A COMBINATORIAL IDENTITY ARISING FROM COBORDISM THEORY

D. GIJSWIJT AND P. MOREE

Dedicated to the memory of Alexander Reznikov

ABSTRACT. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{R}^m_{>0}$. Let $\underline{\alpha}_{i,j}$ be the vector obtained from $\underline{\alpha}$ by deleting the entries α_i and α_j. Besser and Moree [1] introduced some invariants and near invariants related to the solutions $\underline{\epsilon} \in \{\pm 1\}^{m-2}$ of the linear inequality $|\alpha_i - \alpha_j| < \langle \underline{\epsilon}, \underline{\alpha}_{i,j} \rangle < \alpha_i + \alpha_j$, where $\langle , \rangle$ denotes the usual inner product and $\underline{\alpha}_{i,j}$ the vector obtained from $\underline{\alpha}$ by deleting α_i and α_j. The main result of Besser and Moree [1] is extended here to a much more general setting, namely that of certain maps from finite sets to $\{-1, 1\}$.

1. Introduction

Let $m \geq 3$. Let $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{R}^m_{>0}$ and suppose that there is no $\underline{\epsilon} \in \{\pm 1\}^m$ satisfying $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$. Let $1 \leq i < j \leq m$. Let $\underline{\alpha}_{i,j} \in \mathbb{R}^{m-2}_{>0}$ be the vector obtained from $\underline{\alpha}$ by deleting α_i and α_j. Let

$$S_{i,j}(\underline{\alpha}) := \{\underline{\epsilon} \in \{\pm 1\}^{m-2} : |\alpha_i - \alpha_j| < \left\langle \underline{\epsilon}, \underline{\alpha}_{i,j} \right\rangle < \alpha_i + \alpha_j\}.$$

Define $N_{i,j}(\underline{\alpha}) = \sum_{\underline{\epsilon} \in S_{i,j}(\underline{\alpha})} \prod_{k=1}^{m-2} \epsilon_k$. Theorem 2.1 of [1] states that the reduction of $\#S_{i,j}(\underline{\alpha})$ mod 2 depends only on $\underline{\alpha}$ and that in case of m odd, $N_{i,j}(\underline{\alpha})$ depends only on $\underline{\alpha}$. In particular it was shown that for $m \geq 3$ and

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odd we have

$$N_{i,j}(\underline{\alpha}) = -\frac{1}{4} \sum_{\underline{\epsilon}\in\{\pm 1\}^m} \mathrm{sgn}(\langle \underline{\epsilon}, \underline{\alpha}\rangle) \prod_{k=1}^{m} \epsilon_k. \tag{1}$$

From (1) we of course immediately read off that if $m \geq 3$ is odd, $N_{i,j}(\underline{\alpha})$ does not depend on the choice of i and j.

Example 1.1. We take $\underline{\beta}_m = (\log 2, \ldots, \log p_m)$, where $p_1, \ldots, p_m$ denote the consecutive primes and put $Q = p_1 \cdots p_m$. Then it is not difficult to show that, for $1 \leq i < j \leq m$,

$$N_{i,j}(\underline{\beta}_m) = (-1)^m \sum_{\substack{\sqrt{Q/p_i} < n < \sqrt{Q} \\ \gcd(n, p_i p_j)=1,\ P(n) \leq p_m}} \mu(n),$$

where $P(n)$ denotes the largest prime factor of n and μ the Möbius function. For $m \geq 2$ put

$$g(m) = \frac{(-1)^{m+1}}{4} \sum_{d | p_1 \cdots p_m} \mathrm{sgn}\Big(\frac{d^2}{p_1 \cdots p_m} - 1\Big)\mu(d),$$

where sgn denotes the sign function. The fundamental theorem of arithmetic ensures that there is no $\underline{\epsilon} \in \{\pm 1\}^m$ satisfying $\langle \underline{\epsilon}, \underline{\beta}_m \rangle = 0$. By (1) we then infer that if $m \geq 3$ is odd, $N_{i,j}(\underline{\beta}_m) = g(m)$ and so it does not depend on the choice of i and j. By Remark 2.5 of [1] we have $g(m) = 0$ for $m \geq 2$ and even. The first non-trivial values one finds for $g(m)$ are given in the table below.

m	3	5	7	9	11	13	15	17	19	21	23
$g(m)$	1	-1	3	-8	22	-53	158	-481	1471	-4621	14612

(The value given for $m = 15$ corrects the value at p. 471 of [1]. For a computer program to evaluate these values see [2].)

