Acta Math. Univ. Comenianae
Vol. LXXV, 1(2006), pp. 31–41

HOCHSHILD COHOMOLOGY AND EQUIVALENCE OF GRADED STAR PRODUCTS

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Abstract. We study graded star products on dual of finite dimensional Lie algebras. We prove that all graded star products are entirely determined by the expression of $X \star u$ where X belongs to the Lie algebra $\mathfrak{g}$ and u is a polynomial function on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. We also consider the Hochschild cohomology and we prove that all graded differential star products are equivalent.

0. Introduction

Deformations of usual multiplication of functions are called star products. This notion has been introduced by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer [**3**] to give an autonomous phase space formulation of quantum mechanics without operators in the general case of a Poisson manifold. Star products were also used in the representation theory of Lie groups.

The problem of existence of star products has been solved by different steps. In the case of finite dimensional symplectic manifolds, J.Vey has determined the corresponding differential Hochschild cohomology [**11**] and S. Gutt has studied the three first groups in the Chevally cohomology [**6**]. Then, M. Dewilde and P. Lecomte have used these cohomologies to prove the existence of star products on any symplectic manifold [**4**]. The classification of these star products was done by A. Lichnerowicz [**9**] by using the second De Rham cohomological group.

An explicit star product on any Lie algebra was given in [**7**]. In fact S. Gutt constructed a star product on the cotangent bundle of any Lie group G. This star product can be restricted to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. Since then, a geometric construction of a star product has been done by B. F. Fedesov in [**5**].

Recently, M. Kontsevich has entirely solved the problem of existence of star products on any finite dimensional Poisson manifold [8]. He built a star product $\star_\alpha^K$ on $\mathbb{R}^d$ equipped with a general Poisson bracket α. This famous result has been proved by considering some oriented admissible graphs $\overrightarrow{\Gamma}$. In [**1**] D. Arnal proved that in the nilpotent case Gutt's star product coincides with Kontsevich's star product.

Received March 3, 2004.
2000 *Mathematics Subject Classification.* Primary 53D55.
Key words and phrases. Star products, Poisson structures, Lie algebras.

These examples of star products i.e Gutt's star product and Kontsevich's star product are graded star products i.e if u_1 and u_2 are polynomial functions on $\mathbb{R}^d$ homogeneous with degree $|u_1|$ and $|u_2|$, $C_n(\alpha)(u_1, u_2)$ is still a polynomial function, homogeneous with degree $|u_1| + |u_2| - n$.

In the present paper, we study graded star products on dual of finite dimensional Lie algebras and we prove the equivalence of all graded differential star products.

The paper is organized as follows :

In the first section, we prove that a graded star product on dual of Lie algebras is totally determined by the expression of $X \star u$ where X is in $\mathfrak{g}$ and u is in $S(\mathfrak{g})$ (polynomial functions on $\mathfrak{g}^*$). Our proof is original and elementary.

In the second section, we study the Hochschild cohomology on $C^\infty(\mathfrak{g}^*)$ and $S(\mathfrak{g})$.

In the third section, we prove the equivalence of all graded differential star products. The equivalence operator is constructed by induction.

Finally, in the fourth section, we study the case of Kontsevich's star product, we give explicitly the equivalence operator of this star product with Gutt's star product.

1. Graded star products on dual of Lie algebras

Definition 1.1. [**3**] Let W be a differentiable Poisson manifold with a poisson bracket $\{\ ,\ \}$ and E be the space of formal series in the parameter $\hbar$ with coefficients in $C^\infty(W)$.

A star product on $C^\infty(W)$ is defined by a bilinear map from $C^\infty(W) \times C^\infty(W)$ into E :

$$(u, v) \mapsto u \star_\hbar v = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} C^r(u, v) \in E$$

Where :

(i) C^r is a bidifferential operator on $C^\infty(W)$ (of maximum order r $(r > 1)$ in each argument, null on the constants).

(ii) $C^0(u, v) = u.v$; $C^1(u, v) = \{u, v\}$.

(iii) C^r is symmetric (resp skew symmetric) in (u, v) if r is even (resp odd).

