. Let h be a continuous and eventually nondecreasing function such that $|\sigma_n| \leq h(|z_n|)$ for $n = 1, 2, \ldots$ Then there exists an entire function f such that*

$$f(z_n) = \sigma_n, \quad n = 1, 2, \ldots \tag{3.5}$$

and, for any given $\alpha > 1$,

$$\rho(f) \leq \max\left\{\lambda, \limsup_{r\to\infty} \frac{\log\log\left(h(\alpha r)e^{(\alpha r)^q}\right)}{\log r}\right\}. \tag{3.6}$$

Proof. Let $G(z)$ be the canonical product associated with $\{z_n\}$. By q-separability, the points z_n must be simple, hence $\frac{1}{G'(z_n)} \in \mathbb{C}$ for any n. We make use of Theorem 3.1 in the case $c_n = \frac{\sigma_n}{G'(z_n)}$. Indeed, if $H(z)$ denotes the Mittag-Leffler series in (3.1), then, by inspection, $f = GH$ is entire and satisfies (3.5). Hence, it suffices to study the growth of f.

By the assumptions, there exists a constant $C > 1$ such that

$$|c_n| \leq C|z_n|h(|z_n|)e^{|z_n|^q}, \quad n = 1, 2, \ldots$$

Therefore,

$$\frac{\log|c_n|}{\log|z_n|} \leq \frac{\log C + \log\left(h(|z_n|)e^{|z_n|^q}\right)}{\log|z_n|} + 1, \quad n = 1, 2, \ldots$$

By Theorem 3.1, $\rho(H)$ has the upper bound in (3.6). Since $\rho(G) = \lambda$, we conclude that f has the growth rate (3.6). □

4. Methods for solving Problem 1

Suppose $\{z_n\}$ is a (finite or infinite) sequence of distinct points in $\mathbb{C}$ with all (possible) limit points at infinity. We construct an entire function $G(z)$ having zeros exactly at the points z_n as follows. If $\{z_n\}$ is an infinite sequence that has a finite exponent of convergence, let $G(z)$ be the associated canonical product. If $\{z_n\}$ is a finite sequence consisting of N points, say, then we may choose $G(z)$ to be the polynomial

$$G(z) = \prod_{n=1}^{N} (z - z_n). \tag{4.1}$$

In all other cases, we just use Weierstrass' theorem to guarantee the existence of such a function $G(z)$.

We want the function $f = Ge^g$ to be a nontrivial solution of (1.1), where g represents an entire function to be constructed later. This will be done by using two methods.

Modified Bank method. The basic idea is taken from [**1**]: The function $f = Ge^g$ is a solution of (1.1), if and only if,

$$G'' + 2g'G' + ((g')^2 + g'' + A)G = 0. \tag{4.2}$$

At the points z_n, we have

$$\sigma_n := -\frac{G''(z_n)}{2G'(z_n)} = g'(z_n), \quad n = 1, 2, \ldots \tag{4.3}$$

To solve Problem 1, all we have to do is to find an entire function g with the interpolation property (4.3). In [**15**], this step is carried out via Mittag-Leffler theorem. Alternatively, we could apply a more concrete Mittag-Leffler series as in [**16**], see also Section 3. Now, whatever method is used, we can always find an entire function, say h, such that $h(z_n) = \sigma_n$ holds for $n = 1, 2, \ldots$. Note that, in the case when $\{z_n\}$ is a finite sequence, we may choose h to be the Lagrange interpolation polynomial

$$h(z) = \sum_{n=1}^{N} \frac{G(z)}{G'(z_n)} \frac{\sigma_n}{z - z_n},$$

where $G(z)$ is the polynomial in (4.1).

Finally, let g be any primitive function of h, and we see that $f = Ge^g$ solves (1.1), where

$$A = -\frac{G'' + 2g'G'}{G} - (g')^2 - g'' \tag{4.4}$$

is an entire function, see (4.2).

In Section 5 we will estimate the growth of the expressions $|\sigma_n|$. Then, by means of Corollary 3.3, we find an upper bound for $\rho(h)$. By the classical estimate $M(r,g) \le rM(r,h) + O(1)$, it follows that $\rho(g) \le \rho(h)$. Finally, by means of (4.4), we may estimate the growth of $A(z)$. We next state this growth estimate for future reference.

