Acta Math. Univ. Comenianae
Vol. LXXV, 1(2006), pp. 107–117

SOME GENERALIZATIONS OF MINIMAL FUZZY OPEN SETS

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Abstract. Two new minimality concepts are defined and investigated in fuzzy spaces. Relationships between them and the known one are obtained. Several results concerning them are obtained. The results especially deal with products, separation axioms, maps, and fuzzy topologically generated topologies.

1. Introduction

In this paper, the symbol I will denote the unit interval [0,1]. Let X be a non-empty set. A fuzzy set in X [**11**] is a function with domain X and values in I; that is, an element of I^X.

Throughout this paper, for $\lambda, \mu \in I^X$ we write $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$ for all $x \in X$. By $\lambda = \mu$ we mean that $\lambda \leq \mu$ and $\mu \leq \lambda$, i.e., $\lambda(x) = \mu(x)$ for all $x \in X$. If $\{\lambda_j : j \in J\}$ is a collection of fuzzy sets in X, then $(\bigvee \lambda_j)(x) = \sup\{\lambda_j(x) : j \in J\}$, $x \in X$; and $(\bigwedge \lambda_j)(x) = \inf\{\lambda_j(x) : j \in J\}$, $x \in X$. If $r \in [0,1]$ then r_X denotes the fuzzy set given by $r_X(x) = r$ for all $x \in X$; that is, r_X denotes the "constant" fuzzy set of level r.

In this paper we shall follow [**10**] for the definitions of fuzzy topology, first countable fuzzy spaces, the product fuzzy topology, the direct and the inverse images of a fuzzy set under maps, fuzzy continuity and fuzzy openness. A fuzzy set p defined by

$$p(x) = \begin{cases} t & \text{if } x = x_p \\ 0 & \text{if } x \neq x_p \end{cases}$$

where $0 < t < 1$ is called a fuzzy point in X, $x_p \in X$ is called the support of p and $p(x_p) = t$ the value (level) of p [**10**]. Two fuzzy points p and q in X are said to be distinct iff their supports are distinct, i.e., $x_p \neq x_q$. A fuzzy set p defined by

$$p(x) = \begin{cases} 1 & \text{if } x = x_p \\ 0 & \text{if } x \neq x_p \end{cases}$$

is called a fuzzy crisp point in X [**10**]. The set of all fuzzy points in a set X will be denoted by $FP(X)$. A fuzzy set that is either a fuzzy point or a fuzzy crisp point is called fuzzy singleton [**6**].

Received January 8, 2005.
2000 *Mathematics Subject Classification.* Primary 54A40.
Key words and phrases. Fuzzy topological spaces, minimality, product fuzzy topology, fuzzy separation axioms, and fuzzy maps.

In this paper, we follow [**9**] for the definition of 'belonging to'. Namely: A fuzzy point p in X is said to belong to a fuzzy set λ in X (notation: $p \in \lambda$) iff $p(x_p) < \lambda(x_p)$.

If $(X, \Im)$ is a fuzzy space and $a \in [0,1)$, then the set $\{\lambda^{-1}(a,1] : \lambda \in \Im\}$ is a topology on X (see [**8**]). This topology is denoted by $\Im_a$, and is said fuzzy topologically generated.

A bijective function $h : (X, \Im_1) \to (Y, \Im_2)$; where $(X, \Im_1)$ and $(Y, \Im_2)$ are both fuzzy spaces; will be called fuzzy homeomorphism [**9**] iff h and h^{-1} are both fuzzy continuous.

A non-empty open set A of a space (X, τ) is called a minimal open set in X [**7**] if any open set in X which is contained in A is $\emptyset$ or A. The authors in [**7**] gave various results concerning minimal open sets. In [**2**], the author studied minimal preopen sets and characterized them as singleton preopen sets. In [**3**] the authors extended the concept of minimal open sets to include fuzzy spaces. A fuzzy open set λ of a fuzzy space $(X, \Im)$ is called minimal fuzzy open set [**3**] if λ is non-zero and there is no non-zero proper fuzzy open subset of λ, many interesting results concerning this concept are obtained. In this paper, as a generalization of minimal fuzzy open sets we fuzzify minimality at some point by two methods, local minimality at a point in X and local minimality at a fuzzy point in X.

Throughout this paper, for any set X, $|X|$ will denote the cardinality of X. If λ is a fuzzy set in X, then the support of λ is denoted by $S(\lambda)$ and defined by $S(\lambda) = \lambda^{-1}(0,1]$. The set of all minimal fuzzy open subsets of a fuzzy space $(X, \Im)$ will be denoted by $\min(X, \Im)$.

2. LOCAL MINIMAL FUZZY OPEN SETS

We start by the following two main definitions.

Definition 2.1. Let $(X, \Im)$ be a fuzzy space, $x \in X$, and $\lambda \in \Im$ such that $x \in S(\lambda)$. Then λ is called a local minimal fuzzy open set at x if there is no proper fuzzy open subset of λ containing x in its support. The set of all local minimal fuzzy open sets at a point $x \in X$ will be denoted by $\min(X, \Im, x)$.

Definition 2.2. Let $(X, \Im)$ be a fuzzy space, $p \in FP(X)$, and $\lambda \in \Im$ such that $p \in \lambda$. Then λ is called a local minimal fuzzy open set at p if there is no proper fuzzy open subset β of λ with $p \in \beta$. The set of all local minimal fuzzy open sets at a fuzzy point $p \in FP(X)$ will be denoted by $\min(X, \Im, p)$.

