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GROUPS OF PERIODS FOR ARBITRARY MAPS ON GROUPS

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ABSTRACT. We investigate various properties of groups of periods associated to arbitrary maps defined on groups.

1. Introduction

Let G, G' be abelian groups and let $f : G \to G'$ be a homomorphism. In the usual additive notation for the group law, if t belongs to the kernel of f, then

$$f(x+t) = f(x),$$

for any $x \in G$. That is to say, the map f is periodic with period t. The group of periods of f coincides with $\ker f$. If we replace G' by an arbitrary non-empty set S and let f be any map from G to S, the notion of period still make sense, and one can again talk about the group of periods of f. Naturally, one has a richer structure to work with in the case when f is a homomorphism than in the case of a general map from G to an arbitrary set. Nevertheless, there are many important examples of periodic maps defined on groups which are not homomorphisms. For instance, let G be the additive group of real numbers. Trigonometric polynomials are maps of the form

$$f(x) = \sum_{n=-N}^{N} a_n e^{2\pi i n x},$$

where the coefficients a_n are complex numbers, and they play an important role in many problems in number theory (see [**9**], [**5**], [**7**]). If $a_k = 1$ for some k and $a_n = 0$ for $n \neq k$, in other words if $f(x) = e^{2\pi i k x}$, then f is a homomorphism to the multiplicative group of nonzero complex numbers, with kernel $\frac{1}{k}\mathbb{Z}$. A general trigonometric polynomial is not a homomorphism, and yet it has a nonzero group of periods.

Another important class of examples is provided by elliptic functions (see [**1**], [**15**]). Such a function f is meromorphic and doubly periodic. If we let the poles of f be sent to the point at infinity, then f will be defined everywhere on the complex plane $\mathbb{C}$, with values in $\mathbb{C} \cup \{\infty\}$, and will have as group of periods a lattice in $\mathbb{C}$.

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For another example, let K be a number field, which is an abelian extension of the field $\mathbb{Q}$ of rational numbers, and let $G = \mathrm{Gal}(K/\mathbb{Q})$. Any element $\alpha \in K$ gives rise to a natural map $f_\alpha : G \to K$, defined by

$$f_\alpha(\sigma) := \sigma(\alpha),$$

for any $\sigma \in G$. In general f_α is not a homomorphism, although the group of periods of f_α may be nontrivial. To be precise, the group of periods of f_α coincides with the Galois group $\mathrm{Gal}(K/\mathbb{Q}(\alpha))$.

In the present paper we take a general point of view. We consider a group G, which does not need to be abelian, a non-empty set S, a map $f : G \to S$, and investigate some properties of the corresponding groups of periods. Since G is no more assumed to be abelian, we first need to give a precise definition of what we mean by a group of periods in this more general context. There are several subgroups of G that one can consider in this case, namely the groups of left or right periods, as well as their normal and characteristic interior, which will be defined in the following section. An alternative point of view is to define these groups and investigate their properties by considering the partition induced by f on the underlying set of G, and the stabilizers of this partition with respect to the actions of left and right multiplication with elements in G. Groups acting on partitioned sets have been studied by a number of authors (see [**2**], [**3**], [**4**], [**10**], [**13**], [**14**] and [**16**]). Their properties have been extensively used in the computational study of finite permutation groups.

Subgroups appear in many cases in group theory as kernels, images or inverse images of group homomorphisms. Our first purpose is to show how the subgroups of an arbitrary group G may be regarded as groups of periods of arbitrary maps on G. The normal subgroups and the characteristic subgroups of G are then found to be precisely the normal interior and the characteristic interior of such groups of periods, respectively. This could be a source of new examples of subgroups, as well as a tool to study their properties. Another goal is to investigate the groups of periods in the case when G factorizes as a product of two subgroups with trivial intersection. Lastly, we consider modules and rings instead of groups and show how one can describe their submodules and ideals as appropriate kernels of arbitrary maps.

2. Notations and definitions

Let G be a group, $\mathcal{P}(G)$ the set of its non-empty subsets and α, β the actions of G on $\mathcal{P}(G)$ by left and right multiplication, respectively. Let now I be a set of indices and consider a partition $P = \{A_i\}_{i\in I}$ of G, that is

$$G = \bigcup_{i\in I} A_i,$$

where A_i are pairwise disjoint non-empty subsets of G. To any such partition of G we then associate the following four subgroups of G:

$$\begin{aligned}
LS(P) &= \bigcap_{i\in I} \operatorname{Stab}_{\alpha}(A_i),\\
RS(P) &= \bigcap_{i\in I} \operatorname{Stab}_{\beta}(A_i),\\
NS(P) &= \bigcap_{g\in G} g\cdot LS(P)\cdot g^{-1},\\
CS(P) &= \bigcap_{\varphi\in \operatorname{Aut}(G)} \varphi(LS(P)).
\end{aligned}$$

Definition 1. We call these subgroups the *left stabilizer* of P, the *right stabilizer* of P, the *normal stabilizer* of P and the *characteristic stabilizer* of P, respectively.

Definition 2. Let S be a non-empty set. An arbitrary map $f : G \to S$ defines in a natural way a partition of G if we consider $P = \{f^{-1}(s)\}_{s\in \operatorname{Im}(f)}$. In this case we denote the four subgroups associated to P by $LP(f)$, $RP(f)$, $\operatorname{Ker}(f)$ and $\operatorname{Char}(f)$, and call them the *group of left periods of* f, the *group of right periods of* f, the *kernel* of f and the *characteristic kernel* of f, respectively.