Lemma 4.1. *We have* $\rho(A) \le \max\{\lambda, \rho(h)\}$.

Next, let F be an arbitrary entire function. We proceed to solve Problem 1 associated with equation (1.2). Now, we let $g = g_F$ be any primitive function of $h + GF$. Hence, (4.3) clearly holds, and, by the discussion above, $f = f_F = Ge^{g_F}$ solves (1.2), where

$$A_F = -\frac{G'' + 2g_F' G'}{G} - (g_F')^2 - g_F'' \tag{4.5}$$

is an entire function.

Suppose then that we have two entire functions, say F_1 and F_2, such that $F_1(z) \not\equiv F_2(z)$ and yet $A_{F_1}(z) \equiv A_{F_2}(z)$, where $A_{F_1}(z)$ and $A_{F_2}(z)$ both solve Problem 1 related to (1.2). As $g_{F_j}' = h + GF_j$, $j = 1, 2$, elementary computations applied to (4.5) yield

$$(3G' + 2hG)(F_2 - F_1) + G^2(F_2^2 - F_1^2) + G(F_2' - F_1') = 0.$$

Dividing this by $F_2 - F_1 \not\equiv 0$, we get

$$3G' + 2hG + G^2(F_2 + F_1) + G\frac{F_2' - F_1'}{F_2 - F_1} = 0. \tag{4.6}$$

We now conclude that if F_1 and F_2 are entire functions such that $F_1(z) \not\equiv F_2(z)$ and that the function $F_2 - F_1$ has at least one zero not belonging to $\{z_n\}$, then (4.6) leads to a contradiction. Hence, under these assumptions on F_1 and F_2, we have $A_{F_1}(z) \not\equiv A_{F_2}(z)$. We have thus shown that, for a fixed zero sequence, there are infinitely many solutions to Problem 1.

Šeda method. We next review the original ideas of Šeda [**16**] for solving Problem 1 without any restrictions on F_1 and F_2 other than $F_1(z) \not\equiv F_2(z)$. To begin with, we observe that the zeros of a solution f of (1.1) are poles of the corresponding solution $u = \frac{f'}{f}$ of the Riccati differential equation

$$u' = -A(z) - u^2. \tag{4.7}$$

Hence, Problem 1 is equivalent to

Problem 1′. *Let* $\{z_n\}$ *be a given sequence of distinct points in the complex plane* $\mathbb{C}$ *with no finite limit points. Does there exist an entire function* $A(z)$ *such that the differential equation* (4.7) *possesses a solution* u *having poles exactly at the points* z_n*?*

We construct the canonical product $G(z)$ as above. Then define

$$u = \frac{G'}{G} + h, \tag{4.8}$$

where the entire function h will be constructed later. Given a primitive function φ of h, the function u is a logarithmic derivative of the entire function $f = Ge^{\varphi}$, and of functions which are constant multiples of f. Then

$$u' + u^2 = \frac{G'' + 2hG'}{G} + h' + h^2,$$

and hence the function u defined in (4.8) satisfies (4.7) with

$$A(z) = -\frac{G'' + 2hG'}{G} - h' - h^2. \tag{4.9}$$

If it is possible to find an entire function h such that $A(z)$ is an entire function, then Problem $1'$ is solved. Conversely, if equation (4.7) has a solution u with poles exactly at the points z_n, then we can write u in the form (4.8), and the entire coefficient $A(z)$ of the equation in question in the form (4.9).

The function $A(z)$ in (4.9) is entire, if and only if,

$$h(z_n) = -\frac{G''(z_n)}{2G'(z_n)}, \quad n = 1, 2, \ldots$$

Compare this with the similar requirement (4.3). Of course, such a function h can be found by the Mittag-Leffler argument. If h_0 is any such function, then, for an arbitrary entire function F, $h = h_F = h_0 + GF$ is an entire function as well with the required interpolation property. Then the function $A(z) = A_F(z)$ given by equation (4.9) has the form

$$\begin{aligned} A_F(z) &= -\frac{G'' + 2h_F G'}{G} - h_F' - h_F^2 \\ &= Q_0 - (3G' + 2h_0 G)F - GF' - G^2F^2, \end{aligned} \tag{4.10}$$