(iv) $\sum_{r+s=t} (r!s!)^{-1} C^r(C^s(u, v), w)) = \sum_{r+s=t} (r!s!)^{-1} C^r(u, C^s(v, w))$ $(t = 1, 2, \ldots)$.

Let $S(\mathfrak{g})$ be the algebra of polynomial functions on the dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$. The algebra $S(\mathfrak{g})$ is graded. If u is an homogeneous element of $S(\mathfrak{g})$ we will denote $|u|$ its degree. A multilinear function C:

$$C\ :\ S(\mathfrak{g}) \times \ldots \times S(\mathfrak{g}) \to S(\mathfrak{g})$$

is said to be homogeneous with degree $-n$ if for $u_1, \ldots, u_k$ homogeneous elements of $S(\mathfrak{g})$ one has $C(u_1, \ldots, u_k)$ is homogeneous with degree $|u_1| + |u_2| + \ldots + |u_k| - n$.

Definition 1.2. [**2**] Let $S(\mathfrak{g})$ be the algebra of the polynomial functions on $\mathfrak{g}^*$ and S^p be the space of homogeneous polynomials of degree p. A star product on $S(\mathfrak{g})$ is called graded if

$$\forall r, p, q \in \mathbb{N},\ \ \forall (u,v) \in S^p \times S^q, \quad C^r(u,v) \in S^{p+q-r}.$$

Let us show that a graded $\star$-product is totally defined by $X \star u$ where u belongs to $S(\mathfrak{g})$ and X belongs to $\mathfrak{g}$.

Proposition 1.3. *Let C_n be a sequence of bilinear map:*

$$C_n \ : \mathfrak{g} \times S(\mathfrak{g}) \to S(\mathfrak{g})$$

such that

1. $C_0(X,u) = X.u$ *and* $C_1(X,u) = [X,u]$.
2. *If u is homogeneous then $C_n(X,u)$ is homogeneous and its degree is $|u| + 1 - n$.*
3. *Let σ be defined on $\mathfrak{g}^*$ by $\sigma(\xi) = -\xi$.*

We set

$$u^\sigma(\xi) = u(\sigma(\xi)) \ \ \forall\ u \in S(\mathfrak{g}) \quad \textit{and we define} \ \ X \circ u = \sum_{n=0}^{\infty} C_n(X,u) \ .$$

We suppose that

a) $u \circ X = (X^\sigma \circ u^\sigma)^\sigma = \sum_{n=0}^{\infty} (-1)^n C_n(X,u)$.

b) $X \circ (u \circ Y) = (X \circ u) \circ Y \qquad \forall\ X, Y \in \mathfrak{g}\ ,\ \ \forall\ u \in S(\mathfrak{g})$.

c) $X \circ (Y \circ u) - Y \circ (X \circ u) = [X,Y] \circ u \qquad \forall\ X, Y \in \mathfrak{g}\ ,\ \ \forall\ u \in S(\mathfrak{g})$.

Then there exists one and only one star product such that

$$X \star u = X \circ u \quad \forall\ X \in \mathfrak{g}\ ,\ \ \forall\ u \in S(\mathfrak{g})\,.$$

This star product is graded.

Proof. Let $v \in S(\mathfrak{g})$, we define $u \star v$ by induction on the degree of u starting with $1 \star v = v$ and $X \star v = X \circ v$.
If u is an homogeneous polynomial function of the form $u = Xu'$ then there exists a polynomial function u'' such that

$$Xu' = X \circ u' + u'' \text{ and } \ \ |u''| \leq |u| - 1.$$

We suppose $u_1 \star v$ defined for any u_1 such that $|u_1| < |u|$, we suppose also that:

$$u_1 \star (v \star w) = (u_1 \star v) \star w \quad if \quad |u_1| + |v| < |u|\,.$$

Then we set

$$(Xu') \star v = X \circ (u' \star v) + u'' \star v$$

This formula defines without ambiguity $u \star v$. In fact if u has the form

$$\begin{aligned} u = X_1 X_2 w &= X_1 \circ (X_2 \circ w) + u_1'' \\ &= X_2 \circ (X_1 \circ w) + u_2'' \end{aligned}$$

Then

$$X_2 \circ (X_1 \circ w) + u_2'' = X_1 \circ (X_2 \circ w) + u_2'' + [X_2, X_1] \circ w.$$