It is easy to see that these subgroups of G admit the following simple description:

$$\begin{aligned}
LP(f) &= \{h\in G : f(hg) = f(g), \forall g\in G\},\\
RP(f) &= \{h\in G : f(gh) = f(g), \forall g\in G\},\\
\operatorname{Ker}(f) &= \{h\in G : f(g_1hg_1^{-1}\cdot g_2) = f(g_2), \forall g_1, g_2\in G\},\\
\operatorname{Char}(f) &= \{h\in G : f(\varphi(h)\cdot g) = f(g), \forall g\in G, \forall\varphi\in \operatorname{Aut}(G)\}.
\end{aligned}$$

In this definition we may obviously assume that f is a surjective map, and the values taken by f are irrelevant as long as they preserve the same partition $\{f^{-1}(s)\}_{s\in \operatorname{Im}(f)}$ on G.

Remark. One can define the kernel and the characteristic kernel of f in the following equivalent way:

$$\begin{aligned}
\operatorname{Ker}(f) &= \{h\in G : f(g_1hg_2) = f(g_1g_2), \forall g_1, g_2\in G\}\\
&= \{h\in G : f(g_2\cdot g_1hg_1^{-1}) = f(g_2), \forall g_1, g_2\in G\},\\
\operatorname{Char}(f) &= \{h\in G : f(g\cdot\varphi(h)) = f(g), \forall g\in G, \forall\varphi\in \operatorname{Aut}(G)\}.
\end{aligned}$$

The following result shows that the definition of $NS(P)$ and $CS(P)$ does not depend on which action we consider, α or β, and the same obviously holds for the definition of $\operatorname{Ker}(f)$ and $\operatorname{Char}(f)$.

Proposition 1. *For every partition P of a group G we have:*

$$NS(P) = \bigcap_{g\in G} g\cdot RS(P)\cdot g^{-1} \quad \textit{and} \tag{1}$$

$$CS(P) = \bigcap_{\varphi\in \mathrm{Aut}\,(G)} \varphi(RS(P)). \tag{2}$$

Proof. Let the partition of G be $P = \{A_i\}_{i\in I}$, with I the set of indices. We associate to P the map $f : G \to I$ given by $f(g) = i$ for every $g \in A_i$, $i \in I$. Then we have $LS(P) = LP(f)$ and $RS(P) = RP(f)$. By double inclusion it follows easily that

$$\bigcap_{g\in G} g\cdot LP(f)\cdot g^{-1} = \{h\in G : f(g_1hg_1^{-1}\cdot g_2) = f(g_2), \forall g_1, g_2 \in G\},$$

$$\bigcap_{g\in G} g\cdot RP(f)\cdot g^{-1} = \{h\in G : f(g_2\cdot g_1hg_1^{-1}) = f(g_2), \forall g_1, g_2 \in G\},$$

and (1) follows by the previous remark. Similarly,

$$\bigcap_{\varphi\in \mathrm{Aut}\,(G)} \varphi(LP(f)) = \{h\in G : f(\varphi(h)\cdot g) = f(g), \forall g\in G, \forall\varphi\in \mathrm{Aut}\,(G)\},$$

$$\bigcap_{\varphi\in \mathrm{Aut}\,(G)} \varphi(RP(f)) = \{h\in G : f(g\cdot\varphi(h)) = f(g), \forall g\in G, \forall\varphi\in \mathrm{Aut}\,(G)\},$$

from which (2) follows using again the previous remark. □

We therefore see that $NS(P)$ is at the same time the core of $LS(P)$ in G and the core of $RS(P)$ in G. Similarly, $CS(P)$ is both the characteristic interior of $LS(P)$ in G and the characteristic interior of $RS(P)$ in G.

Remarks. 1. If S is a group and $f : G \to S$ is a group homomorphism, then $LP(f)$, $RP(f)$ and $\mathrm{Ker}\,(f)$ coincide with the usual kernel of f, and $\mathrm{Char}\,(f) = \bigcap_{\varphi\in\mathrm{Aut}\,(G)} \varphi(\mathrm{Ker}\,(f))$, the characteristic interior of $\mathrm{Ker}\,(f)$.

2. For an arbitrary map $f : G \to S$ we have the following inclusions:

$$\mathrm{Char}\,(f) \subseteq \mathrm{Ker}\,(f) \subseteq LP(f) \cap RP(f),$$

and for $h \in LP(f)$ or $h \in RP(f)$ we have $f(h) = f(1)$, so all these subgroups are contained in the set $f^{-1}(1)$.

3. In general $LP(f) \neq RP(f)$. To see this we consider the dihedral group $G = \{1, x, x^2, y, xy, x^2y\}$ with $x^3 = y^2 = 1$ and $yx = x^2y$, and a set S with 3 elements: $S = \{a, b, c\}$. For the map $f : G \to S$ given by

$$\begin{aligned} f(1) &= f(y) = a \\ f(x) &= f(x^2y) = b \\ f(x^2) &= f(xy) = c \end{aligned}$$

we have $LP(f) = \{1, y\}$ and $RP(f) = \{1\}$.

4. If f is an injective map we have $LP(f) = RP(f) = \mathrm{Ker}\,(f) = \mathrm{Char}\,(f) = 1$, and obviously $\mathrm{Char}\,(f) = G$ if and only if f is constant.

5. If G is an abelian group, then $LP(f) = RP(f) = \mathrm{Ker}\,(f)$, which is the group of periods of f, if we consider the additive notation for the group law.

6. If $G/\mathrm{Ker}\,(f)$ is abelian, then f is a central map and $LP(f) = RP(f) = \mathrm{Ker}\,(f)$.

7. For a group G and a partition $P = \{A_i\}_{i\in I}$ of G we may consider $N(P) = \cap_{i\in I} N_G(A_i)$ and call it the *normalizer* of the partition P. Here $N_G(A_i)$ stands for the normalizer of A_i in G. For a finite group G, a non-empty set S and an arbitrary map $f : G \to S$, the set

$$N(f) = \{h \in G : f(hg) = f(gh), \forall g \in G\}$$

is a subgroup of G. Obviously $N(f)$ is closed under multiplication, $1 \in N(f)$, and for $h \in N(f)$ we have

$$\begin{aligned} f(h^{-1}g) &= f(hh^{-2}g) = f(h^{-2}gh) = f(hh^{-3}gh) = f(h^{-3}gh^2) = \dots \\ &= f(h^{-o(h)}gh^{o(h)-1}) = f(gh^{-1}), \end{aligned}$$

where $o(h)$ is the order of h. This shows that $h^{-1} \in N(f)$. It is easy to see that $N(f)$ is actually the normalizer of the partition $P = \{f^{-1}(s)\}_{s\in \mathrm{Im}\,(f)}$. We obviously have the inclusions $LP(f) \cap RP(f) \subseteq N(f)$ and $Z(G) \subseteq N(f)$.