where $Q_0 := -\frac{G''+2h_0G'}{G} - h_0' - h_0^2$. Similarly, we write (4.8) in the form

$$u_F = \frac{G'}{G} + h_0 + GF. \tag{4.11}$$

Suppose next that F_1 and F_2 are arbitrary entire functions such that $A_{F_1}(z) \equiv A_{F_2}(z)$. Then equation (4.7) has solutions u_{F_1} and u_{F_2} of the form (4.11) both with the same poles z_n. Therefore, by the basic uniqueness theorem of differential equations applied to $v_j = 1/u_{F_j}$, $j = 1, 2$, we have $u_{F_1} = u_{F_2}$, and, by (4.11), it follows that $G(F_1 - F_2) = 0$. But this is possible only when $F_1(z) \equiv F_2(z)$. Finally, we conclude that, for a fixed pole sequence, there exist infinitely many solutions to Problem $1'$.

5. Results of Bank and Sauer in terms of q-separated sequences, and Problem 3(a)

Neither one of the papers [**16**] or [**17**] took into consideration the growth of $A(z)$ with respect to the growth of the given zero-sequence(s) of solutions of (1.1), say in terms of the exponent of convergence. As for Problem 3(a), the first contribution in this line was due to Bank in [**1**]. He gave a necessary condition for a sequence having a finite exponent of convergence to be the zero-sequence of a solution of (1.1), where $A(z)$ is an entire function of finite order, by proving

Theorem B. *Let $\{z_n\}$ be an infinite sequence of distinct non-zero complex points having a finite exponent of convergence and no finite limit points. Let p denote the genus of $\{z_n\}$, and define*

$$\lambda_k := \sum_{n \neq k} \left(\frac{z_k}{z_n} \right)^p (z_n - z_k)^{-1}. \tag{5.1}$$

If the sequence $\{z_n\}$ is the zero-sequence of a solution of equation (1.1)*, where $A(z)$ is an entire function of finite order, then there must exist a real number $b > 0$ and a positive integer k_0, such that*

$$|\lambda_k| \leq \exp\left(|z_k|^b\right) \tag{5.2}$$

for all $k \geq k_0$.

Further, [**1**, Corollary 1] shows that not every sequence of finite exponent of convergence is a zero sequence of a solution of (1.1), where $A(z)$ is of finite order of growth. The counterexample constructed in [**1**] is precisely the sequence discussed in Example 2.4, which we know not to be q-separated for any $q \geq 0$.

To illustrate the reverse part of Theorem B (and Problem 3(a)), Bank proved the following result, see [**1**, Theorem 2]:

Theorem C. *Let $K > 1$ be a real number, and let $\{z_n\}$ be a sequence of non-zero complex points satisfying*

$$|z_{n+1}| \geq K|z_n|, \quad n = 1, 2, \ldots \tag{5.3}$$

Then there exists an entire transcendental function $A(z)$ of order zero, such that (1.1) *possesses a solution whose zero-sequence is $\{z_n\}$.*

Remark. The exponent of convergence and the genus of the zero sequence $\{z_n\}$ in Theorem C are, clearly, both equal to zero. We next show that $\{z_n\}$ is also 0-separated. Indeed, condition (5.3) implies that $|z_j| \geq K^{j-m}|z_m|$ for $m < j$. Therefore,

$$\begin{aligned}
\prod_{n \neq k} \left| 1 - \frac{z_k}{z_n} \right| &= \prod_{n<k} \left| 1 - \frac{z_k}{z_n} \right| \prod_{n>k} \left| 1 - \frac{z_k}{z_n} \right| \\
&\geq \prod_{n<k} \left(\left| \frac{z_k}{z_n} \right| - 1 \right) \prod_{n>k} \left(1 - \left| \frac{z_k}{z_n} \right| \right) \\
&\geq (k-1)(K-1) \prod_{n>k} \left(1 - \left(\frac{1}{K} \right)^{n-k} \right)
\end{aligned}$$

for $k \geq 2$. Similarly for the case $k = 1$. It follows, by the convergence of the geometric series $\displaystyle\sum_n \left(\frac{1}{K} \right)^n$ and by Lemma 2.2, that $\{z_n\}$ is 0-separated, and we are done. It should also be noted that there are 0-separated sequences that do not satisfy (5.3) for any $K > 1$ — take, for instance, the sequence in Example 2.3. Therefore, we conclude that it is more restrictive for a sequence $\{z_n\}$ to satisfy condition (5.3) than to be 0-separated.