Example 1.2. Put $Q(n) = \sum_{d|n,\, d\le\sqrt{n}} \mu(d)$.
The sequence $\{Q(0), Q(1), Q(2), \dots\}$ is the sequence A068101 of OEIS [3].
Let $n > 1$ be a squarefree integer having k distinct prime divisors $q_1, \dots, q_k$ with $k \ge 2$.

Note that in the previous example we used only that $p_1, \dots, p_m$ are distinct primes. If we replace them by $q_1, \dots, q_k$ we infer, proceeding as in the previous example, that

$$g_n(k) := \frac{(-1)^{k+1}}{4} \sum_{d|n} \operatorname{sgn}\left(\frac{d^2}{n} - 1\right) \mu(d)$$

is an integer that equals zero if k is even. On using that $\sum_{d|n} \mu(d) = 0$ it is seen that $g_n(k) = \frac{(-1)^k}{2} Q(n)$, whence the following result is inferred:

Proposition 1. *Let $n > 1$ be a squarefree number having k distinct prime divisors. Then*

$$Q(n) = \begin{cases} 1 & if\ n\ is\ a\ prime; \\ 0 & if\ k\ is\ even; \\ \text{even} & if\ k \ge 3\ is\ odd. \end{cases}$$

2. General setup

We consider a more general quantity $N_\sigma(a, b)$ similar to $N_{i,j}(\underline{\alpha})$ so that the latter is a special case of the former.

Let X be a finite set. Suppose that we have a map $\sigma : 2^X \to \{-1, 1\}$ such that $\sigma(X \backslash A) = \sigma(A)$ for all $A \subseteq X$. We will call such a map σ *even*. Let $u, v \in X$ with $u \neq v$. Define

$$N_\sigma(u, v) := \sum_{\substack{A \subseteq X,\ u \in A,\ v \notin A \\ \sigma(A) = \sigma(A+v)}} \sigma(A), \tag{2}$$

where the summation is over all subsets A of X such that $u \in A$, $v \notin A$ and $\sigma(A) = \sigma(A + v)$.

Theorem 1. *Let σ be an even map from $X \to \{-1, 1\}$. Then*

$$N_\sigma(u, v) = \frac{1}{4} \sum_{A \subseteq X} \sigma(A)$$

and thus in particular $N_\sigma(u, v)$ does not depend on the choice of u and v.

Proof. We have

$$\begin{aligned} 2N_\sigma(u, v) &= \sum_{\substack{A \subseteq X,\ u \in A,\ v \notin A \\ \sigma(A) = \sigma(A+v)}} (\sigma(A) + \sigma(A + v)) = \sum_{\substack{A \subseteq X \\ u \in A, v \notin A}} (\sigma(A) + \sigma(A + v)) \\ &= \sum_{\substack{A \subseteq X \\ u \in A}} \sigma(A) = \frac{1}{2} \sum_{\substack{A \subseteq X \\ u \in A}} (\sigma(A) + \sigma(X \backslash A)), \\ &= \frac{1}{2} (\sum_{\substack{A \subseteq X \\ u \in A}} \sigma(A) + \sum_{\substack{A \subseteq X \\ u \notin A}} \sigma(A)) = \frac{1}{2} \sum_{A \subseteq X} \sigma(A), \end{aligned}$$

where we used that there is a bijection between the sets containing u and those not containing u, the bijection being taking complementary sets. $\square$

Remark. In case the cardinality of X is odd, we can alternatively consider a map $\tau : 2^X \to \{-1, 1\}$ such that $\tau(X\backslash A) = -\tau(A)$ for all $A \subseteq X$. Then the map σ defined by $\sigma(A) = (-1)^{\#A}\tau(A)$ is even and the conditions of Proposition 1 are satisfied.

3. Examples

We present three applications of Theorem 1.

Example 3.1. Suppose $X = \{x_1, \dots, x_m\}$ and $m \geq 3$. Let f be a map such that $f(x_j) = \pm 1$ for $1 \leq j \leq m$. Consider the map $\sigma : 2^X \to \{-1, 1\}$ defined by $\sigma(A) = \prod_{a \in A} f(a)$ for $A \subseteq X$. Let us assume that $\prod_{x \in X} f(x) = 1$ (so that σ is an even map). Theorem 1 then gives that