The following two results characterize the concepts in the above two definitions respectively.

Proposition 2.1. *Let $(X, \Im)$ be a fuzzy space, $\lambda \in \Im$ and $x \in S(\lambda)$. Then $\lambda \in \min(X, \Im, x)$ iff for each $\beta \in \Im$ such that $x \in S(\beta)$ we have $\lambda \leq \beta$.*

Proof. $\Longrightarrow$) Suppose that $\lambda \in \min(X, \Im, x)$ and let $\beta \in \Im$ with $x \in S(\beta)$. Then $x \in S(\lambda) \cap S(\beta) = S(\lambda \wedge \beta)$. Since $\lambda \wedge \beta \leq \lambda$, it follows that $\lambda \wedge \beta = \lambda$. Hence $\lambda \leq \beta$.

$\Longleftarrow$) Obvious. □

Proposition 2.2. *Let* $(X, \Im)$ *be a fuzzy space,* $\lambda \in \Im$ *with* $p \in \lambda$ *and* $p \in FP(X)$. *Then* $\lambda \in \min(X, \Im, p)$ *iff for each* $\beta \in \Im$ *such that* $p \in \beta$ *we have* $\lambda \leq \beta$.

Proof. The proof is similar to that used in Proposition 2.1. □

Proposition 2.3. *Let* $(X, \Im)$ *be a fuzzy space,* $x \in X$ *and* $p \in FP(X)$. *Then* $|\min(X, \Im, x)| \leq 1$ *and* $|\min(X, \Im, p)| \leq 1$.

Proof. Suppose that $\lambda, \beta \in \min(X, \Im, x)$. Then by Proposition 2.1, it follows that $\lambda \leq \beta$ and $\beta \leq \lambda$. Thus $\lambda = \beta$. Hence, $|\min(X, \Im, x)| \leq 1$. Similarly we can show $|\min(X, \Im, p)| \leq 1$. □

Theorem 2.1. *Let* $(X, \Im)$ *be a fuzzy space,* $\lambda \in \Im$ *and* $x \in X$. *Then the following are equivalent.*

(i) $\min(X, \Im, x) = \{\lambda\}$,
(ii) *For each* $p \in \lambda$ *with* $x_p = x$, $\min(X, \Im, p) = \{\lambda\}$.

Proof. (i) $\Longrightarrow$ (ii). Let $p \in \lambda$ with $x_p = x$ and let $\beta \in \Im$ such that $p \in \beta$. Then $x \in S(\beta)$. Since $\min(X, \Im, x) = \{\lambda\}$, then $\lambda \leq \beta$. Therefore, $\min(X, \Im, p) = \{\lambda\}$.

(ii) $\Longrightarrow$ (i) Let $\beta \in \Im$ such that $x \in S(\beta)$. Consider $p \in FP(X)$ with $x_p = x$ and $p(x_p) = \min\left\{\frac{\lambda(x)}{2}, \frac{\beta(x)}{2}\right\}$. Then $p \in \lambda \wedge \beta$ and hence by (ii), it follows that $\min(X, \Im, p) = \{\lambda\}$. Therefore, $\lambda \leq \beta$ and hence $\min(X, \Im, x) = \{\lambda\}$. □

Example 2.1. Let $X = \{a, b\}$ with the fuzzy topology $\Im = \{0_X, 1_X, \lambda, \beta, (0.3)_X\}$ where $\lambda = \{(a, 0.3), (b, 1)\}$ and $\beta = \{(a, 1), (b, 0.3)\}$. Consider $p, q \in FP(X)$ with $x_p = x_q = b$, $p(x_p) = 0.5$ and $q(x_q) = 0.2$. Then $\min(X, \Im, p) = \{\lambda\}$, while $\min(X, \Im, q) \neq \{\lambda\}$.

In Theorem 2.1 Example 2.1 shows that the statement "For each $p \in \lambda$ with $x_p = x$, $\min(X, \Im, p) = \{\lambda\}$" cannot be replaced by the statement "for some $p \in \lambda$ with $x_p = x$, $\min(X, \Im, p) = \{\lambda\}$".

Theorem 2.2. *Let* $(X, \Im)$ *be a fuzzy space,* $\lambda \in \Im$ *and* $p \in FP(X)$. *Then the following are equivalent.*

(i) $\min(X, \Im, p) = \{\lambda\}$.
(ii) $\min(X, \Im, q) = \{\lambda\}$ *for every* $q \in FP(X)$ *with* $q \in \lambda$, $x_q = x_p$ *and* $p(x_p) \leq q(x_q)$.

Proof. (i) $\Longrightarrow$ (ii) Let $q \in FP(X)$ with $q \in \lambda$, $x_q = x_p$ and $p(x_p) \leq q(x_q)$. Let $\beta \in \Im$ with $q \in \beta$. Then $p(x_p) \leq q(x_q) < \beta(x_q) = \beta(x_p)$ and so $p \in \beta$. Therefore, by (i) it follows that $\lambda \leq \beta$ and hence $\min(X, \Im, q) = \{\lambda\}$.