Related to the reverse part of Theorem B (and Problem 3(a)), Sauer proved the following sufficient condition, see [**15**, Theorem 1]:

Theorem D. *Let $\{z_n\}$ be an infinite sequence of distinct non-zero complex points having a finite exponent of convergence and no finite limit points. Let p denote the genus of $\{z_n\}$, and define*

$$\mu_k := \prod_{n \neq k} \left(1 - \frac{z_k}{z_n}\right)^{-1} e_p\left(\frac{z_k}{z_n}\right)^{-1}, \tag{5.4}$$

where $e_p(z)$ denotes the Weierstrass convergence factor. If there exists a real number $b > 0$ and a positive integer k_0, such that

$$|\mu_k| \leq \exp\left(|z_k|^b\right) \tag{5.5}$$

for all $k \geq k_0$, then $\{z_n\}$ is the zero-sequence of a solution of an equation (1.1) *with transcendental entire $A(z)$ of finite order of growth.*

Remarks.

1. By Lemma 2.2, the sequence $\{z_n\}$ is b-separated, if and only if, condition (5.5) holds.
2. To estimate the growth of the Mittag-Leffler series arising in the proof of Theorem D in [**15**], Sauer refers to [**14**]. Indeed, if λ denotes the (finite) exponent of convergence of the zero sequence, if b is the constant in (5.5), and if $\varepsilon > 0$ is fixed, then we can find an entire function $A(z)$ as in Problem 1 such that

$$\rho(A) \leq \max\{2\lambda, b + \lambda\} + \varepsilon = \lambda + \max\{\lambda, b\} + \varepsilon. \tag{5.6}$$

See Remark (a) in [**15**, p. 1147] for details.

In the reasoning below, we estimate the growth of the Mittag-Leffler series by using the reasoning in Section 3. This approach enables us to improve the growth estimate in (5.6). To this end, we restate Theorem D in terms of q-separated sequences, including growth estimates for the coefficient function $A(z)$.

Theorem 5.1. *Let $\{z_n\}$ be an infinite sequence of non-zero complex points having a finite exponent of convergence λ, a finite genus p, and no finite limit points. Let $G(z)$ be the canonical product associated with $\{z_n\}$.*

(a) *Suppose that $\{z_n\}$ is q-separable for some $q \geq 0$, and that $\varepsilon > 0$ is arbitrary. Then $\{z_n\}$ is the zero-sequence of a solution of an equation* (1.1) *with transcendental entire $A(z)$ such that*

$$\rho(A) \leq \max\{\lambda + \varepsilon, q\}. \tag{5.7}$$

(b) *Suppose that $\{z_n\}$ is q-separable for some $q \geq 0$, that $\lambda > 0$, and that $G(z)$ is of finite type. Then $\{z_n\}$ is the zero-sequence of a solution of an equation* (1.1) *with transcendental entire $A(z)$ such that*

$$\rho(A) \leq \max\{\lambda, q\}. \tag{5.8}$$

Remarks.

1. The growth estimate (5.7) is clearly better than the corresponding estimate in (5.6), since $b = q$.
2. If $\lambda = q = 0$ in Part (a) of Theorem 5.1, then we can construct a transcendental entire function $A(z)$ solving Problem 1 such that $\rho(A) \leq \varepsilon$, where $\varepsilon > 0$ is any preassigned fixed constant. Therefore, Part (a) is a natural extension of Theorem C, see Remark following Theorem C.

Proof of Theorem 5.1. By the discussion in the beginning of Section 4, we need to construct an entire function g such that g' satisfies (4.3). This will be done by means of Corollary 3.3.