Examples. 1. For the power functions $f_n : G \to G$ given by $f_n(g) = g^n$, $n \in N$, we have:

$$\begin{aligned} \mathrm{Ker}\,(f_n) &= \{h \in G : (g_1hg_2)^n = (g_1g_2)^n, \forall g_1, g_2 \in G\} \\ &= \{h \in G : (hg_2g_1)^{n-1}hg_2 = (g_2g_1)^{n-1}g_2, \forall g_1, g_2 \in G\} \\ &= \{h \in G : (hg_2g_1)^{n-1}hg_2g_1 = (g_2g_1)^n, \forall g_1, g_2 \in G\} \\ &= \{h \in G : (hg_2g_1)^n = (g_2g_1)^n, \forall g_1, g_2 \in G\} = LP(f_n) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ker}\,(f_n) &= \{h \in G : (g_1hg_2)^n = (g_1g_2)^n, \forall g_1, g_2 \in G\} \\ &= \{h \in G : (hg_2g_1)^{n-1}h = (g_2g_1)^{n-1}, \forall g_1, g_2 \in G\} \\ &= \{h \in G : g_2g_1(hg_2g_1)^{n-1}h = (g_2g_1)^n, \forall g_1, g_2 \in G\} \\ &= \{h \in G : (g_2g_1h)^n = (g_2g_1)^n, \forall g_1, g_2 \in G\} = RP(f_n). \end{aligned}$$

Moreover, for $h \in \mathrm{Ker}\,(f_n)$ and $\varphi \in \mathrm{Aut}\,(G)$ we have $(\varphi(h)\varphi(g))^n = (\varphi(g))^n$, for all $g \in G$, and therefore $\varphi(h) \in \mathrm{Ker}\,(f_n)$. This shows that for every n, $\mathrm{Ker}\,(f_n)$ is a characteristic subgroup of G. We therefore have

$$\begin{aligned} \mathrm{Char}\,(f_n) &= \mathrm{Ker}\,(f_n) = RP(f_n) = LP(f_n) \\ &= \{h \in G : (hg)^n = (g)^n, \forall g \in G\}. \end{aligned}$$

Note that the order of any element belonging to $\mathrm{Ker}\,(f_n)$ must be a divisor of n. It is then easily seen that $\mathrm{Ker}\,(f_2)$ is the subgroup of involutions of $Z(G)$.

For two natural numbers m and n we have:

$$\begin{aligned}\mathrm{Ker}\,(f_m) \cap \mathrm{Ker}\,(f_n) &= \mathrm{Ker}\,(f_{\gcd(m,n)}),\\ \mathrm{Ker}\,(f_m) \cdot \mathrm{Ker}\,(f_n) &\subseteq \mathrm{Ker}\,(f_{lcm(m,n)}),\end{aligned}$$

Thus if m divides n we have $\mathrm{Ker}\,(f_m) \subseteq \mathrm{Ker}\,(f_n)$, and if G is a finite group of exponent e, we have $\mathrm{Ker}\,(f_n) = \mathrm{Ker}\,(f_{\gcd(n,e)})$.

2. Let x be a fixed element of a group G. For the commutator map given by $f_x(g) = gxg^{-1}x^{-1}$ we have:

$$LP(f_x) = \mathrm{Ker}\,(f_x) = C_G(C_x), \qquad RP(f_x) = C_G(x),$$

$$\mathrm{Char}\,(f_x) = \bigcap_{\varphi \in \mathrm{Aut}\,(G)} \varphi(C_G(C_x)),$$

where C_x is the conjugacy class of x.

3. Groups of periods

The methods to prove that a given subset of a group is a subgroup are omnipresent tools and can be found in all the classical texts of group theory. It is worth-mentioning a less known result due to G. Horrocks (see [**12**, p. 42]) stating that if a finite set $X = \{x_1, \ldots, x_n\}$ of a group G has the property that $x_i x_j \in X$ whenever $1 \le i \le j \le n$, then it is necessarily a subgroup of G.

In what follows we prove that the subgroups, the normal subgroups and the characteristic subgroups of an arbitrary group may be regarded as groups of periods, kernels and characteristic kernels of arbitrary maps, respectively.

Theorem 1. *A non-empty subset H of a group G is a subgroup (a normal subgroup, or a characteristic subgroup) of G if and only if there exist a set S and a map $f : G \to S$ such that $H = LP(f)$ ($H = \mathrm{Ker}\,(f)$, or $H = \mathrm{Char}\,(f)$, respectively). The same characterization for the subgroups of G holds if we replace $LP(f)$ by $RP(f)$.*

Proof. Let H be a subgroup of G and S a set with at least two elements, say a and b. If $H = G$, we take the constant map $f : G \to S$, $f(g) = a$ for all $g \in G$ and obviously $H = G = LP(f)$.