$$N_\sigma(u, v) = \begin{cases} 2^{\#X-2} & if\ f(x_j) = 1\ for\ 1 \leq j \leq m; \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.2. We reprove the main result from [1] which is reproduced in the present note as (1), where we now drop the requirement that $\alpha_j > 0$ for $1 \leq j \leq m$. Let $X = \{\alpha_1, \dots, \alpha_m\}$ be a set of cardinality m consisting of real numbers such that there is no $\underline{\epsilon} \in \{\pm 1\}^m$ satisfying $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$. Let A be any subset of X. To A we associate $\underline{\epsilon} = (\epsilon_1, \dots, \epsilon_m)$, where $\epsilon_j = -1$ if $\alpha_j \in A$ and $\epsilon_j = 1$ otherwise. Let $\sigma(A) = \operatorname{sgn}(\langle \underline{\epsilon}, \underline{\alpha} \rangle)\epsilon_1 \cdots \epsilon_m$. By assumption $\langle \underline{\epsilon}, \underline{\alpha} \rangle \neq 0$ and hence $\sigma(A) \in \{-1, 1\}$. Let $i \neq j$. We evaluate $N_\sigma(\alpha_i, \alpha_j)$ according to the definition (2). We obtain that $N_\sigma(\alpha_i, \alpha_j) = \sum' \operatorname{sgn}(\langle \underline{\epsilon}, \underline{\alpha} \rangle) \prod_{k=1}^m \epsilon_k$, where the dash indicates that we sum over those $\underline{\epsilon} \in \{\pm 1\}^m$, where $\epsilon_i = -1$, $\epsilon_j = 1$ and

$$-\operatorname{sgn}(\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \rangle - \alpha_i + \alpha_j) = \operatorname{sgn}(\langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \rangle - \alpha_i - \alpha_j).$$

Note that the latter condition is satisfied iff $\alpha_i - |\alpha_j| < \langle \underline{\epsilon}_{i,j}, \underline{\alpha}_{i,j} \rangle < \alpha_i + |\alpha_j|$. If $\underline{\epsilon} \in \{\pm 1\}^m$ satisfies the latter inequality, $\epsilon_i = -1$ and $\epsilon_j = 1$, then

$$\operatorname{sgn}(\langle \underline{\epsilon}, \underline{\alpha} \rangle) \prod_{k=1}^{m} \epsilon_k = -\operatorname{sgn}(\alpha_j) \prod_{\substack{k=1 \\ k \neq i,j}}^{m} \epsilon_k.$$

We infer that

$$N_\sigma(\alpha_i, \alpha_j) = -\operatorname{sgn}(\alpha_j) \sum_{\substack{\underline{\epsilon} \in \{\pm 1\}^{m-2} \\ \alpha_i - |\alpha_j| < \langle \underline{\epsilon}, \underline{\alpha}_{i,j} \rangle < \alpha_i + |\alpha_j|}} \prod_{k=1}^{m-2} \epsilon_k.$$

In case m is odd, σ is even and Theorem 1 can be applied (note that $N_\sigma(\alpha_i, \alpha_j) = -\mathcal{N}_{i,j}(\underline{\alpha})$) to give the following corollary.

Corollary 1. *Let $\underline{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{R}^m$ and suppose that there is no $\underline{\epsilon} \in \{\pm 1\}^m$ satisfying $\langle \underline{\epsilon}, \underline{\alpha} \rangle = 0$. Let $1 \leq i < j \leq m$. Put*

$$\mathcal{S}_{i,j}(\underline{\alpha}) := \{\underline{\epsilon} \in \{\pm 1\}^{m-2} : \alpha_i - |\alpha_j| < \left\langle \underline{\epsilon}, \underline{\alpha}_{i,j} \right\rangle < \alpha_i + |\alpha_j|\}.$$

Define $\mathcal{N}_{i,j}(\underline{\alpha}) = \operatorname{sgn}(\alpha_j) \sum_{\underline{\epsilon} \in \mathcal{S}_{i,j}(\underline{\alpha})} \prod_{k=1}^{m-2} \epsilon_k$. If $m \geq 3$ and m is odd, then

$$\mathcal{N}_{i,j}(\underline{\alpha}) = -\frac{1}{4} \sum_{\underline{\epsilon} \in \{\pm 1\}^m} \operatorname{sgn}(\langle \underline{\epsilon}, \underline{\alpha} \rangle) \prod_{k=1}^{m} \epsilon_k = h(\underline{\alpha}),$$

does not depend on i and j. If one of the entries of $\underline{\alpha}$ is zero, then $h(\underline{\alpha}) = 0$.