Thus

$$u_1'' = u_2'' + [X_2, X_1] \circ w$$

and one has

$$\begin{aligned} X_1 \circ & ((X_2 \circ w) \star v) + u_1'' \star v \\ &= X_1 \circ (X_2 \circ (w \star v)) + u_2'' \star v + ([X_2, X_1] \circ w) \star v \\ &= X_1 \circ (X_2 \circ (w \star v)) + [X_2, X_1] \circ (w \star v) + u_2'' \star v \\ &= X_2 \circ (X_1 \circ (w \star v)) + u_2'' \star v \\ &= X_2 \circ ((X_1 \circ w) \star v) + u_2'' \star v. \end{aligned}$$

The homogeneous term of the maximum degree in $u \star v$ is $C_0(u, v) = u \cdot v$ then we set

$$C_n(u, v) = \text{ the homogeneous term of degree } |u| + |v| - n.$$

A simple calculation shows that $\star$ is a star product in fact let us first show that the $C_1(u, v) = \{u, v\}$:

It is clear that the term of degree $|u| + |v| - 1$ in $u'' \star v$ is $-\{X, u'\}\, v$ (coming from $u'' \cdot v$) and in $X \circ (u' \star v)$ is $X\{u', v\} + \{X, u' \cdot v\}$. Thus the term of degree $|u| + |v| - 1$ in $u \star v$ is the sum of these two terms

$$\begin{aligned} X\{u', v\} + \{X, u'v\} - \{X, u'\}\, v = {} & X\{u', v\} + u'\{X, v\} \\ = {} & \{X\, u', v\} = \{u, v\} \end{aligned}$$

Now one has

$$\begin{aligned} (u \circ X) \circ Y - (u \circ Y) \circ X &= (Y^\sigma \circ (u \circ X)^\sigma)^\sigma - ((X^\sigma \circ (u \circ Y)^\sigma)^\sigma \\ &= (Y^\sigma \circ (X^\sigma \circ u^\sigma))^\sigma - (X^\sigma \circ (Y^\sigma \circ u^\sigma))^\sigma \\ &= ([Y^\sigma, X^\sigma] \circ u^\sigma)^\sigma = ([X, Y]^\sigma \circ u^\sigma)^\sigma \\ &= u \circ [X, Y] \end{aligned}$$

Using the same construction, we can then define similarly $u \star' v$ by induction “on the right side” on the degree of v. Then $\star$ and $\star'$ coincide in fact if $|u| = |v| = 1$ then by using a/ we deduce $u \star' v = u \star v$. Now suppose that they coincide for u' and v' such that $|u'| + |v'| < |u| + |v|$, then

$$\begin{aligned} (X'u') \star (v' \circ Y) = X'(u' \star (v' \circ Y)) &= X \circ (u' \star' (v' \circ Y)) \\ &= X \circ ((u' \star' v') \circ Y) = (X \circ (u' \star' v')) \circ Y \\ &= ((X \circ u') \star' v') \circ Y = (X \circ u') \star' (v' \circ Y) \end{aligned}$$

Finally, by construction, $u \star' v = (v^\sigma \star u^\sigma)^\sigma$ then

$$C_n(u, v) = (-1)^n C_n(v, u).$$

By induction on the degree of u and w, we can show that $\star$ is associative.

By definition $u \star (v \star w) = (u \star v) \star w$ if $|u| \leq 1$ and $|w| \leq 1$. Then by induction on $|u|$ the same holds if $|w| \leq 1$ since

$$\begin{aligned}((X \circ u') \star v) \star w &= (X \circ (u' \star v)) \star w = X \circ ((u' \star v) \star w) \\ &= X \circ (u' \star (v \star w)) = (X \circ u') \star (v \star w)\end{aligned}$$

and similarly for any w' since:

$$\begin{aligned}u \star (v \star (w' \circ Y)) &= u \star ((v \star w') \circ Y) \\ &= ((u \star v) \star w') \circ Y) = (u \star v) \star (w' \circ Y)\end{aligned}$$

Finally $\star$ is a graded star product by construction. □

Thus we can conclude the following theorem.