Part (a). To begin with, we estimate the growth of the expressions $|\sigma_n|$, where the points σ_n are defined in (4.3). Since $\{z_n\}$ is q-separable, and since $\rho(G'') = \rho(G) = \lambda$, we have, for every $\varepsilon > 0$, that there exist constants $C > 0$ and $N \in \mathbb{N}$ such that

$$|\sigma_n| \leq C|z_n| \exp\left(|z_n|^{\lambda+\varepsilon} + |z_n|^q\right), \quad n \geq N. \tag{5.9}$$

Since $z_n \neq 0$ for all n, we may increase the constant C, if necessary, to conclude that the inequality in (5.9) holds for all $n = 1, 2, \ldots$ By Corollary 3.3, an entire function g can be found such that $g'(z_n) = \sigma_n$, and that $\rho(g) = \rho(g') \leq \max\{\lambda + \varepsilon, q\}$. Finally, by means of Lemma 4.1, we have the desired estimate (5.7).

It remains to show that we may choose a Mittag-Leffler series H of the form (3.1), where the points z_n are as in the statement of the assertion, such that $A(z)$ is transcendental. But this fact is proved in [**15**, p. 1146].

Part (b). The functions $G(z)$ and $G''(z)$ have the same finite type by the assumption and [**4**, Theorem 2.1.4]. (Note that $2r$ has to be replaced by $r + 1$ in the proof of [**4**, Theorem 2.1.4], see Errata in [**4**].) Since $\rho(G) = \rho(G'') = \lambda$, there exists constants $D_1 > 0$, $D_2 > 0$ such that $|G''(z_n)| \leq D_1 \exp\left(D_2|z_n|^\lambda\right)$ for $n = 1, 2, \ldots$ The assertion follows by modifying the proof of Part (a). □

6. Methods for solving Problem 2

We solve Problem 2 by using two methods, and show that they both yield infinitely many solutions.

Šeda method. We first approach Problem 2 by using the method due to Šeda, see [**16**]. Let P and R be entire functions (canonical products) with simple zeros exactly at the points a_n and b_n, respectively. By Weierstrass' theorem, such functions do exist. We proceed to construct two entire functions

$$f_1 = Pe^g \quad \text{and} \quad f_2 = Re^h, \tag{6.1}$$

which will be linearly independent solutions of (1.1). Of course, the functions g and h must be entire satisfying certain interpolation conditions. Moreover, we may assume that

$$W(f_1, f_2) = f_1 f_2' - f_1' f_2 = 1 \tag{6.2}$$

by multiplying f_1, say, by a suitable constant factor. If the expressions in (6.1) are substituted in (6.2), an equivalent condition for f_1 and f_2 will be

$$(PR' - P'R + PR(h' - g'))e^{g+h} = 1. \tag{6.3}$$

Denote

$$I = h' - g' \quad \text{and} \quad J = h + g. \tag{6.4}$$

Clearly, the functions g and h are entire, if and only if, the functions I and J are entire. Condition (6.3) may be written as

$$(PR' - P'R + PRI)e^J = 1. \tag{6.5}$$

We proceed to show that there exist entire functions I and J so that (6.5) is satisfied. Solving first (6.5) for I, we get

$$I = \frac{e^{-J} - (PR' - P'R)}{PR}. \tag{6.6}$$

The function I determined by this equation is entire, if and only if, each zero of PR is a zero of $e^{-J} - (PR' - P'R)$. Therefore, I is entire, if and only if,

$$J(z) = \begin{cases} \log\left(-\frac{1}{P'(a_n)R(a_n)}\right), & \text{if } z = a_n, \\ \log\left(\frac{1}{P(b_n)R'(b_n)}\right), & \text{if } z = b_n. \end{cases} \tag{6.7}$$

Observe that any branch of the logarithm in (6.7) may be chosen by the periodic nature of the exponential function.

An entire function J satisfying (6.7) certainly exists. If J_0 is any such function, then the general form of the function is $J = J_F = J_0 + PRF$, where F is an arbitrary entire function. From (6.6), we then define I_F corresponding to J_F. Finally, by means of (6.4), we get the functions $g = g_F$ and $h = h_F$ as follows:

$$\begin{aligned} g &= \frac{1}{2}\left(J - \int^z \frac{e^{-J} - (PR' - P'R)}{PR}\, d\zeta - C\right), \\ h &= \frac{1}{2}\left(J + \int^z \frac{e^{-J} - (PR' - P'R)}{PR}\, d\zeta + C\right), \end{aligned} \tag{6.8}$$

where the integrals represent primitive functions, and $C \in \mathbb{C}$ is a constant, completing the first part of the Šeda approach. Indeed, by [**10**, Proposition 1.4.7][2] and