If $H \ne G$, we consider the indicator map of H given by

$$f(g) = \begin{cases} a & \text{if } g \in H \\ b & \text{if } g \notin H \end{cases}. \tag{3}$$

For $h \in LP(f)$ we have $f(hg) = f(g)$ for all $g \in G$ and in particular for $g \in H$ we have $f(hg) = a$, which according to the definition of f means that $hg \in H$, that is $h \in H$. Therefore we have $LP(f) \subseteq H$. Conversely, for $h \in H$ we have

$$\begin{aligned} f(hg) &= \begin{cases} a & \text{if } hg \in H \ (\Leftrightarrow g \in H) \\ b & \text{if } hg \notin H \ (\Leftrightarrow g \notin H) \end{cases} \\ &= \begin{cases} a & \text{if } g \in H \\ b & \text{if } g \notin H \end{cases} = f(g), \end{aligned}$$

for all $g \in G$, which shows that $h \in LP(f)$. Therefore we have $H = LP(f)$. The proof is similar if we consider $RP(f)$ instead of $LP(f)$.

If H is a proper normal subgroup of G we consider again the indicator map of H given by (3). For $h \in \mathrm{Ker}\,(f)$ we have $f(g_1hg_2) = f(g_1g_2)$ for all $g_1, g_2 \in G$. In particular, for $g_1, g_2 \in H$ we have $f(g_1hg_2) = a$, which shows according to the definition of f that $g_1hg_2 \in H$, that is $h \in H$. Therefore we have $\mathrm{Ker}\,(f) \subseteq H$. Conversely, for $h \in H$ we have

$$\begin{aligned} f(g_1hg_2) &= \begin{cases} a & \text{if } g_1hg_2 \in H \ (\Leftrightarrow g_1hg_1^{-1}g_1g_2 \in H) \\ b & \text{if } g_1hg_2 \notin H \ (\Leftrightarrow g_1hg_1^{-1}g_1g_2 \notin H) \end{cases} \\ &= \begin{cases} a & \text{if } g_1g_2 \in H \\ b & \text{if } g_1g_2 \notin H \end{cases} = f(g_1g_2), \end{aligned}$$

for all $g_1, g_2 \in G$, and therefore $h \in \mathrm{Ker}\,(f)$, that is $H = \mathrm{Ker}\,(f)$.

Finally, consider a proper characteristic subgroup H of G and f given by (3). For $h \in \mathrm{Char}\,(f)$ we have $f(\varphi(h)g) = f(g)$ for all $g \in G$ and all $\varphi \in \mathrm{Aut}\,(G)$. In particular, for $g \in H$ and $\varphi = 1_G$ we have $f(hg) = f(g) = a$, which by (3) shows that $hg \in H$, that is $h \in H$. Therefore we have $\mathrm{Char}\,(f) \subseteq H$. Conversely, for $h \in H$ we have

$$\begin{aligned} f(\varphi(h)g) &= \begin{cases} a & \text{if } \varphi(h)g \in H \ (\Leftrightarrow g \in H) \\ b & \text{if } \varphi(h)g \notin H \ (\Leftrightarrow g \notin H) \end{cases} \\ &= \begin{cases} a & \text{if } g \in H \\ b & \text{if } g \notin H \end{cases} = f(g), \end{aligned}$$

for all $g \in G$ and all $\varphi \in \mathrm{Aut}\,(G)$. Thus $H = \mathrm{Char}\,(f)$, which completes the proof. $\square$

This theorem (as well as its proof) may be alternatively rephrased in terms of partitions of G as follows:

Theorem 1′. *A non-empty subset H of a group G is a subgroup (a normal subgroup, or a characteristic subgroup) of G if and only if there exist a partition P of G such that $H = LS(P)$ ($H = NS(P)$, or $H = CS(P)$, respectively). The same characterization for the subgroups of G holds if we replace $LS(P)$ by $RS(P)$.*

We denote by $\{G/LP(f)\}_l$ and $\{G/RP(f)\}_l$ the sets of left cosets of $LP(f)$ and $RP(f)$ in G, respectively. The following result may be regarded as an analogue for arbitrary maps of the fundamental theorem on homomorphisms.

Proposition 2. Let G be a group, S a non-empty set and $f : G \to S$ an arbitrary map. Then $|G/\mathrm{Ker}\,(f)| \geq \mathrm{card}\,\{\mathrm{Im}\,(f)\}$, and moreover we have $\mathrm{card}\,\{G/LP(f)\}_l \geq \mathrm{card}\,\{\mathrm{Im}\,(f)\}$ and $\mathrm{card}\,\{G/RP(f)\}_l \geq \mathrm{card}\,\{\mathrm{Im}\,(f)\}$.

Proof. Consider $\phi : G/\mathrm{Ker}\,(f) \to \mathrm{Im}\,(f)$ given by $\phi(g\,\mathrm{Ker}\,(f)) = f(g)$. The map ϕ is well defined: indeed, if $g_1\,\mathrm{Ker}\,(f) = g_2\,\mathrm{Ker}\,(f)$ then $g_2^{-1}g_1 \in \mathrm{Ker}\,(f)$, which means that $f(x_1g_2^{-1}g_1x_1^{-1}x_2) = f(x_2)$ for all $x_1, x_2 \in G$.

In particular, for $x_1 = x_2 = g_2$ we find $f(g_1) = f(g_2)$. Since obviously ϕ is a surjective map, we have $|G/\operatorname{Ker}(f)| \geq \operatorname{card}\{\operatorname{Im}(f)\}$. For the remaining two inequalities we consider the maps $\phi_1 : \{G/LP(f)\}_l \to \operatorname{Im}(f)$ and $\phi_2 : \{G/RP(f)\}_l \to \operatorname{Im}(f)$ given by $\phi_1(gLP(f)) = f(g^{-1})$ and $\phi_2(gRP(f)) = f(g)$, which are also well defined and surjective. Hence, if G is a finite group, we have

$$
\begin{aligned}
|\operatorname{Ker}(f)| \cdot \operatorname{card}\{\operatorname{Im}(f)\} &\leq |G|, \\
|LP(f)| \cdot \operatorname{card}\{\operatorname{Im}(f)\} &\leq |G| \qquad \text{and} \\
|RP(f)| \cdot \operatorname{card}\{\operatorname{Im}(f)\} &\leq |G|,
\end{aligned}
\tag{4}
$$

or, equivalently:

$$
\begin{aligned}
|NS(P)| \cdot \operatorname{card}\{I\} &\leq |G|, \\
|LS(P)| \cdot \operatorname{card}\{I\} &\leq |G| \qquad \text{and} \\
|RS(P)| \cdot \operatorname{card}\{I\} &\leq |G|,
\end{aligned}
$$

if we consider the same problem in terms of partitions of G. □

Inequalities (4) show that if we try to find maps f having nontrivial kernels or groups of periods, then we have to ask for $\operatorname{card}\{\operatorname{Im}(f)\}$ to be "small". For instance, if $|G| = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$ with $p_1 < p_2 < \ldots < p_k$ prime numbers, $n_1 \geq 1, \ldots, n_k \geq 1$ and $\operatorname{card}\{\operatorname{Im}(f)\} > |G|/p_1$, then $LP(f) = RP(f) = \operatorname{Ker}(f) = 1$. In particular, if we choose f such that $\operatorname{card}\{\operatorname{Im}(f)\} > |G|/2$, then necessarily $LP(f) = RP(f) = \operatorname{Ker}(f) = 1$.

For finite groups we can also establish the following connection between $|LS(P)|$, $|RS(P)|$, $|NS(P)|$, $|CS(P)|$ and $\{\operatorname{card}\{A_i\}\}_{i\in I}$.

Proposition 3. *Let G be a finite group and $P = \{A_i\}_{i=1}^n$ a partition of G. Then $|LS(P)|$, $|RS(P)|$, $|NS(P)|$ and $|CS(P)|$ are divisors of* $\gcd(\operatorname{card}\{A_1\}, \ldots, \operatorname{card}\{A_n\})$.

Proof. It will be sufficient to prove this assertion for $|LS(P)|$. Denote by γ the action of $LS(P)$ on G by left multiplication. The length of the orbit of each element with respect to γ equals $|LS(P)|$. Since $LS(P)$ acts on G and stabilizes each one of the A_i's, it turns out that each A_i is a union of distinct orbits with respect to γ. Hence $|LS(P)|$ divides $\operatorname{card}\{A_i\}$ for every i, which completes the proof. □

This proposition shows that a nontrivial subgroup H of a finite group G can be a left or a right stabilizer only for maps partitioning G into parts each of whose length is divisible by $|H|$.

Some properties of $LP(f)$, $RP(f)$, $\operatorname{Ker}(f)$ and $\operatorname{Char}(f)$ which are immediate from the definition are given by the following:

Proposition 4. (i) *Let $f_i : G \to S_i$, $i = 1, \dots, n$ be arbitrary maps. For the map $f : G \to S_1 \times \cdots \times S_n$ given by $f(g) = (f_1(g), \dots, f_n(g))$ we have:*

$$\begin{aligned} LP(f) &= \bigcap_{i=1}^{n} LP(f_i), & RP(f) &= \bigcap_{i=1}^{n} RP(f_i), \\ \mathrm{Ker}\,(f) &= \bigcap_{i=1}^{n} \mathrm{Ker}\,(f_i), & \mathrm{Char}\,(f) &= \bigcap_{i=1}^{n} \mathrm{Char}\,(f_i). \end{aligned}$$

(ii) *Let $f_i : G_i \to S_i$, $i = 1, \dots, n$ be arbitrary maps. For the map $f : G_1 \times \cdots \times G_n \to S_1 \times \cdots \times S_n$ given by $f(g_1, \dots, g_n) = (f_1(g_1), \dots, f_n(g_n))$ we have:*

$$LP(f) = \prod_{i=1}^{n} LP(f_i), \qquad RP(f) = \prod_{i=1}^{n} RP(f_i)$$

$$\mathrm{Ker}\,(f) = \prod_{i=1}^{n} \mathrm{Ker}\,(f_i).$$

Let us consider now the situation when G has subgroups H and K such that $G = K \cdot H$ and $H \cap K = 1$. Since H and K are not assumed to be normal subgroups of G, one might not expect to obtain an immediate correspondent of Proposition 4, ii). Nevertheless, since every element $g \in G$ may be expressed in a unique way as a product of an element $k \in K$ and an element $h \in H$, we may consider the two projections $\pi : G \to K$ and $\rho : G \to H$ given by $\pi(g) = k$ and $\rho(g) = h$, which are not necessarily group homomorphisms, but still play an important role when we study the subgroups of G. We proceed now to describe the groups of periods and the kernels of these projections. For this we first recall a construction introduced by M. Takeuchi in [**17**], which characterizes in terms of group actions the groups which can be expressed as internal product of two subgroups with trivial intersection. His construction has also nice applications in the study of Hopf algebras structure, developed in [**8**].

The fact that for every element $g \in G$ there exists a unique pair $(k, h) \in K \times H$ such that $g = k \cdot h$ allows one to define the maps $\alpha : H \times K \to K$ and $\beta : K \times H \to H$ by

$$\alpha(h, k) = z \text{ and } \beta(k, h) = y, \tag{5}$$

where $(z, y) \in K \times H$ is the unique pair such that $h \cdot k = z \cdot y$. Then, the associativity relations

$$(h \cdot h') \cdot k = h \cdot (h' \cdot k) h \cdot (k \cdot k') = (h \cdot k) \cdot k'$$

and the unit properties $h \cdot 1 = 1 \cdot h$ and $1 \cdot k = k \cdot 1$ show that α is a left action of H on the set K and β is a right action of K on the set H, satisfying the following conditions:

$$\alpha(h, k \cdot k') = \alpha(h, k) \cdot \alpha(\beta(k, h), k') \tag{6}$$

$$\beta(k, h \cdot h') = \beta(\alpha(h', k), h) \cdot \beta(k, h') \tag{7}$$

and

$$\alpha(h,1) = 1\,, \tag{8}$$
$$\beta(k,1) = 1\,. \tag{9}$$

The group law in G may be then regarded as

$$(k_1h_1)\cdot(k_2h_2) = k_1\alpha(h_1,k_2)\cdot\beta(k_2,h_1)h_2, \tag{10}$$

and the inverse of an element kh is easily seen to be $\alpha(h^{-1},k^{-1})\cdot\beta(k^{-1},h^{-1})$.