(i) $\Longrightarrow$ (ii) Clear. □

Definition 2.3. A fuzzy space $(X, \Im)$ is said to be

(i) minimal at a point $x \in X$ if $(\bigwedge\{\beta \in \Im : x \in S(\beta)\})(x) = 0$.
(ii) minimal at a fuzzy point $p \in FP(X)$ if $(\bigwedge\{\beta \in \Im : p \in \beta\})(x_p) = p(x_p)$.

Theorem 2.3. *Let* $(X,\Im)$ *be a fuzzy space,* $x \in X$*. Then the following are equivalent:*

(i) $\min(X,\Im,x) \neq \emptyset$.
(ii) $(X,\Im)$ *is not minimal at* x *and* $\bigwedge\{\beta \in \Im : x \in S(\beta)\} \in \Im$.
(iii) $\min(X,\Im,x) = \{\bigwedge\{\beta \in \Im : x \in S(\beta)\}\}$.

Proof. (i) $\Longrightarrow$ (ii) Let $\min(X,\Im,x) = \{\lambda\}$, then for each $\beta \in \Im$ with $x \in S(\beta)$, $\lambda \leq \beta$. Therefore $\lambda \leq \bigwedge\{\beta \in \Im : x \in S(\beta)\}$. On the other hand, since $x \in S(\lambda)$ then $\bigwedge\{\beta \in \Im : x \in S(\beta)\} \leq \lambda$. Thus, $\lambda = \bigwedge\{\beta \in \Im : x \in S(\beta)\}$. Hence $(X,\Im)$ is not minimal at x and $\bigwedge\{\beta \in \Im : x \in S(\beta)\} \in \Im$.

(ii) $\Longrightarrow$ (iii) Let $\gamma = \bigwedge\{\beta \in \Im : x \in S(\beta)\}$, then $\gamma \in \Im$. Also, since $(X,\Im)$ is not minimal at x, it follows that $x \in S(\gamma)$. If $\lambda \in \Im$ with $x \in S(\lambda)$, then it is clear that $\gamma \leq \lambda$. Therefore, $\gamma \in \min(X,\Im,x)$ and hence $\min(X,\Im,x) = \{\gamma\}$.

(iii) $\Longrightarrow$ (i) Clear. □

Corollary 2.1. *Let* $(X,\Im)$ *be a fuzzy space with* $|\Im| < \infty$*. Then for each* $x \in X$*,* $\min(X,\Im,x) \neq \emptyset$.

The proof of the following result is similar to that used in Theorem 2.3.

Theorem 2.4. *Let* $(X,\Im)$ *be a fuzzy space,* $p \in FP(X)$*. Then the following are equivalent:*

(i) $\min(X,\Im,p) \neq \emptyset$.
(ii) $(X,\Im)$ *is not minimal at* p *and* $\bigwedge\{\beta \in \Im : p \in \beta\} \in \Im$.
(iii) $\min(X,\Im,p) = \{\bigwedge\{\beta \in \Im : p \in \beta\}\}$

Corollary 2.2. *Let* $(X,\Im)$ *be a fuzzy space with* $|\Im| < \infty$*. Then for each* $p \in FP(X)$*,* $\min(X,\Im,p) \neq \emptyset$.

The following example shows that the condition "$|\Im| < \infty$" cannot be dropped neither from Corollary 2.1 nor Corollary 2.2.

Example 2.2. Let X be any non-empty set with the fuzzy topology $\Im = \{0_X\} \bigcup \{\lambda : \lambda(X) \subseteq (\frac{1}{2}, 1]\}$. If $p \in FP(X)$ with $p(x_p) = \frac{1}{2}$, then neither $(X,\Im)$ is not minimal at p nor $\bigwedge\{\beta \in \Im : p \in \beta\} \in \Im$, thus by Theorem 2.4 we conclude strongly that $\min(X,\Im,p) = \emptyset$. Therefore, by Theorem 2.1 we must also have $\min(X,\Im,x) = \emptyset$ for each $x \in X$.

Example 2.3. Let $X = \{a,b\}$ with the fuzzy topology $\Im = \{0_X, 1_X\} \bigcup \{\lambda : 0 < \lambda(a) \leq 1 \text{ and } \lambda(b) = 0\}$. Then $\min(X,\Im,b) = \{1_X\}$, but $\min(X,\Im,a) = \emptyset$.

Example 2.3 shows that if a fuzzy set is locally minimal at some point in a fuzzy space, then in general it is not true that each point on its support must have a local minimal fuzzy open set. However, we have the following result.

Theorem 2.5. *Let* $(X,\Im)$ *be a fuzzy space and let* $\lambda \in \Im$*. Then the following are equivalent:*

(i) $\lambda \in \min(X,\Im)$.
(ii) *For each* $x \in S(\lambda)$*,* $\min(X,\Im,x) = \{\lambda\}$.

(iii) *For each* $p \in FP(X)$ *with* $p \in \lambda$, $\min(X, \Im, p) = \{\lambda\}$.

Proof. (i) $\Longrightarrow$ (ii). Let $x \in S(\lambda)$ and $\beta \in \Im$ such that $x \in S(\beta)$. Then $x \in S(\lambda) \cap S(\beta) = S(\lambda \wedge \beta)$ and so $\lambda \wedge \beta \neq 0_X$. Since $\lambda \wedge \beta \leq \lambda \in \min(X, \Im)$, it follows that $\lambda \wedge \beta = \lambda$ and hence $\lambda \leq \beta$. Therefore, $\min(X, \Im, x) = \{\lambda\}$.