Theorem 1.4. *Let $\star$ be a graded star product, then $\star$ is totally determined by the mapping from $\mathfrak{g} \times S(\mathfrak{g})$ into $S(\mathfrak{g})$:*

$$\mathfrak{g} \times S(\mathfrak{g}) \to S(\mathfrak{g}) \qquad (X, u\ ) \mapsto X \star u.$$

2. Hochschild cohomology

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and $\mathfrak{g}^*$ its dual. Let M be the space $C^\infty(\mathfrak{g}^*)$ of C^∞ functions on $\mathfrak{g}^*$, or the space $S(\mathfrak{g})$ of polynomial functions on $\mathfrak{g}^*$.

If β is a multi index we denote D^β the differential operator

$$D^\beta = (\partial_1)^{b_1} \dots (\partial_n)^{b_n} \qquad \text{if} \ \ \beta = (b_1, \dots, b_n), \ \ b_i \in \mathbb{N}$$

A multi differential operator C on M is a p-linear application from $M \times M \times \ldots \times M$ to M such that

$$C(u_1, \dots, u_p) = \sum_{\beta_1 \dots \beta_p} C_{\beta_1 \dots \beta_p}\ D^{\beta_1} u_1\ D^{\beta_2} u_2 \dots D^{\beta_p} u_p$$

where the $C_{\beta_1 \dots \beta_p}$ are all in M and the sum is finite.

Now let us recall the Hochschild cohomology for p-cochain. We consider here only differential, vanishing on constants cochains i.e a p-cochain is a p-differential operator C:

$$C = \sum_{\beta_1 \dots \beta_p} C_{\beta_1 \dots \beta_p}\ D^{\beta_1} \cup\ D^{\beta_2} \cup \ldots \cup\ D^{\beta_p}$$

such that $C(u_0, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_p) = 0\ \ \forall\ i$ or $|\beta_i| > 0\ \ \forall\ i$.

The coboundary operator δ is given by :

$$\begin{aligned}(\delta C)(u_0, \dots, u_p) &= u_0 \cdot C(u_1, \dots, u_p) - C(u_0 \cdot u_1, u_2, \dots, u_p) + \dots \\ &\quad + (-1)^{i+1} C(u_0, \dots, u_{i-1}, u_i \cdot u_{i+1}, \dots, u_p) + \dots \\ &\quad + (-1)^{p-1} C(u_0, \dots, u_{p-1}) \cdot u_p\end{aligned}$$

Remark 2.1. Let us remark that we can write the associativity of a star product by using the coboundary operator δ. In fact a bilinear map $u \star v = \sum_{n\geq 0} C_n(u,v)$ such that $C_0(u,v) = u\cdot v$, defines an associative law at the order n if

$$\sum_{p+q=n} C_p(C_q(u,v),w) = \sum_{p+q=n} C_p(u, C_q(v,w))$$

and if we know $C_0, C_1, \ldots, C_{n-1}$ this relation can be written

$$\delta C_n(u,v,w) = \sum_{p=1}^{n-1} C_p(C_{n-p}(u,v),w) - C_p(u, C_{n-p}(v,w)).$$

Remark 2.2. We also remark that if $\star$ and $\star'$ are two star products coinciding up to the order $n-1$,then we can change $\star'$ by an other star product $\star''$ equivalent coinciding with $\star$ up to the order n if and only if there exists an operator T_n such that

$$C_n(u,v) = C'_n(u,v) + (\delta T_n)(u,v).$$

Now we will define the graded cohomology.

Definition 2.3. Let $n \in \mathbb{N}$, $n \geq 2$. A bilinear map C_n

$$C_n : \mathfrak{g} \times S(\mathfrak{g}) \to S(\mathfrak{g})$$

such that $C_n(X,u)$ is homogeneous of degree $1 + |u| - n$ for all homogeneous polynomial u is said to be a 2-cocycle if

$$\begin{aligned}&\delta C_n(X,u,Y)\\ &\quad = (-1)^n X C_n(Y,u) - (-1)^n C_n(Y,Xu) + C_n(X,uY) - C_n(X,u)Y \equiv 0\end{aligned}$$

and

$$\begin{aligned}&\delta C_n(X,Y,u) - \delta C_n(Y,X,u)\\ &\qquad = X C_n(Y,u) + C_n(X,Yu) - Y C_n(X,u) - C_n(Y,Xu) \equiv 0\end{aligned}$$