Conversely, if α is a left action and β a right action satisfying (6) – (9), then the direct product set $K\times H$ acquires the structure of a group denoted $K_\beta \bowtie_\alpha H$, when we define the multiplication law by:

$$(k_1,h_1)\cdot(k_2,h_2) = (k_1\cdot\alpha(h_1,k_2), \beta(k_2,h_1)\cdot h_2)\,.$$

The unit element is $(1,1)$ and the inverse of the element (k,h) is $(\alpha(h^{-1},k^{-1}),\beta(k^{-1},h^{-1}))$. Using the injective homomorphisms $i_1 : K\to K_\beta\bowtie_\alpha H$ and $i_2 : H \to K_\beta \bowtie_\alpha H$ sending k to $(k,1)$ and h to $(1,h)$, we can identify the groups K and H with $K_1 = i_1(K)$ and $H_1 = i_2(H)$ respectively, and thus we have $K_\beta\bowtie_\alpha H = K_1\cdot H_1$ and $K_1\cap H_1 = (1,1)$. Moreover, one can prove that if $G = K\cdot H$ with $K\cap H = 1$, then G is isomorphic to $K_\beta\bowtie_\alpha H$, with α and β given by (5) (the map $\theta : K_\beta\bowtie_\alpha H\to G$ given by $\theta(k,h) = kh$ is an isomorphism).

We have the following description for the groups of periods and the kernels of π and ρ:

Lemma 1. *Let H, K be subgroups of G such that $G = K\cdot H$, $K\cap H = 1$, and let π and ρ be the projections of G onto K and H respectively. Then $RP(\pi) = H$, $LP(\pi) = \mathrm{Ker}\,(\pi) = \mathrm{Ker}\,(\alpha) = H_G$ and $LP(\rho) = K$, $RP(\rho) = \mathrm{Ker}\,(\rho) = \mathrm{Ker}\,(\beta) = K_G$, with α, β given by* (5)*.*

Proof. According to the definition, $RP(\pi)$ consists of those elements $k_2\cdot h_2\in G$ for which $\pi(k_1h_1\cdot k_2h_2) = \pi(k_1h_1)$ for all the elements $k_1\cdot h_1\in G$. Thus, by (10) we search for the elements $k_2\cdot h_2$ such that $k_1\alpha(h_1,k_2) = k_1$ for all $k_1\cdot h_1\in G$. In particular, for $h_1 = 1$ we find $k_2 = 1$, which shows that $RP(\pi) = H$. Then we obviously have

$$\mathrm{Ker}\,(\pi) = \bigcap_{g\in G} g\cdot RP(\pi)\cdot g^{-1} = H_G.$$

Similarly, $LP(\pi)$ consists of those elements $k_1\cdot h_1\in G$ for which $\pi(k_1h_1\cdot k_2h_2) = \pi(k_2h_2)$ for all the elements $k_2\cdot h_2\in G$. Thus, by (10) we search for the elements $k_1\cdot h_1$ such that $k_1\alpha(h_1,k_2) = k_2$ for all $k_2\in K$. In particular, if we put $k_2 = k_1$, we must have $\alpha(h_1,k_1) = 1$. Applying now $\alpha(h_1,\cdot)$, we find $k_1 = \alpha(h_1^{-1},1)$, which by (8) is equal to 1. We therefore see that $LP(\pi)$ consists of those h_1 for which $\alpha(h_1,k_2) = k_2$ for all $k_2\in K$, that is $LP(\pi) = \mathrm{Ker}\,(\alpha)\trianglelefteq H$. By taking the normal interior in both sides, we see that $\mathrm{Ker}\,(\pi) = \mathrm{Ker}\,(\alpha)_G$. So in order to prove that $\mathrm{Ker}\,(\alpha) = H_G$ we have to check that $\mathrm{Ker}\,(\alpha)$ is actually a normal subgroup of G.

Let $h_2 \in \mathrm{Ker}\,(\alpha)$ and let $k_1 \cdot h_1$ be an arbitrary element of G. Then we have

$$\begin{aligned}(k_1h_1) \cdot h_2 \cdot (k_1h_1)^{-1} &= (k_1h_1h_2) \cdot (\alpha(h_1^{-1}, k_1^{-1})\beta(k_1^{-1}, h_1^{-1})) \\ &= k_1\alpha(h_1h_2, \alpha(h_1^{-1}, k_1^{-1})) \\ &\quad \cdot \beta(\alpha(h_1^{-1}, k_1^{-1}), h_1h_2)\beta(k_1^{-1}, h_1^{-1}) \quad \text{(by (10))} \\ &= \beta(\alpha(h_1^{-1}, k_1^{-1}), h_1h_2)\beta(k_1^{-1}, h_1^{-1}) \quad (h_2 \in \mathrm{Ker}\,(\alpha)) \\ &= \beta(k_1^{-1}, h_1h_2h_1^{-1}), \quad \text{(by (7))}\end{aligned}$$

and for an arbitrary $k \in K$ we find

$$\begin{aligned}\alpha(\beta(k_1^{-1}, h_1h_2h_1^{-1}), k) &= \alpha(h_1h_2h_1^{-1}, k_1^{-1})^{-1} \cdot \alpha(h_1h_2h_1^{-1}, k_1^{-1}k) \quad \text{(by (7))} \\ &= k,\end{aligned}$$

since $h_2 \in \mathrm{Ker}\,(\alpha)$ and $\mathrm{Ker}\,(\alpha) \trianglelefteq H$. Therefore $\mathrm{Ker}\,(\alpha) \trianglelefteq G$ and $\mathrm{Ker}\,(\pi) = LP(\pi) = \mathrm{Ker}\,(\alpha) = H_G$.