(ii) $\Longrightarrow$ (iii). Let $p \in FP(X)$ with $p \in \lambda$ and let $\beta \in \Im$ such that $p \in \beta$. Then $x_p \in S(\lambda) \cap S(\beta)$. Since by (ii), $\min(X, \Im, x_p) = \{\lambda\}$, it follows that $\lambda \leq \beta$ and hence $\min(X, \Im, p) = \{\lambda\}$.

(iii) $\Longrightarrow$ (ii). Let $\beta \in \Im - \{0_X\}$ such that $\beta \leq \lambda$. Choose $x_\circ \in X$ such that $\beta(x_\circ) > 0$. Then $\lambda(x_\circ) \geq \beta(x_\circ) > 0$. Consider the fuzzy point p with $x_p = x_\circ$ and $p(x_p) = \frac{\beta(x_\circ)}{2}$. Then $p \in \lambda \wedge \beta$. Since by (iii) $\min(X, \Im, p) = \{\lambda\}$ it follows that $\lambda \leq \beta$. Therefore, $\lambda = \beta$ and hence $\lambda \in \min(X, \Im)$. □

Recall that a subset A of a topological space (X, τ) is called minimal open set at $x \in X$ if there is no open proper subset of A containing x. The set of all minimal open sets at a point $x \in X$ will be denoted by $\min(X, \tau, x)$.

Theorem 2.6. *Let* $(X, \Im)$ *be a fuzzy space and* $p \in FP(X)$*. If* $\min(X, \Im, p) = \{\lambda\}$*, then for every* $a \in [p(x_p), \lambda(x_p))$, $\lambda^{-1}(a, 1] \in \min(X, \Im_a, x_p)$

Proof. Suppose that $\min(X, \Im, p) = \{\lambda\}$ and let $a \in [p(x_p), \lambda(x_p))$, then $x_p \in \lambda^{-1}(a, 1]$. Let $V \in \Im_a$ with $x_p \in V \subseteq \lambda^{-1}(a, 1]$. Choose $\beta \in \Im$ such that $V = \beta^{-1}(a, 1]$. Since $p(x_p) \leq a$ and $\beta(x_p) > a$, then $p \in \beta$. Since $\min(X, \Im, p) = \{\lambda\}$ it follows that $\lambda \leq \beta$ and hence $\lambda^{-1}(a, 1] \subseteq \beta^{-1}(a, 1] = V$. Therefore, $V = \lambda^{-1}(a, 1]$ and hence $\lambda^{-1}(a, 1] \in \min(X, \Im_a, x)$. □

Theorem 2.7. *Let* $(X, \Im)$ *be a fuzzy space,* $x \in X$*. If* $\min(X, \Im, x) = \{\lambda\}$*, then for every* $a \in [0, \lambda(x))$, $\lambda^{-1}(a, 1] \in \min(X, \Im_a, x)$.

Proof. The proof is similar to that used in Theorem 2.6. □

Corollary 2.3. *Let* $(X, \Im)$ *be a fuzzy space,* $x \in X$, $\min(X, \Im, x) = \{\lambda\}$*, and* $a \in [0, 1)$*. Then the following are equivalent:*

(i) $\lambda^{-1}(a, 1] \in \min(X, \Im_a, x)$.
(ii) $a \in [0, \lambda(x))$.

Proof. (i) $\Longrightarrow$ (ii) Clear.
(ii) $\Longrightarrow$ (i) Theorem 2.7. □

The following example shows that the converse of both Theorems 2.6 and 2.7 is not true in general.

Example 2.4. Let $X = \{b, c\}$ with the fuzzy topology $\Im = \{0_X, 1_X, \lambda, \beta\}$ where $\lambda = \{(b, 0.3), (c, 1)\}$ and $\beta = \{(b, 0), (c, 1)\}$. Consider $p \in FP(X)$ with $x_p = c$ and $p(x_p) = 0.5$. If $a \in [0, 0.3)$, then $\Im_a = \{\emptyset, X\}$ and so $\lambda^{-1}(a, 1] = X \in \min(X, \Im_a) \subseteq (X, \Im_a, c)$. Also, if $a \in [0.3, 1)$, then $\Im_a = \{\emptyset, X, \{c\}\}$ and so $\lambda^{-1}(a, 1] = \{c\} \in \min(X, \Im_a) \subseteq (X, \Im_a, c)$. Since $p \in \beta < \lambda$, then $\lambda \notin \min(X, \Im, p)$ and hence $\lambda \notin \min(X, \Im, c)$.

3. Products, separation axioms, and maps

From now on $\Im_{\text{prod}}$ will denote the product fuzzy topology of $\Im_1$ and $\Im_2$.

We begin this section by the following main product theorem.