If n is even, C_n is said to be a 2-cobord if there exists a linear map b_n from $S(\mathfrak{g})$ to $S(\mathfrak{g})$ which verifies $b_n(u)$ is homogeneous of degree $|b_n(u)| = |u| - n$ and such that

$$C_n(X,u) = (\delta b_n)(X,u) = X b_n(u) - b_n(Xu)$$

It is not difficult to verify that the space B^2_n of 2-cobords is a subspace of the space Z^2_n of 2-cocycles. Then we set

$$H^2_n = \frac{Z^2_n}{B^2_n}$$

The space H^2_n is called the second cohomology group graded of degree n (if n is odd then $H^2_n = Z^2_n$).

More generally, let M be the space $C^\infty(\mathfrak{g}^*)$ or $S(\mathfrak{g})$. If $C^p(M)$ is the space of p-cochain, the space of p-cocycle $Z^p(M,\delta)$ is the kernel of δ in $C^p(M)$, the space of p-coboundary $B^p(M,\delta)$ is $\delta(C^{p-1}(M))$ and the cohomology group $H^p(M,\delta)$ is the quotient

$$\frac{Z^p(M,\delta)}{B^p(M,\delta)}.$$

These groups were studied by J. Vey [**11**]. The p-cochains considered are multidifferential operators with action on C^∞ functions $u_1, u_2, \dots, u_p$.

Theorem 2.4. [**11**] *Let $\mathfrak{g}$ be a Lie algebra and let M the space of C^∞ functions on $\mathfrak{g}^*$. The P^{th} cohomology group $H^p(M,\delta)$ is isomorphic to the space of contravariant totally antisymmetric p-tensors γ*

$$H^p(M,\delta) \simeq \mathrm{Hom}\,(\wedge^p\mathfrak{g},M)$$

The isomorphism is given by:
To $[C] \in H^p(M,\delta)$, we associate $c \in \mathrm{Hom}\,(\wedge^p\mathfrak{g},M)$ defined by

$$c\ (X_1,\dots,X_p) = \sum_{\sigma\in\Sigma_p} \epsilon(\sigma) C(x_{\sigma(1)},\dots x_{\sigma(p)})$$

Σ_p is the group of permutations of $\{1,\dots,p\}$.

Now we can show the following proposition.

Proposition 2.5. *Let C be a 2-cocycle homogeneous.*
(1) *If C is antisymmetric of degree $-n$ $(n > 2)$, then $C = 0$.*
(2) *If C is symmetric of degree $-n$ $(n \geq 0)$, then, there exists an operator T such that $C = \delta T$.*

Proof. By the above theorem $H^p(M,\delta)$ is isomorphic to the space of contravariant totally antisymmetric p-tensors γ. Then if C is a p-differential cocycle, there exists a differential operator T homogeneous of degree p such that

$$C = C_\gamma + \delta T,$$

where C_γ is a p-tensor contravariant totally antisymmetric

$$C_\gamma(u_1,\dots,u_p) = \sum_{i_1\dots i_p} \gamma^{i_1\dots i_p}\ \partial_{i_1}u_1 \dots\ \partial_{i_p}u_p$$

A multidifferential operator is totally determined by its restriction to polynomial functions. Such operator sends $S\,(\mathfrak{g}) \times S\,(\mathfrak{g}) \times \dots \times S\,(\mathfrak{g})$ to $S\,(\mathfrak{g})$ if and only if its coefficients are polynomials.

In particular let C be a 2-cocycle homogeneous antisymmetric of degree $-n$ $(n > 2)$. δT being symmetric, we can deduce, from the equality $C = C_\gamma + \delta T$, that $C = C_\gamma$. But for all i,j we have $C_\gamma(x_i,x_j) = \gamma^{ij}$ is an homogeneous polynomial of degree $2-n$.
Then we obtain

$$C(x_i,x_j) = \gamma^{ij} = 0.$$

Thus we can deduce that $C = 0$.

Now let C be a 2-cocycle homogeneous symmetric, C_γ being antisymmetric then $C = \delta T$.