In case $\underline{\alpha} \in \mathbb{R}^m_{>0}$ it is not immediately clear that this result implies (1). To see that this is nevertheless true it suffices to show that under the conditions of Corollary 1 we have $\mathcal{N}_{i,j}(\underline{\alpha}) = N_{i,j}(\underline{\alpha})$. If $\alpha_j \leq \alpha_i$ this is obvious, so assume that $\alpha_j > \alpha_i$. Notice that $\underline{\epsilon} \in \{\pm 1\}^{m-2}$ is in $\mathcal{S}_{i,j}(\underline{\alpha}) \backslash S_{i,j}(\underline{\alpha})$ iff $\alpha_i - \alpha_j < \left\langle \underline{\epsilon}, \underline{\alpha}_{i,j} \right\rangle < \alpha_j - \alpha_i$. But if $\underline{\epsilon}$ satisfies the latter inequality, so does $-\underline{\epsilon}$ and both are counted with opposite sign in $\mathcal{N}_{i,j}(\underline{\alpha}) - N_{i,j}(\underline{\alpha})$ and consequently $\mathcal{N}_{i,j}(\underline{\alpha}) = N_{i,j}(\underline{\alpha})$.

Example 3.3. Corollary 1 can be generalised to a higher dimensional setting. Instead of numbers $\alpha_1, \ldots, \alpha_m$ we can consider points $\underline{\alpha}_1, \ldots, \underline{\alpha}_m$ with $\underline{\alpha}_i \in \mathbb{R}^n$ and $n \geq 2$. We assume that $\pm\underline{\alpha}_1 \pm \cdots \pm \underline{\alpha}_m \neq \underline{0}$. Let us define B to be the matrix with $\underline{\alpha}_j$ as jth row for $1 \leq j \leq m$. Choose a hyperplane H through the origin not containing any of the points $\pm\underline{\alpha}_1 \pm \cdots \pm \underline{\alpha}_m$ (the assumption that $\pm\underline{\alpha}_1 \pm \cdots \pm \underline{\alpha}_m \neq \underline{0}$ ensures that this is possible). Let $\underline{n} \notin H$ be on the normal of this hyperplane. Let A be any subset of X. To A we associate $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_m)$, where $\epsilon_j = -1$ if $\underline{\alpha}_j \in A$ and $\epsilon_j = 1$ otherwise. Let $\sigma(A) = \operatorname{sgn}(\langle \underline{n}, \underline{\epsilon}B\rangle)\epsilon_1 \cdots \epsilon_m$. The assumption on H implies that $\langle \underline{n}, \underline{\epsilon}B\rangle \neq 0$ and hence $\sigma(A) \in \{-1, 1\}$. Choose two points $\underline{\alpha}_i$ and $\underline{\alpha}_j$, $i \neq j$. Let V be the hyperplane with normal $\underline{n}$ containing $\underline{\alpha}_i - \underline{\alpha}_j$ and W be the hyperplane with normal $\underline{n}$ containing $\underline{\alpha}_i + \underline{\alpha}_j$. We define the weight $w(\underline{\alpha})$ of a point $\underline{\alpha}$ of the form $\underline{\alpha} = \sum_{\substack{1 \leq k \leq m \\ k \neq i,\ k \neq j}} \epsilon_k \underline{\alpha}_k$ with $\underline{\epsilon}_{i,j} \in \{\pm 1\}^{m-2}$ to be $\prod_{\substack{1 \leq k \leq m \\ k \neq i,\ k \neq j}} \epsilon_k$. Note that our choice of $\underline{n}$ ensures that none of these points is in V or W. Then let $M(i,j)$ be the sum of the weights of all points $\sum_{\substack{1 \leq k \leq m \\ k \neq i,\ k \neq j}} \epsilon_k \underline{\alpha}_k$ that are in between V and W and for which $\underline{\epsilon}_{i,j} \in \{\pm 1\}^{m-2}$. If $m \geq 3$ is odd, then σ is an even map. It is not difficult to show that $N_\sigma(\underline{\alpha}_i, \underline{\alpha}_j) = \pm M(i,j)$, where the sign is independent of i and j. Theorem 1 applies and we infer that $M(i,j)$ is independent of the choice of i and j.

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1. Besser A. and Moree R., *On an invariant related to a linear inequality*, Arch. Math. **79** (2002), 463–471.
2. Noe T., *Sequence A086596, On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.
3. Quet L., *Sequence A068101, On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences/`.

D. Gijswijt, Korteweg-de Vries Institute, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands,
e-mail: `gijswijt@science.uva.nl`

P. Moree, Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany,
e-mail: `moree@mpim-bonn.mpg.de`