Theorem 3.1. *Let $(X, \Im_1)$ and $(Y, \Im_2)$ be two fuzzy spaces, $(x, y) \in X \times Y$. Then*

(i) *If $\min(X \times Y, \Im_{\text{prod}}, (x, y)) = \{\gamma\}$ then there exist $\lambda \in \Im_1$ and $\beta \in \Im_2$ such that $\gamma = \lambda \times \beta$.*
(ii) *If $\min(X, \Im_1, x) = \{\lambda\}$ and $\min(X, \Im_2, y) = \{\beta\}$, then*

$$\min(X \times Y, \Im_{\text{prod}}, (x, y)) = \{\lambda \times \beta\}.$$

Proof. (i) Suppose that $\min(X \times Y, \Im_{\text{prod}}, (x, y)) = \{\gamma\}$. Consider $p \in FP(X \times Y)$ with $x_p = (x, y)$ and $p(x_p) = \frac{\lambda(x,y)}{2}$. Then $p \in \gamma$ and so there exist $\lambda \in \Im_1$ and $\beta \in \Im_2$ such that $p \in \lambda \times \beta \leq \gamma$. Now $(\lambda \times \beta)(x, y) > \frac{\lambda(x,y)}{2} > 0$ and so $(x, y) \in S(\lambda \times \beta)$. Since $\min(X \times Y, \Im_{\text{prod}}, (x, y)) = \{\gamma\}$, it follows that $\gamma \leq \lambda \times \beta$. Therefore, $\gamma = \lambda \times \beta$.

(ii) Suppose that $\min(X, \Im_1, x) = \{\lambda\}$ and $\min(X, \Im_2, y) = \{\beta\}$. Since $x \in S(\lambda)$ and $y \in S(\beta)$ it follows that $(x, y) \in S(\lambda \times \beta)$. Let $\gamma \in \Im_{\text{prod}}$ such that $(x, y) \in S(\gamma)$. Consider $p \in FP(X \times Y)$ with $x_p = (x, y)$ and $p(x_p) = \frac{\lambda(x,y)}{2}$. Then $p \in \lambda$ and so there exist $\lambda_\circ \in \Im_1$ and $\beta_\circ \in \Im_2$ such that $p \in \lambda_\circ \times \beta_\circ \leq \gamma$. It is clear that $x \in S(\lambda_\circ)$ and $y \in S(\beta_\circ)$. Thus, $\lambda \leq \lambda_\circ$ and $\beta \leq \beta_\circ$, hence $\lambda \times \beta \leq \lambda_\circ \times \beta_\circ \leq \gamma$. Therefore, $\min(X \times Y, \Im_{\text{prod}}, (x, y)) = \{\lambda \times \beta\}$. □

In Theorem 3.1 (i) it is not necessarily true that $\min(X, \Im_1, x) = \{\lambda\}$ and $\min(X, \Im_2, y) = \{\beta\}$. This can be noted easily in Example 2.19 of [**3**].

Theorem 3.2. *Let $(X, \Im_1)$ and $(Y, \Im_2)$ be two fuzzy spaces, $s \in FP(X \times Y)$ with $x_s = (x_\circ, y_\circ)$, and $\min(X \times Y, \Im_{\text{prod}}, s) = \{\gamma\}$. If $p \in FP(X)$, $q \in FP(Y)$ with $x_p = x_\circ$, $x_q = y_\circ$ and $p(x_p) = q(x_q) = s(x_s)$, then there exist $\lambda \in \Im_1$ and $\beta \in \Im_2$ such that $p \in \lambda$, $q \in \beta$, and $\gamma = \lambda \times \beta$.*

Proof. Since $s \in \gamma$, there exist $\lambda \in \Im_1$ and $\beta \in \Im_2$ such that $s \in \lambda \times \beta \leq \gamma$. Since $\min(X \times Y, \Im_{\text{prod}}, s) = \{\gamma\}$, then $\gamma \leq \lambda \times \beta$ and hence $\gamma = \lambda \times \beta$. On the other hand, it is easy to see that $p \in \lambda$ and $q \in \beta$. □

The following lemma will be needed in the proof of the next main result. Its proof follows easily and is left to the reader.

Lemma 3.1. *Let X and Y be two non-empty sets, λ and β be two fuzzy subsets of X and Y respectively. Suppose there exist $p \in FP(X)$ and $q \in FP(Y)$ such that $p(x_p) = q(x_q)$ and the fuzzy point $p \times q \in \lambda \times \beta$. Then $p \in \lambda$ and $q \in \beta$.*

The following example shows that the condition "$p(x_p) = q(x_q)$" in Lemma 3.1 cannot be dropped.

Example 3.1. Let $X=Y=\{a,b\}$. Let $\lambda=\{(a,1),(b,0)\}$, $\beta=\left\{\left(a,\frac{1}{2}\right),(b,0)\right\}$, $p\in FP(X)$ with $x_p=a$ and $p(x_p)=\frac{1}{4}$, $q\in FP(Y)$ with $x_q=a$ and $q(x_q)=\frac{3}{4}$. Then it is easy to see that $p\times q\in\lambda\times\beta$ while $q\notin\beta$.

Theorem 3.3. *Let $(X,\Im_1)$ and $(Y,\Im_2)$ be two fuzzy spaces and let $p\in FP(X)$, $q\in FP(Y)$ such that $p(x_p)=q(x_q)$. If $\min(X,\Im_1,p)=\{\lambda\}$ and $\min(X,\Im_2,q)=\{\beta\}$, then $\min(X\times Y,\Im_{\text{prod}},p\times q)=\{\lambda\times\beta\}$.*

Proof. Since $p\in\lambda$ and $q\in\beta$, it is clear that $p\times q\in\lambda\times\beta$. If $\gamma\in\Im_{\text{prod}}$ such that $p\times q\in\gamma$ then there exists $\lambda_1\in\Im_1$ and $\beta_1\in\Im_2$ such that $p\times q\in\lambda_1\times\beta_1\leq\gamma$. Thus by Lemma 3.1, it follows that $p\in\lambda_1$ and $q\in\beta_1$. Since $\min(X,\Im_1,p)=\{\lambda\}$ and $\min(X,\Im_2,q)=\{\beta\}$, then $\lambda\leq\lambda_1$ and $\beta\leq\beta_1$. Therefore, $\lambda\times\beta\leq\lambda_1\times\beta_1\leq\gamma$ and hence $\min(X\times Y,\Im_{\text{prod}},p\times q)=\{\lambda\times\beta\}$. ☐

The following definition is needed in the sequel.