In a similar way one can prove that $LP(\rho) = K$ and $\mathrm{Ker}\,(\rho) = RP(\rho) = \mathrm{Ker}\,(\beta) = K_G$. □

Proposition 5. *Let H, K be subgroups of G such that $G = K \cdot H$, $K \cap H = 1$, and let π and ρ be the projections of G onto K and H respectively. Let S_1, S_2 be non-empty sets, $f_1 : K \to S_1$, $f_2 : H \to S_2$ arbitrary maps and $f : G \to S_1 \times S_2$ given by $f(g) = (f_1(k), f_2(h))$, with $k \in K$, $h \in H$ uniquely determined by $g = k\cdot h$. Then*

(i) *If $\rho(LP(f)) \subseteq H_G$, then $LP(f) \subseteq LP(f_1) \cdot LP(f_2)$ (in particular this holds if $H \trianglelefteq G$); Conversely, if $LP(f_2) = H_G$, then $LP(f_1) \cdot LP(f_2) \subseteq LP(f)$;*

(ii) *If $\pi(RP(f)) \subseteq K_G$, then $RP(f) \subseteq RP(f_1) \cdot RP(f_2)$ (in particular this holds if $K \trianglelefteq G$); Conversely, if $H \trianglelefteq G$ and $RP(f_1) \subseteq K_G$, then $RP(f_1) \cdot RP(f_2) \subseteq RP(f)$.*

Proof. (i) Let $x = k_1 \cdot h_1 \in LP(f)$. Then for every $k_2 \cdot h_2 \in G$ we have $f(k_1h_1 \cdot k_2h_2) = f(k_2h_2)$, which in view of (10) gives

$$\begin{aligned}f_1(k_1\alpha(h_1, k_2)) &= f_1(k_2) \text{ and} \\ f_2(\beta(k_2, h_1)h_2) &= f_2(h_2).\end{aligned}$$

Our assumption that $\rho(LP(f)) \subseteq H_G$ shows that $h_1 \in H_G$, which according to Lemma 1 equals $\mathrm{Ker}\,(\alpha)$. Therefore the first equation becomes $f_1(k_1k_2) = f_1(k_2)$ for all $k_2 \in K$, which shows that $k_1 \in LP(f_1)$. Choosing $k_2 = 1$, the second equation above shows that $h_1 \in LP(f_2)$. Assume now $LP(f_2) = H_G = \mathrm{Ker}\,(\alpha)$ and let $k_1 \in LP(f_1)$ and $h_1 \in LP(f_2)$. Then for arbitrary $k_2 \cdot h_2 \in G$ one finds

$$\begin{aligned}f(k_1h_1 \cdot k_2h_2) &= (f_1(k_1\alpha(h_1, k_2)), f_2(\beta(k_2, h_1)h_2)) \\ &= (f_1(\alpha(h_1, k_2)), f_2(\beta(k_2, h_1)h_2)) \qquad (\text{since } k_1 \in LP(f_1)) \\ &= (f_1(k_2), f_2(\beta(k_2, h_1)h_2)) \qquad (\text{since } h_1 \in \mathrm{Ker}\,(\alpha)) \\ &= (f_1(k_2), f_2(h_2)),\end{aligned}$$

since by the definition of α and β one has $h_1 \cdot k_2 = \alpha(h_1, k_2) \cdot \beta(k_2, h_1)$, which for $h_1 \in \mathrm{Ker}\,(\alpha)$ becomes $\beta(k_2, h_1) = k_2^{-1}h_1k_2 \in \mathrm{Ker}\,(\alpha) = LP(f_2)$.

(ii) The first assertion follows in a similar way. For the second one we use the fact that $H \trianglelefteq G$ forces α to be trivial. $\square$

In the finite case, an additional result relating the groups of periods of f_1, f_2 and f will be derived in Corollary 1, by using again the projections π and ρ. In the case when G is a direct product, these projections play an important role in the study of the structure of its subgroups, as shown by the well-known:

Theorem (Remak [**11**], Klein, Fricke [**6**]). *Let K and H be normal subgroups of G such that $G = K \times H$, and let π and ρ be the corresponding projections of G onto K and H, respectively. Let L be a subgroup of G. Then*

(i) $(L\cap K) \trianglelefteq \pi(L) \leq K$, $(L\cap H) \trianglelefteq \rho(L) \leq H$, *and* $\pi(L)/(L\cap K) \simeq \rho(L)/(L\cap H)$*;*

(ii) $L = (L \cap K) \times (L \cap H)$ *if and only if* $\pi(L) = L \cap K$ *(or if and only if* $\rho(L) = L \cap H$*).*

For finite groups this result can be extended in the following way:

Theorem 2. *Let H, K be subgroups of a finite group G such that $G = K \cdot H$, $K \cap H = 1$, and let π and ρ be the projections of G onto K and H respectively. Let L be a subgroup of G. Then $L \cap K \subseteq \pi(L)$, $L \cap H \subseteq \rho(L)$ and*

(i) $\mathrm{card}\,(\pi(L))/|L \cap K| = \mathrm{card}\,(\rho(L))/|L \cap H| = |L|/(|L \cap K| \cdot |L \cap H|)$*;*

(ii) $L = (L \cap K) \cdot (L \cap H)$ *if and only if* $\pi(L) = L \cap K$ *(or if and only if* $\rho(L) = L \cap H$*).*