[2]Obviously, Šeda did not refer to [**10**]. We have merely given a reference from which an interested reader can easily find this classical result.

the requirement (6.2), the coefficient function $A(z)$ now has to be of the form

$$A(z) = -\begin{vmatrix} f_1'' & f_2'' \\ f_1' & f_2' \end{vmatrix}.$$

Remark. If both of the zero sequences $\{a_n\}$ and $\{b_n\}$ are empty, i.e., if we are searching for two linearly independent zero-free solutions of (1.1), then simply define $P(z) \equiv 1$ and $R(z) \equiv 1$. In this case, we may choose the entire function J arbitrarily, since the function I in (6.6) will always be entire. See also [**2**, p. 356] for another method of how to construct two linearly independent zero-free solutions of (1.1).

Finally, if one of the zero sequences, say $\{a_n\}$, is empty, while $\{b_n\}$ is non-empty, then choose $P(z) \equiv 1$, and adjust the method above accordingly, or, simply note that the task is now reduced to solving Problem 1.

Following Šeda, we now proceed to show that there are infinitely many solutions to Problem 2. In the method above, both of the solutions f_1 and f_2 depend on the entire parameter function F. Therefore, the coefficient function $A(z)$ of (1.1) certainly depends on F as well. So, in fact, we are dealing with equation (1.2). We now express the dependence on the function F as follows:

$$A_F(z) = -\begin{vmatrix} f_{1,F}'' & f_{2,F}'' \\ f_{1,F}' & f_{2,F}' \end{vmatrix}. \tag{6.9}$$

It suffices to prove

Statement 1. *Let F_1 and F_2 be entire functions that do not differ by an integer multiple of $2\pi i$. Then $A_{F_1}(z) \not\equiv A_{F_2}(z)$.*

First proof of Statement 1. We show that if $A_{F_1}(z) \equiv A_{F_2}(z)$, then $F_1 - F_2 \equiv k2\pi i$ for some integer k. In this case, we have two solution bases $\{f_{1,F_1}, f_{2,F_1}\}$ and $\{f_{1,F_2}, f_{2,F_2}\}$ such that the functions f_{1,F_1} and f_{1,F_2} (resp. the functions f_{2,F_1} and f_{2,F_2}) share the same zero sequence $\{a_n\}$ (resp. $\{b_n\}$). Therefore, the functions f_{1,F_1} and f_{1,F_2} (resp. the functions f_{2,F_1} and f_{2,F_2}) must be linearly dependent, as one may see by looking at the corresponding Wronskian determinants. It follows that

$$\frac{f_{1,F_1}}{f_{1,F_2}} = e^{g_{F_1} - g_{F_2}} = C_1 \quad \text{and} \quad \frac{f_{2,F_1}}{f_{2,F_2}} = e^{h_{F_1} - h_{F_2}} = C_2,$$

where $C_1, C_2 \in \mathbb{C}$ are some constants. This means that

$$g_{F_1} - g_{F_2} = c_1 \quad \text{and} \quad h_{F_1} - h_{F_2} = c_2 \tag{6.10}$$

for some constants $c_1, c_2 \in \mathbb{C}$. Using the expressions for g and h in (6.8), with $J = J_0 + PRF$, in (6.10), and then summing these two equations, it follows that $PR(F_1 - F_2) = c_1 + c_2$.

If the function $PR(F_1 - F_2)$ has at least one zero, then $PR(F_1 - F_2) \equiv 0$, and hence $F_1 \equiv F_2$. If the function $PR(F_1 - F_2)$ has no zeros, then PR has no zeros, and we define $P(z) \equiv 1$ and $R(z) \equiv 1$, see Remark above. We may choose

$J_0(z) \equiv 0$, since the function J_0 need not satisfy any further properties. Now, subtracting the first equation from the second in (6.10), we get

$$\int^z \left(e^{-F_1} - e^{-F_2}\right) d\zeta = c_2 - c_1.$$

After differentiation, we have $e^{-F_1} = e^{-F_2}$. This implies $F_1 \equiv F_2 + k2\pi i$ for some integer k, and the proof is completed. □

Modified Shen method. The rest of this section is devoted to approaching Problem 2 by modifying the reasoning due to Shen in [**17**], involving the Bank-Laine formula (1.3). More precisely, an arbitrary entire function F is involved just as in [**16**], originally not applied in [**17**]. This approach will be used in Section 7, where we obtain some information on the growth of the coefficient function $A_F(z)$.