Definition 3.1. [**5**] A fuzzy space $(X,\Im)$ is said to be:
1. T_0 iff for any two distinct fuzzy points p,q in X, there exists an open fuzzy set μ such that ($p\in\mu$ and $\mu\cap q=0$ (i.e., $\mu(x_q)=0$)) or ($q\in\mu$ and $\mu\cap p=0$ (i.e., $\mu(x_p)=0$)).
2. T_1 iff for any two distinct fuzzy points p,q in X, there exist open fuzzy sets μ_1 and μ_2 such that $p\in\mu_1,\mu_1\cap q=0$ (i.e., $\mu_1(x_q)=0$), $\mu_2\cap\, p=0$ (i.e., $\mu_2(x_p)=0$) and $q\in\mu_2$.

Theorem 3.4. *Let $(X,\Im)$ be a T_0 fuzzy space, $\lambda\in\Im$ and $p,q\in FP(X)$. If $\min(X,\Im,p)=\min(X,\Im,q)=\{\lambda\}$, then $x_p=x_q$.*

Proof. Suppose on the contrary that $x_p\neq x_q$. Then p and q are distinct fuzzy points in the T_0 fuzzy space $(X,\Im)$, so there exists $\mu\in\Im$ such that ($p\in\mu$ and $\mu(x_q)=0$) or ($q\in\mu$ and $\mu(x_p)=0$). Without loss of generality we may assume that $p\in\mu$ and $\mu(x_q)=0$. Since $\min(X,\Im,p)=\{\lambda\}$, then $\lambda\leq\mu$. Since $q\in\lambda$, then $\lambda(x_q)>0$ and so $\mu(x_q)>0$, a contradiction. ☐

Corollary 3.1. *Let $(X,\Im)$ be a T_0 fuzzy space, $\lambda\in\Im$ and $x,y\in X$. If $\min(X,\Im,x)=\min(X,\Im,y)=\{\lambda\}$, then $x=y$.*

Proof. Consider $p,q\in FP(X)$ with $x_p=x$, $x_q=y$ and $p(x_p)=q(x_q)=\min\left\{\frac{\lambda(x)}{2},\frac{\lambda(y)}{2}\right\}$. Then $p,q\in\lambda$ and by Theorem 2.1, it follows that $\min(X,\Im,p)=\min(X,\Im,q)=\{\lambda\}$. Therefore, by Theorem 3.4 we conclude that $x_p=x_q$. Hence $x=y$. ☐

Corollary 3.2. *Let $(X,\Im)$ be a T_0 fuzzy space. If $\lambda\in\min(X,\Im)$, then λ is a fuzzy singleton.*

Proof. Suppose to the contrary there are $x,y\in S(\lambda)$ with $x\neq y$. Then by Theorem 2.5, it follows that $\min(X,\Im,x)=\min(X,\Im,y)=\{\lambda\}$. Thus by corollary 3.1 it follows that $x=y$, a contradiction. ☐

The following example shows that the condition "T_0 fuzzy space" cannot be dropped in Corollaries 3.1, 3.2 and Theorem 3.4.

Example 3.2. Let $X = \{a, b\}$ with the indiscrete fuzzy topology. Let $p, q \in FP(X)$ with $x_p = a$, $x_q = b$ and $p(x_p) = q(x_q) = \frac{1}{2}$. Then $\min(X, \Im, x) = \min(X, \Im, y) = \min(X, \Im, p) = \min(X, \Im, q) = \min(X, \Im) = \{1_X\}$.

The discrete fuzzy space shows that the converse of Corollary 3.2 is not true in general. However, we have the following result.

Corollary 3.3. *Let $(X, \Im)$ be a T_0 fuzzy space. If $\min(X, \Im, x) = \{\lambda\}$, then the following are equivalent:*

(i) $\lambda \in \min(X, \Im)$.
(ii) λ *is a fuzzy singleton.*

Proof. (i) $\Longrightarrow$ (ii) Corollary 3.2 and (ii) $\Longrightarrow$ (i) Theorem 2.3. □

Theorem 3.5. *Let $(X, \Im)$ be a T_1 fuzzy space and let $p \in FP(X)$. If*

$$\min(X, \Im, p) = \{\lambda\},$$

then λ is a fuzzy singleton.

