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THE SHERMAN-MORRISON FORMULA AND EIGENVALUES OF A SPECIAL BORDERED MATRIX

G. TRENKLER AND D. TRENKLER

ABSTRACT. The article of Ding and Pye [**3**] is reconsidered and extended. In contrast to their assertion, we find that the Sherman-Morrison formula is well suited to prove certain statements about a class of bordered matrices.

1. INTRODUCTION

In their paper, Ding and Pye [**3**] were interested in finding the eigenvalues and the pseudoinverse of the $(n+1)\times(n+1)$ bordered matrix

$$\mathbf{A} = \begin{pmatrix} 1 & \mathbf{p}^{\mathrm{T}} \\ \mathbf{q} & \mathbf{u}\mathbf{v}^{\mathrm{T}} \end{pmatrix}, \tag{1}$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$ and $\mathbf{q}$ are real n-dimensional vectors. The authors reported that in the special case in which $\mathbf{p} = \mathbf{D}\mathbf{v}$ and $\mathbf{q} = \mathbf{D}^{-1}\mathbf{u}$, where $\mathbf{D}$ is a diagonal matrix and corresponding components of $\mathbf{u}$ and $\mathbf{v}$ are reciprocals of each other, the characteristic polynomial for $\mathbf{A}$ can be obtained with the help of the Sherman-Morrison formula. However, this approach would be "tedious and may not be applicable for general bordered matrix $\mathbf{A}$ as given in (1)".

Opposing this point of view, in the following we demonstrate that the Sherman-Morrison formula is a useful tool to calculate the eigenvalues of the matrix $\mathbf{A}$. Furthermore, we investigate a slightly more general class of matrices and allow for complex entries. Finally, we do not assume linear independence of the occurring vectors in order to treat the general case later by a continuity argument.

2. A NEW CLASS OF MATRICES AND ITS SPECTRUM

Let us consider the $(n+1)\times(n+1)$ matrix

$$\mathbf{B} = \begin{pmatrix} 0 & \mathbf{s}^{*} \\ \mathbf{r} & \mathbf{0}_{n\times n} \end{pmatrix}, \tag{2}$$

where $\mathbf{r}$ and $\mathbf{s}$ are $n\times 1$ vectors with complex entries, $\mathbf{s}^{*}$ denoting the conjugate transpose of $\mathbf{s}$. We are interested in finding the eigenvalues of the matrix

$$\mathbf{M} = \mathbf{B} + \mathbf{b}\mathbf{c}^{*}, \tag{3}$$

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where $\mathbf{b}$ and $\mathbf{c}$ are complex vectors with $n+1$ components. Observe that the matrix $\mathbf{A}$ from (1) can be represented in the form of (3) by choosing

$$\mathbf{B} = \begin{pmatrix} 0 & \mathbf{p}^{\mathsf{T}} - \mathbf{v}^{\mathsf{T}} \\ \mathbf{q} - \mathbf{u} & \mathbf{0}_{n\times n} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix}.$$

Note that in the real case the notions transpose and conjugate transpose coincide.

The Moore-Penrose inverse $\mathbf{B}^{\dagger}$ of $\mathbf{B}$ is readily seen to be

$$\mathbf{B}^{\dagger} = \begin{pmatrix} 0 & \mathbf{r}^{\dagger} \\ \mathbf{s}^{\dagger *} & \mathbf{0}_{n\times n} \end{pmatrix}, \tag{4}$$

where $\mathbf{r}^{\dagger} = \mathbf{r}^*/\mathbf{r}^*\mathbf{r}$ if $\mathbf{r} \neq \mathbf{0}$ and $\mathbf{r}^{\dagger} = \mathbf{0}$ otherwise. The Moore-Penrose inverse of $\mathbf{M} = \mathbf{B} + \mathbf{bc}^*$ can be easily calculated from $\mathbf{B}^{\dagger}$ with some additive correction terms (see [**2**, Ch. 3], or [**1**]). In their paper, Ding and Pye [**3**, Theorem 3.2] computed the Moore-Penrose inverse of $\mathbf{A}$ on the basis of the eigenvalues of $\mathbf{AA}^{\mathsf{T}}$.

It is easy to see that the rank of $\mathbf{B}$ cannot exceed 2. This follows from $\mathrm{rk}(\mathbf{B}) = \mathrm{rk}(\mathbf{B}^{\dagger}\mathbf{B}) = \mathrm{tr}(\mathbf{B}^{\dagger}\mathbf{B}) = \mathbf{s}^{\dagger}\mathbf{s} + \mathbf{r}^{\dagger}\mathbf{r} \leq 2$.

Let us first calculate the eigenvalues of $\mathbf{B}$. For this purpose put

$$\mathbf{B}(\lambda) = \mathbf{B} - \lambda \mathbf{I}_{n+1} = \begin{pmatrix} -\lambda & \mathbf{s}^* \\ \mathbf{r} & -\lambda \mathbf{I}_n \end{pmatrix}, \tag{5}$$

and assume $n \geq 2$.