An explicit calculation of such T is given by Gutt in [6] . If $u = x_1^{a_1} \dots x_d^{a_d}$ then

$$(T(u))(\xi) = \sum_{i=1}^{d} \sum_{j=1}^{a_{i-1}} C(x_i,\ x_1^{a_1} \dots\ x_{i-1}^{a_{i-1}}\ x_i^j)_{|\xi}\ \xi_i^{a_{i-j-1}}\ \xi_{i+1}^{a_{i+1}} \dots\ \xi_d^{a_d}\ \ , \forall \xi \in \mathfrak{g}^*.$$

This operator T is differential and homogeneous of degree n. □

3. Equivalence of graded star products

Let us recall the definition of the equivalence of two star products.

Definition 3.1. Two graded star products $\star$ and $\star'$ are said to be equivalent if there exists a linear map T

$$T : S(\mathfrak{g}) \to S(\mathfrak{g})$$
$$u \mapsto T(u) = \sum_{n=0}^{\infty} T_n(u)$$

such that

(1) $T_0(u) = u$.
(2) If u is homogeneous then $T_n(u)$ is also homogeneous and its degree is $|u| - n$.
(3) $T(u \star v) = (T(u)) \star' (T(v))$.

Now we can show the following theorem.

Theorem 3.2. *Two differential graded star products on $S(\mathfrak{g})$ are equivalent by an operator of the form*

$$T = Id + \sum_{k=1}^{\infty} T_{2k}$$

where T_{2k} is a differential operator homogeneous of degree $-2k$.

Proof. Let $\star$ and $\star'$ be two differential graded star products on $S(\mathfrak{g})$. We shall construct the operator T by induction. Assume that there exist $2(k-1)$ differential operators $T_0, \dots, T_{2(k-1)}$ for $k \geq 1$ such that $T_0 = \mathrm{Id}$ and each T_{2j} $(0 \leq j \leq k-1)$ is homogeneous of degree $-2j$.

Then the following star product $\star''$ defined by

$$u \star'' v = (T_0 + \dots + T_{2(k-1)})^{-1}((T_0 + \dots + T_{2(k-1)})u) \star' ((T_0 + \dots + T_{2(k-1)})v),$$

satisfies

$$C''_j(u, v) = C_j(u, v) \qquad \text{for } \ \forall j \leq 2(k-1).$$

By construction $\star''$ is a differential star product, then by using the associativity of $\star''$, we obtain

$$\delta(C''_{2k-1} - C_{2k-1}) = 0\ .$$

Now, if $k-1=0$, then we have

$$C_1''(u,v) = C_1(u,v) = \{u,v\}\,.$$

If $k \geq 2$, then $2k-1 > 2$ and $2k-1$ is odd, we obtain by (1) of the above proposition that:

$$C_{2k-1}'' = C_{2k-1}.$$

Thus by the associativity condition, we deduce that

$$\delta(C_{2k}'' - C_{2k}) = 0,$$

Finally (2) of the Proposition 2.5, proves that there exists a differential operator homogeneous of degree $-2k$ such that

$$C_{2k}'' - C_{2k} = \delta(T_{2k}).$$

We continue the construction by induction. □

In the next section,we will give an example of the equivalence operator T.

4. The case of Kontsevich star product and Gutt star product

We recall that the star product $\star_\alpha^G$ constructed by Gutt in [**7**] and the star product $\star_\alpha^K$ built by Kontsevich in [**8**] are graded. Thus by the Theorem 3.2 these two star products are equivalent. We can give explicitly the equivalence operator between these two well known star products. In fact, Gutt's star product has an integral formula given for u_1, u_2 polynomial functions on $\mathfrak{g}^*$ or u_1, u_2 such that $\hat{u}_1, \hat{u}_2$ are smooth functions with sufficiently small support

$$u_1 \star_\alpha^G u_2(\xi) = \int_{\mathfrak{g}^2} \hat{u}_1(X)\hat{u}_2(Y) e^{2i\pi\langle X\times_\alpha Y,\xi\rangle} dXdY\ ,$$

if

$$\hat{u}(X) = \int_{\mathfrak{g}^*} u(\xi) e^{-2i\pi\langle X,\xi\rangle} d\xi\ .$$