Proof. Suppose to the contrary that there exists $y \in S(\lambda)$ with $y \neq x_p$. Consider $q \in FP(X)$ with $x_q = y$ and $q(x_q) = \frac{1}{2}$. Since $(X, \Im)$ is a T_1 fuzzy space, there exists $\mu \in \Im$ such that $p \in \mu$ and $\mu(x_q) = 0$. Since $\min(X, \Im, p) = \{\lambda\}$ and $p \in \mu$, it follows that $\lambda \leq \mu$, but $\lambda(y) > 0$ and $\mu(y) = 0$, a contradiction. □

Corollary 3.4. *Let $(X, \Im)$ be a T_1 fuzzy space and let $x \in X$. If $\min(X, \Im, x) = \{\lambda\}$, then λ is a fuzzy singleton.*

Proof. Consider $p \in FP(X)$ with $x_p = x$ and $p(x_p) = \frac{\lambda(x)}{2}$. Then by Theorem 2.1, it follows that $\min(X, \Im, p) = \{\lambda\}$. Therefore, by Theorem 3.5, it follows that λ is a fuzzy singleton. □

Theorem 3.6. *Let $(X, \Im_1)$ and $(Y, \Im_2)$ be two fuzzy spaces and let $f : (X, \Im_1) \longrightarrow (Y, \Im_2)$ be fuzzy continuous map. If $x_0 \in X$ such that $\min(X, \Im_1, x_0) = \{\lambda\}$ and $f(\lambda) \in \Im_2$ then $\min(Y, \Im_2, f(x_0)) = \{f(\lambda)\}$.*

Proof. Since $\lambda(x_0) > 0$, then $(f(\lambda))(f(x_0)) = \sup\{\lambda(x) : f(x) = f(x_0)\} \geq \lambda(x_0) > 0$ and so $f(x_0) \in S(f(\lambda))$. Suppose that for some $\beta \in \Im_2$, $f(x_0) \in S(\beta)$, then $x_0 \in S(f^{-1}(\beta))$. Since $\min(X, \Im_1, x_0) = \{\lambda\}$, then $\lambda \leq f^{-1}(\beta)$ and so $f(\lambda) \leq f(f^{-1}(\beta)) \leq \beta$. Therefore, $\min(Y, \Im_2, f(x_0)) = \{f(\lambda)\}$. □

Theorem 3.7. *Let $(X, \Im_1)$ and $(Y, \Im_2)$ be two fuzzy spaces and let $f : (X, \Im_1) \longrightarrow (Y, \Im_2)$ be an injective open fuzzy map. If $\min(Y, \Im_2, y_0) = \{\lambda\}$, $x_0 \in f^{-1}(\{y_0\})$, and $f^{-1}(\lambda) \in \Im_1$ then $\min(X, \Im_1, x_0) = \{f^{-1}(\lambda)\}$.*

Proof. Since $(f^{-1}(\lambda))(x_0) = \lambda(f(x_0)) = \lambda(y_0) > 0$, $x_0 \in S(f^{-1}(\lambda))$. Suppose for some $\beta \in \Im_1$, $x_0 \in S(\beta)$, then $f(\beta) \in \Im_2$ and $y_0 \in S(f(\beta))$. Since $\min(X, \Im_2, y_0) = \{\lambda\}$, it follows that $\lambda \leq f(\beta)$. Hence $f^{-1}(\lambda) \leq f^{-1}(f(\beta))$. Since f is an injection, $f^{-1}(f(\beta)) = \beta$. Therefore, $f^{-1}(\lambda) \leq \beta$ and hence $\min(X, \Im_1, x_0) = \{f^{-1}(\lambda)\}$. □

The following result follows immediately either from Theorem 3.6 or Theorem 3.7.

Corollary 3.5. *Let* $(X, \Im_1)$ *and* $(Y, \Im_2)$ *be two fuzzy spaces and let* $f : (X, \Im_1) \longrightarrow (Y, \Im_2)$ *be a fuzzy homeomorphism. Then for each* $x \in X$*,* $\min(X, \Im_1, x) = \{\lambda\}$ *iff* $\min(Y, \Im_2, f(x)) = \{f(\lambda)\}$.

Definition 3.2. [**4**] A fuzzy space $(X, \Im)$ is called homogeneous if for any two points $x, y \in X$ there exists a fuzzy homeomorphism $h : (X, \Im) \longrightarrow (X, \Im)$ such that $h(x) = y$.

Theorem 3.8. *Let* $(X, \Im)$ *be a homogeneous fuzzy space. Then the following are equivalent:*

(i) $\min(X, \Im, x) \neq \emptyset$ *for some* $x \in X$.
(ii) $\min(X, \Im, x) \neq \emptyset$ *for each* $x \in X$ *and whenever* $\min(X, \Im, x) = \{\lambda\}$ *and* $\min(X, \Im, y) = \{\beta\}$ *then* $\lambda(x) = \beta(y)$.

Proof. (i) $\Longrightarrow$ (ii) Suppose for some $x \in X$, $\min(X, \Im, x) \neq \emptyset$ and let $\min(X, \Im, x) = \{\lambda\}$. Let $y \in X$. Since $(X, \Im)$ is homogeneous, there exists a fuzzy homeomorphism $h : (X, \Im) \longrightarrow (X, \Im)$ such that $h(x) = y$. Therefore by Corollary 3.5, it follows that $\min(X, \Im, y) = \{h(\lambda)\}$. Also, $(h(\lambda))(y) = \lambda\left(h^{-1}(y)\right) = \lambda(x)$.

(ii) $\Longrightarrow$ (i) Obvious. □

Definition 3.3. Let $(X, \Im)$ be a fuzzy space and let $t \in (0, 1]$. Then $(X, \Im)$ is called first countable of level t if it is first countable at each $p \in FP(X)$ with $p(x_p) < t$.

It is clear that a fuzzy space is first countable iff it is first countable of level 1.