Theorem 1. *The characteristic polynomial of* $\mathbf{B}$ *is*

$$p_{\mathbf{B}}(\lambda) = (-\lambda)^{n-1}(\lambda^2 - \mathbf{s}^*\mathbf{r}).$$

Proof. We have $p_{\mathbf{B}}(\lambda) = \det \mathbf{B}(\lambda)$. By a well-known formula (see e.g. [**4**, Ch. 2]) for $\lambda \neq 0$ we get

$$\begin{aligned} \det \mathbf{B}(\lambda) &= \det(-\lambda \mathbf{I}_n) \det(-\lambda - \mathbf{s}^*(-\lambda \mathbf{I}_n)^{-1}\mathbf{r}) \\ &= (-\lambda)^n \left(-\lambda + \frac{1}{\lambda}\mathbf{s}^*\mathbf{r}\right) \\ &= (-\lambda)^{n-1}(\lambda^2 - \mathbf{s}^*\mathbf{r}). \end{aligned}$$

□

Observe that Ding and Pye [**3**] define the characteristic polynomial in a different, but equivalent way.

The preceding result shows that $\lambda = 0$ is an eigenvalue of algebraic multiplicity $n-1$. The other two potentially nonzero eigenvalues of $\mathbf{B}$ are $\lambda = \pm\sqrt{\mathbf{s}^*\mathbf{r}}$. It easily follows that in this case the inverse of $\mathbf{B}(\lambda)$ is given by

$$\mathbf{B}(\lambda)^{-1} = \frac{1}{\lambda} \begin{pmatrix} \varphi\lambda & \varphi \mathbf{s}^* \\ \varphi \mathbf{r} & \frac{\varphi}{\lambda}\mathbf{rs}^* - \mathbf{I}_n \end{pmatrix}, \tag{6}$$

where $\varphi = \lambda/(\mathbf{s}^*\mathbf{r} - \lambda^2)$.

Let us now turn to the problem of finding the eigenvalues of $\mathbf{M} = \mathbf{B} + \mathbf{bc}^*$, whose rank cannot exceed $\mathrm{rk}(\mathbf{B}) + 1$. For this purpose we consider $\mathbf{M}(\lambda) = \mathbf{M} - \lambda \mathbf{I}_{n+1}$, i.e. $\mathbf{M}(\lambda) = \mathbf{B}(\lambda) + \mathbf{bc}^*$.

To determine the eigenvalues of $\mathbf{M}$, the Sherman-Morrison formula will be used. It says that if $\mathbf{F}$ is nonsingular and $\mathbf{g}$ and $\mathbf{h}$ are suitable vectors such that $1 + \mathbf{h}^*\mathbf{F}^{-1}\mathbf{g} \neq 0$ then the sum $\mathbf{F} + \mathbf{g}\mathbf{h}^*$ is nonsingular, and

$$(\mathbf{F} + \mathbf{g}\mathbf{h}^*)^{-1} = \mathbf{F}^{-1} - \frac{1}{1 + \mathbf{h}^*\mathbf{F}^{-1}\mathbf{g}}\mathbf{F}^{-1}\mathbf{g}\mathbf{h}^*\mathbf{F}^{-1}$$

(see [**4**, Ch. 3]).

Case 1: $\lambda \neq 0$, $\lambda^2 \neq \mathbf{s}^*\mathbf{r}$

Then $\det \mathbf{M}(\lambda) = \det(\mathbf{B}(\lambda) + \mathbf{b}\mathbf{c}^*) = \det \mathbf{B}(\lambda)[1 + \mathbf{c}^*\mathbf{B}(\lambda)^{-1}\mathbf{b}]$, see [**4**, Ch. 6]. Partitioning $\mathbf{c} = (\overline{c_0}, \mathbf{c}_1^*)^*$ and $\mathbf{b} = (\overline{b_0}, \mathbf{b}_1^*)^*$ from (6) we get

$$\begin{aligned}\mathbf{c}^*\mathbf{B}(\lambda)^{-1}\mathbf{b} = \frac{1}{\lambda^2}[&\overline{c_0}\varphi\lambda^2 b_0 + \overline{c_0}\varphi\lambda(\mathbf{s}^*\mathbf{b}_1) \\ &+ \varphi\lambda b_0(\mathbf{c}_1^*\mathbf{r}) + \varphi(\mathbf{c}_1^*\mathbf{r})(\mathbf{s}^*\mathbf{b}_1) - \lambda(\mathbf{c}_1^*\mathbf{b}_1)].\end{aligned} \tag{7}$$

Hence from Theorem 1 we obtain

$$\begin{aligned}\det \mathbf{M}(\lambda) = (-\lambda)^{n-3}(\lambda^2 - \mathbf{s}^*\mathbf{r})[&\overline{c_0}\varphi\lambda^2 b_0 + \overline{c_0}\varphi\lambda(\mathbf{s}^*\mathbf{b}_1) \\ &+ \varphi\lambda b_0(\mathbf{c}_1^*\mathbf{r}) + \varphi(\mathbf{c}_1^*\mathbf{r})(\mathbf{s}^*\mathbf{b}_1) - \lambda(\mathbf{c}_1^*\mathbf{b}_1) + \lambda^2].\end{aligned}$$