So, let F be an arbitrary entire function. Modifying the reasoning in [**17**], the idea is to construct an entire function E_F of the form $E_F = GJ_F$, where G is a canonical product associated with the sequence $\{z_n\} = \{a_n\} \cup \{b_n\}$, and J_F is an entire function with no zeros satisfying certain interpolation properties, to be described later on.

The functions E_F and $A_F(z)$ are connected via the Bank-Laine formula (1.3). To apply Lemma A, we have to ensure that E_F defined above is a BL-function. Since $E_F' = G'J_F + GJ_F'$, it suffices to construct an entire function J_F with no zeros and satisfying

$$J_F(z) = \begin{cases} -\frac{1}{G'(a_n)}, & \text{if } z = a_n, \\ \frac{1}{G'(b_n)}, & \text{if } z = b_n. \end{cases}$$

After such a function J_F is constructed, E_F will be a BL-function, and

$$E_F(z) = \begin{cases} -1, & \text{if } z = a_n, \\ 1, & \text{if } z = b_n. \end{cases} \tag{6.11}$$

By the reasoning in [**17**], E_F takes the zeros $\{a_n\}$ for one solution $f_{1,F}$ of (1.2), and the zeros $\{b_n\}$ for another solution $f_{2,F}$, linearly independent with $f_{1,F}$. Indeed, if D denotes a Euclidean disc such that E_F doesn't vanish there, then $f_{1,F}$ and $f_{2,F}$ have the local representations

$$f_{1,F} = (E_F)^{1/2} \exp\left(-\frac{1}{2}\int^z \frac{d\zeta}{E_F}\right) \tag{6.12}$$

and

$$f_{2,F} = (E_F)^{1/2} \exp\left(\frac{1}{2}\int^z \frac{d\zeta}{E_F}\right) \tag{6.13}$$

in D in terms of a primitive function of $1/E_F$ in D, see [**17**].

Since J_F has no zeros, we may write $J_F = e^{h_F}$, where h_F is an entire function depending on F and satisfying

$$h_F(z) = \begin{cases} \log\left(-\frac{1}{G'(a_n)}\right), & \text{if } z = a_n, \\ \log\left(\frac{1}{G'(b_n)}\right), & \text{if } z = b_n. \end{cases} \tag{6.14}$$

Just as in (6.7), any branch of the logarithm in (6.14) may be chosen by the periodic nature of the exponential function.

By the standard Mittag-Leffler theorem (see, e.g., [**9**, Theorem 7.1.2]) or, alternatively, by Theorem 3.1, we may now construct a meromorphic function H having poles exactly at the points z_n with principal parts

$$\frac{1}{G'(a_n)} \log\left(-\frac{1}{G'(a_n)}\right) \frac{1}{z - a_n}$$

at $z = a_n$ and

$$\frac{1}{G'(b_n)} \log\left(\frac{1}{G'(b_n)}\right) \frac{1}{z - b_n}$$

at $z = b_n$. If one of the sequences $\{a_n\}$ or $\{b_n\}$ is empty, say $\{a_n\}$, we simply construct H such that it has poles at the points b_n only.

We now claim that

$$h_F = G(H + F)$$

satisfies the desired properties. To prove this, we first observe that GH is entire and satisfies

$$G(z)H(z) = \begin{cases} \log\left(-\frac{1}{G'(a_n)}\right), & \text{if } z = a_n, \\ \log\left(\frac{1}{G'(b_n)}\right), & \text{if } z = b_n. \end{cases}$$

Hence, h_F is entire and clearly satisfies (6.14). Problem 2 is now solved by incorporating the entire parameter function F in the reasoning in [**17**].