Proof. (i) By (10) we see that π and ρ satisfy the relations

$$\pi(g_1 \cdot g_2) = \pi(g_1) \cdot \alpha(\rho(g_1), \pi(g_2)) \tag{11}$$

$$\rho(g_1 \cdot g_2) = \beta(\pi(g_2), \rho(g_1)) \cdot \rho(g_2) \tag{12}$$

with α and β given by (5). We obviously have

$$(\pi|_L)^{-1}(1) = \{l \in L : \pi(l) = 1\} = L \cap H \text{ and} \tag{13}$$

$$(\rho|_L)^{-1}(1) = \{l \in L : \rho(l) = 1\} = L \cap K. \tag{14}$$

The set $\pi(L)$ is not necessarily a group, but we can prove that $[L : L \cap H] = \mathrm{card}\,(\pi(L))$. Let $\{L/L \cap H\}_l$ be the set of left cosets of $L \cap H$ in H and $\varphi : \{L/L\cap H\}_l \to \pi(L)$ given by $\varphi(g \cdot L\cap H) = \pi(g)$. To check that φ is a well defined map, assume that $g_1 \cdot L\cap H = g_2 \cdot L\cap H$, with $g_1, g_2 \in L$. Then $g_1^{-1} g_2 \in L\cap H$, so by (13) we have $\pi(g_1^{-1} g_2) = 1$, which by (11) gives $1 = \pi(g_1^{-1}) \cdot \alpha(\rho(g_1^{-1}), \pi(g_2))$. This shows that $\pi(g_2) = \alpha(\rho(g_1^{-1})^{-1}, \pi(g_1^{-1})^{-1})$. On the other hand, we have $\pi(1) = 1$, which by (11) gives $1 = \pi(g_1^{-1} g_1) = \pi(g_1^{-1}) \cdot \alpha(\rho(g_1^{-1}), \pi(g_1))$, or furthermore $\pi(g_1) = \alpha(\rho(g_1^{-1})^{-1}, \pi(g_1^{-1})^{-1})$. We therefore have $\pi(g_1) = \pi(g_2)$, so φ is a well defined map.

The fact that φ is an injective map follows exactly in the reverse order, since if we assume $\pi(g_1) = \pi(g_2)$, with $g_1, g_2 \in L$, then by (11) we must have $1 = \pi(g_1^{-1}) \cdot \alpha(\rho(g_1^{-1}), \pi(g_2))$, that is $\pi(g_1^{-1} g_2) = 1$, again by (11). Since φ is obviously a surjective map, we must have $[L : L \cap H] = \mathrm{card}\,(\pi(L))$. Similarly, using (12) and (14) we find $[L : L \cap K] = \mathrm{card}\,(\rho(L))$. Then

$$\frac{\mathrm{card}\,(\pi(L))}{|L \cap K|} = \frac{\mathrm{card}\,(\rho(L))}{|L \cap H|} = \frac{|L|}{|L \cap K| \cdot |L \cap H|},$$

which also gives the proof of (ii), since $(L \cap K) \cdot (L \cap H) \subseteq L \subseteq \pi(L) \cdot \rho(L)$. □

Corollary 1. *Let H, K be subgroups of a finite group G such that $G = K \cdot H$, $K \cap H = 1$, and let π and ρ be the projections of G onto K and H respectively. Let S_1, S_2 be non-empty sets, $f_1 : K \to S_1$, $f_2 : H \to S_2$ arbitrary maps and $f : G \to S_1 \times S_2$ given by $f(g) = (f_1(k), f_2(h))$, with $k \in K$, $h \in H$ uniquely determined by $g = k \cdot h$. Then*
(i) *$LP(f) = (LP(f) \cap K) \cdot (LP(f) \cap H)$ if and only if $\pi(LP(f)) = LP(f_1)$;*
(ii) *$RP(f) = (RP(f) \cap K) \cdot (RP(f) \cap H)$ if and only if $\rho(RP(f)) = RP(f_2)$.*

Proof. We use the fact that $(LP(f) \cap K) = LP(f_1)$ and $(RP(f) \cap H) = RP(f_2)$. □

We end by mentioning some similar results which allow one to describe submodules and ideals as apropriate "kernels" of arbitrary maps. Thus, if R is a ring with unit, ${}_RM$ a left R-module, S a non-empty set and $f : M \to S$ an arbitrary map, we define

$$\mathrm{Ker}\,(f) = \{x \in M : f(\alpha x + y) = f(y),\ \forall y \in M,\ \forall \alpha \in R\}.$$

Similarly, if we replace ${}_RM$ by a right R-module M_R we define

$$\mathrm{Ker}\,(f) = \{x \in M : f(x\alpha + y) = f(y),\ \forall y \in M,\ \forall \alpha \in R\}$$

and have the following:

Proposition 6. *A non-empty subset N of a module M is a submodule of M if and only if there exists a non-empty set S and a map $f : M \to S$ such that $N = \mathrm{Ker}\,(f)$.*

In particular, if we replace R by a commutative field and M by a vector space V we obtain a similar description for the subspaces of V. We note that if S is a topological space and $f : V \to S$ is a continuous map, then $\mathrm{Ker}\,(f)$ is a closed subspace of V.

Finally, if R is a ring with unit, S a non-empty set and $f : R \to S$ an arbitrary map, we define:

$$\begin{aligned}
\mathrm{Ker}_l(f) &= \{x \in R : f(ax + b) = f(b),\ \forall\ a, b \in R\},\\
\mathrm{Ker}_r(f) &= \{x \in R : f(xa + b) = f(b),\ \forall\ a, b \in R\},\\
\mathrm{Ker}\,(f) &= \{x \in R : f(a_1 x a_2 + b) = f(b),\ \forall\ a_1, a_2, b \in R\},
\end{aligned}$$

the left kernel, the right kernel and the kernel of f, respectively. These ideals obviously coincide if R is a commutative ring. We then have:

Proposition 7. *Let R be a ring with unit and I a proper non-empty subset of R. Then I is a left (right, two-sided) ideal of R if and only if there exists a set S with at least two elements and a map $f : R \to S$ such that $I = \mathrm{Ker}_l(f)$ ($I = \mathrm{Ker}_r(f)$, $I = \mathrm{Ker}\,(f)$, respectively).*

The proof of these results is similar to the one of Theorem 1 and uses again the indicator map of the corresponding subset.

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