Since $\varphi(\lambda^2 - \mathbf{s}^*\mathbf{r}) = -\lambda$ it follows that

$$\begin{aligned}\det \mathbf{M}(\lambda) = (-\lambda)^{n-2}[&-\lambda^3 + (\mathbf{c}^*\mathbf{b})\lambda^2 + (\overline{c_0}(\mathbf{s}^*\mathbf{b}_1) + b_0(\mathbf{c}_1^*\mathbf{r}) + (\mathbf{s}^*\mathbf{r}))\lambda \\ &+ (\mathbf{c}_1^*\mathbf{r})(\mathbf{s}^*\mathbf{b}_1) - (\mathbf{c}_1^*\mathbf{b}_1)(\mathbf{s}^*\mathbf{r})],\end{aligned} \tag{8}$$

where use is made of the identity $\mathbf{c}^*\mathbf{b} = \overline{c_0}b_0 + \mathbf{c}_1^*\mathbf{b}_1$.

Case 2: $\lambda \neq 0$, $\lambda^2 = \mathbf{s}^*\mathbf{r}$

Then $\det \mathbf{M}(\lambda) = \det \mathbf{B}(\lambda) + \mathbf{c}^*\mathbf{B}(\lambda)^{\#}\mathbf{b}$, where $\mathbf{B}(\lambda)^{\#}$ is the adjoint of $\mathbf{B}(\lambda)$ i.e. the transpose of the matrix of cofactors of $\mathbf{B}(\lambda)$, see e.g. [**5**, Ch. 6]. However, since $\lambda = \mathbf{s}^*\mathbf{r}$, by Theorem 1 we have $\det \mathbf{B}(\lambda) = 0$, and consequently $\det \mathbf{M}(\lambda) = \mathbf{c}^*\mathbf{B}(\lambda)^{\#}\mathbf{b}$. Some direct calculations show that

$$\mathbf{B}(\lambda)^{\#} = (-\lambda)^{n-2}\begin{pmatrix}\lambda^2 & \lambda\mathbf{s}^* \\ \lambda\mathbf{r} & \mathbf{r}\mathbf{s}^*\end{pmatrix}.$$

Thus we obtain

$$\begin{aligned}\det \mathbf{M}(\lambda) &= (-\lambda)^{n-2}[\overline{c_0}b_0\lambda^2 + (\overline{c_0}(\mathbf{s}^*\mathbf{b}_1) + b_0(\mathbf{c}_1^*\mathbf{r}))\lambda + (\mathbf{c}_1^*\mathbf{r})(\mathbf{s}^*\mathbf{b}_1)] \\ &= (-\lambda)^{n-2}[(\overline{c_0}(\mathbf{s}^*\mathbf{b}_1) + b_0(\mathbf{c}_1^*\mathbf{r}))\lambda + \overline{c_0}b_0(\mathbf{s}^*\mathbf{r}) + (\mathbf{c}_1^*\mathbf{r})(\mathbf{s}^*\mathbf{b}_1)]\end{aligned} \tag{9}$$

by using $\mathbf{s}^*\mathbf{r} = \lambda^2$ again. However, formula (9) coincides with formula (8) when inserting $\mathbf{s}^*\mathbf{r} = \lambda^2$ in the latter.

This gives us the main result.

Theorem 2. *The characteristic polynomial of* $\mathbf{M} = \mathbf{B} + \mathbf{b}\mathbf{c}^*$ *is*

$$\begin{aligned}p_{\mathbf{M}}(\lambda) = \det \mathbf{M}(\lambda) = (-\lambda)^{n-2}[&-\lambda^3 + (\mathbf{c}^*\mathbf{b})\lambda^2 + (\overline{c_0}(\mathbf{s}^*\mathbf{b}_1) + b_0(\mathbf{c}_1^*\mathbf{r}) \\ &+ (\mathbf{s}^*\mathbf{r}))\lambda + (\mathbf{c}_1^*\mathbf{r})(\mathbf{s}^*\mathbf{b}_1) - (\mathbf{c}_1^*\mathbf{b}_1)(\mathbf{s}^*\mathbf{r})].\end{aligned}$$

3. Final Remark

The eigenspaces corresponding to the eigenvalues λ of $\mathbf{M} = \mathbf{B} + \mathbf{bc}^*$ can be found by solving the linear homogenous equations $\mathbf{M}(\lambda)\mathbf{x} = \mathbf{0}$. Its explicit solution $\mathbf{x} = [\mathbf{I}_{n+1} - \mathbf{M}(\lambda)^\dagger \mathbf{M}(\lambda)]\mathbf{z}$ is obtainable from List 2 in [**1**].

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G. Trenkler, Department of Statistics, University of Dortmund, Vogelpothsweg 87, D-44221 Dortmund, Germany, *e-mail*: `trenkler@statistik.uni-dortmund.de`

D. Trenkler, Department of Economics, University of Osnabrück, Rolandstraße 8, D-49069 Osnabrück, Germany, *e-mail*: `dtrenkler@nts6.oec.uni-osnabrueck.de`