Here $X \times_\alpha Y$ is the Baker-Campbell-Hausdorff formula for any Lie algebra $\mathfrak{g}$ equipped with a linear Poisson bracket α and any X and Y in $\mathfrak{g}$

$$\exp X \cdot \exp Y = \exp(X \times_\alpha Y).$$

In [**10**], Shoiket compared the Kontsevich star product and the Duflo formula in the case of linear Poisson bracket. From his results, we can deduce that the Kontsevich star product has the following universal integral formula

$$(u_1 \star_\alpha^K u_2)(\xi) = \int_{\mathfrak{g}^2} \hat{u}_1(X)\hat{u}_2(Y) \frac{J(X)J(Y)}{J(X\times_\alpha Y)} e^{2i\pi\langle X\times_\alpha Y,\xi\rangle} dXdY\ ,$$

where

$$J(X) = \left[\det\left(\frac{\mathrm{sh}(\mathrm{ad}\frac{X}{2})}{\mathrm{ad}\frac{X}{2}}\right)\right]^{\frac{1}{2}}.$$

Proposition 4.1. *The Kontsevich star product is equivalent to the Gutt star product through the equivalence operator T defined by*

$$T(u_1)(\xi) = \int_{\mathfrak{g}} \hat{u}_1(X) J(X) e^{2i\pi\langle X,\xi\rangle} dX \ .$$

Proof. Let T be the operator defined by

$$\left(T(u_1)\right)^{\wedge}(X) = \hat{u}_1(X)J(X) \ .$$

We have:

$$\left(T\ (u_1 \star_\alpha^K u_2)\right)^{\wedge}\ (X \times_\alpha Y) = (\widehat{u_1 \star_\alpha^K u_2})\ (X \times_\alpha Y) J(X \times_\alpha Y)$$

Now, by the integral formula, we obtain

$$\begin{aligned}
(\widehat{u_1 \star_\alpha^K u_2})(X \times_\alpha Y)J(X \times_\alpha Y) &= \hat{u}_1(X)\hat{u}_2(Y)J(X)J(Y) \\
&= \left(T(u_1)\right)^{\wedge}(X)\left(T(u_2)\right)^{\wedge}(Y) \\
&= \left(T(u_1) \star_\alpha^G T(u_2)\right)^{\wedge}(X \times_\alpha Y)
\end{aligned}$$

Thus

$$\left(T\ (u_1 \star_\alpha^K u_2)\right)^{\wedge} = \left(T(u_1) \star_\alpha^G T(u_2)\right)^{\wedge}$$

This proves our proposition. □

This equivalence operator T is a formal series of differential operators T_k

$$T = Id + \sum_{n=1}^{\infty} \sum_{|k_1+k_2+\ldots k_p|=n} a_{k_1\ldots k_p}\ T_{k_1} \circ T_{k_2} \circ \ldots \circ T_{k_p}.$$

Here T_k is the operator

$$T_k(u)(\xi) = (2i\pi)^k \int_{\mathfrak{g}} \hat{u}(X) Tr(adX)^k e^{2i\pi\langle X,\xi\rangle} dX.$$

These operators T_k are "wheel operators". In fact, each T_k is associated to a graph Γ_k called by Kontsevich a "wheel". Γ_k has k vertices of the first kind $p_1, \ldots, p_k$, one vertex of the second kind q_1 and the edges of Γ_k are

$$\{\ \overrightarrow{p_1p_2},\ \overrightarrow{p_1q_1},\ \overrightarrow{p_2p_3},\ \overrightarrow{p_2q_1}, \ldots,\ \overrightarrow{p_{k-1}p_k},\ \overrightarrow{p_{k-1}q_1},\ \overrightarrow{p_kp_1},\ \overrightarrow{p_kq_1}\}$$

T_k can be written

$$T_k(u) = \sum_{i_1\ldots i_k} \sum_{j_1\ldots j_k} C_{i_1 j_1}^{j_2}\ C_{i_2\ j_2}^{j_3} \ldots C_{i_k\ j_k}^{j_1} \partial_{i_1\ldots i_k} u,$$

where the C_{ij}^k are the structure constants of the Lie algebra $\mathfrak{g}$.

Acknowledgement. We want to thank professor Didier Arnal for advices and suggestions.

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