Theorem 3.9. *Let* $(X, \Im)$ *be a homogeneous fuzzy space. If for some* $x \in X$ $\min(X, \Im, x) = \{\lambda\}$*, then* $(X, \Im)$ *is a first countable fuzzy space of level* $\lambda(x)$.

Proof. Let $p \in FP(X)$ with $p(x_p) < \lambda(x)$. Then by Theorem 3.8, it follows that there exists $\beta \in \Im$ such that $\min(X, \Im, x_p) = \{\beta\}$ and $\beta(x_p) = \lambda(x)$. Since $p(x_p) < \lambda(x) = \beta(x_p)$, then $p \in \beta$. It is not difficult to see that $\{\beta\}$ is a countable local base at p. Therefore, $(X, \Im)$ is a first countable fuzzy space of level $\lambda(x)$. □

Theorem 3.10. *Let* $(X, \Im_1)$ *and* $(Y, \Im_2)$ *be two fuzzy spaces and let* $f : (X, \Im_1) \longrightarrow (Y, \Im_2)$ *be fuzzy continuous map. If* $p \in FP(X)$ *with* $\min(X, \Im_1, p) = \{\lambda\}$ *and* $f(\lambda) \in \Im_2$ *then* $\min(Y, \Im_2, f(p)) = \{f(\lambda)\}$.

Proof. The proof is similar to that used in Theorem 3.6. □

Lemma 3.2. *Let* $(X, \Im_1)$ *and* $(Y, \Im_2)$ *be two fuzzy spaces and let* $f : (X, \Im_1) \longrightarrow (Y, \Im_2)$ *be a fuzzy map. If* $p \in FP(Y)$ *with* $f^{-1}(\{y_p\}) = \{x_0\}$ *then* $f^{-1}(p) \in FP(X)$ *with support* $x_{f^{-1}(p)} = x_0$ *and level* $p(y_p)$.

Proof. Straightforward. □

Theorem 3.11. *Let $(X, \Im_1)$ and $(Y, \Im_2)$ be two fuzzy spaces and let $f : (X, \Im_1) \longrightarrow (Y, \Im_2)$ be an injective open fuzzy map and $p \in FP(Y)$. If $\min(X, \Im_2, p) = \{\lambda\}$, $f^{-1}(\{y_p\}) = \{x_0\}$, and $f^{-1}(\lambda) \in \Im_1$ then $\min\left(X, \Im_1, f^{-1}(p)\right) = \left\{f^{-1}(\lambda)\right\}$.*

Proof. Let $q = f^{-1}(p)$, then by Lemma 3.1 it follows that $q \in FP(X)$ with support $x_q = x_0$ and level $q(x_q) = p(y_p)$. To complete the proof just mimic the proof of Theorem3.7. $\square$

The following result follows immediately either from Theorem 3.10 or Theorem 3.11.

Corollary 3.6. *Let $(X, \Im_1)$ and $(Y, \Im_2)$ be two fuzzy spaces and let $f : (X, \Im_1) \longrightarrow (Y, \Im_2)$ be a fuzzy homeomorphism. Then for every $p \in FP(X)$,*

$$\min(X, \Im_1, p) = \{\lambda\}$$

iff $\min(Y, \Im_2, f(p)) = \{f(\lambda)\}$.

The following lemma will be used in the proof of the following result.

Lemma 3.3. [**1**] *Let $h : X \to Y$ be a bijective function and let p, q be any two fuzzy points of X. Then $h(p) = q$ iff $p(x_p) = q(x_q)$ and $h(x_p) = x_q$.*

Theorem 3.12. *Let $(X, \Im)$ be a homogeneous fuzzy space and $t \in (0, 1)$. The following are equivalent:*

(i) $\min(X, \Im, p) \neq \emptyset$ *for some $p \in FP(X)$ with $p(x_p) = t$,*
(ii) $\min(X, \Im, p) \neq \emptyset$ *for each $p \in FP(X)$ with $p(x_p) = t$.*

Proof. (i) $\Longrightarrow$ (ii). Suppose for some $p \in FP(X)$ with $p(x_p) = t$, $\min(X, \Im, p) \neq \emptyset$ and let $\min(X, \Im, p) = \{\lambda\}$. Let $q \in FP(X)$ with $q(x_q) = t$. Since $(X, \Im)$ is homogeneous, there exists a fuzzy homeomorphism $h : (X, \Im) \longrightarrow (X, \Im)$ such that $h(x_p) = x_q$. Thus by Lemma 3.3, it follows that $h(p) = q$. Therefore by Corollary 3.6, it follows that $\min(X, \Im, q) = \{h(\lambda)\}$.

(ii) $\Longrightarrow$ (i). Obvious. $\square$

The following example shows that the condition "homogeneous" on the fuzzy space $(X, \Im)$ in Theorems 3.8 and 3.12 cannot be dropped.

Example 3.3. Let $X = \{a, b\}$ with the fuzzy topology $\Im = \{0_X, 1_X, \lambda\}$ where $\lambda = \{(a, 0), (b, 1)\}$. Then $\min(X, \Im, b) = \{\lambda\}$ but $\min(X, \Im, a) = \emptyset$. Consider $p, q \in FP(X)$ with $x_p = a$, $x_q = b$, and $p(x_p) = q(x_q) = \frac{1}{2}$. Then $\min(X, \Im, p) = \{\lambda\}$ but $\min(X, \Im, q) = \emptyset$.

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