We proceed to show that this second method also yields infinitely many solutions to Problem 2. This immediately results from the following:

Second proof of Statement 1. As in the first proof, we assume that $A_{F_1}(z) \equiv A_{F_2}(z)$. We recall that F_1 and F_2 both determine solution bases $\{f_{1,F_1}, f_{2,F_1}\}$ and $\{f_{1,F_2}, f_{2,F_2}\}$ for (1.2) such that the functions f_{1,F_1} and f_{1,F_2} (resp. f_{2,F_1} and f_{2,F_2}) are linearly dependent, since they share the same zeros. Therefore, $f_{1,F_1}/f_{1,F_2}$ (resp. $f_{2,F_1}/f_{2,F_2}$) is a constant function. Choose a Euclidean disc D' such that E_{F_1} and E_{F_2} do not vanish there. By the local representation (6.12), we see that

$$\frac{f_{1,F_1}}{f_{1,F_2}} = \left(\frac{E_{F_1}}{E_{F_2}}\right)^{1/2} \exp\left(\frac{1}{2}\int^z \frac{d\zeta}{E_{F_2}} - \frac{1}{2}\int^z \frac{d\zeta}{E_{F_1}}\right) = \text{constant} \tag{6.15}$$

in D'. Differentiating (6.15) results in

$$W(E_{F_1}, E_{F_2}) = E_{F_1} - E_{F_2}, \tag{6.16}$$

which is true in D'. Similarly, by the local representation (6.13) for $f_{2,F_1}/f_{2,F_2}$, we get

$$W(E_{F_1}, E_{F_2}) = E_{F_2} - E_{F_1} \tag{6.17}$$

in D'. Equations (6.16) and (6.17) yield $E_{F_1} = E_{F_2}$ in D', hence in $\mathbb{C}$ by the standard uniqueness theorem of analytic functions.

By our notation, the identity $E_{F_1} = E_{F_2}$ yields $h_{F_1} = h_{F_2} + k2\pi i$ for some integer k. Further,

$$G(F_1 - F_2) = k2\pi i \tag{6.18}$$

must hold for all $z \in \mathbb{C}$. If the function G has at least one zero, then $F_1 = F_2$ and $k = 0$. If G has no zeros, we can define $G(z) \equiv 1$ (and J_F can be any entire function), which yields the assertion. □

7. Problem 3(b)

In this section we shortly discuss the growth of the coefficient functions $A(z)$ as related to Problem 2, using the method due to Shen discussed in Section 6. Indeed, the growth of $A(z)$ is strongly related to the growth of the function $E(z)$ by means of the BL-formula (1.3). In this direction, Elzaidi [**6**] and Langley [**12**] made the first contributions quite recently. Indeed, the essential question may be formulated as follows:

Problem 4. *For which infinite sequences of distinct complex points having a finite exponent of convergence we may find a BL-function of finite order having zeros exactly at these points?*

Following [**6**], such sequences are called as *Bank-Laine sequences* (BL-sequences). For example, the set of all non-zero integers (which clearly has an exponent of convergence equal to zero) does not form a BL-sequence, see [**6**, Theorem 2.1]. In terms of equation (1.1), we may rephrase this as follows. If we choose the prescribed zero sequences as $a_n = n$ and $b_n = -n$, then we may construct $A(z)$ as required in Problem 2, but the function $A(z)$ will always be of infinite order of growth by (1.3). This suggests that the coefficient function $A(z)$ related to Problem 2 is typically of infinite order of growth.

Of course, one may ask about the iterated order of $A(z)$. All we need is a growth estimate for the BL-function $E(z)$ related to $A(z)$, and then use (1.3). We know that $E(z)$ is a product of a canonical product and a certain exponential function determined by the related interpolation problem (6.14). Indeed, we could form a sequence $\{z_n\}$ by setting

$$z_{2n-1} = a_n \quad \text{and} \quad z_{2n} = b_n,$$

assume that the sequence so obtained is q-separable, and then apply Corollary 3.3 to construct the entire function $h = h_F$ satisfying (6.14). Indeed, by taking $\sigma_n = \log((-1)^n/G'(z_n))$ in Corollary 3.3, we obtain an estimate for $|\sigma_n|$ by q-separability. This then gives a growth estimate for h_F and finally for the coefficient function $A(z) = A_F(z)$. We omit these details.

We finally note that the recent investigations on BL-sequences and BL-functions are closely related to the well-known Bank-Laine conjecture claiming that for any two linearly independent solutions f_1, f_2 of (1.1), $\max(\lambda(f_1), \lambda(f_2) = \infty$, provided $A(z)$ is of finite non-integer order, see, e.g., [**11**] for more details.

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