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AN ASSOCIATED MOB OF A TOPOLOGICAL GROUP

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Abstract. Here a typical topological semigroup $C(G)$ is studied. A partial equivalence [**1**] is defined on $C(G)$ compatible with its semigroup structure. Also a uniformity is constructed giving the Vietoris topology [**3**] on $C(G)$.

1. Introduction

If G is a group then the product of subsets of G can be defined in a natural way to produce a semigroup of subsets; we present here the semigroup $C(G)$ of all compact subsets of a topological group G endowed with the Vietoris topology [**3**]. Actually, this semigroup is a subsemigroup of the semigroup of all subsets; the construction of this semigroup is quite obvious in view of the fact that for compact subsets A and B of a topological group G, AB is compact.

In the second section we introduce a partial equivalence [**1**] on the semigroup consisting of all compact subsets of a topological group G, compatible with the semigroup structure and determine its several classes.

In the third article we show that the Vietoris topology [**3**] on the collection of aforesaid subsets of a topological group is compatible with the algebraic structure.

Lastly, we determine a uniformity giving the topology of the above space.

2. Construction of $C(G)$

Let G be a topological group which is assumed to be a Hausdorff space. We consider the collection of all nonempty compact subsets of G and denote this collection by $C(G)$.

For $A, B \in C(G)$ we define, $AB = \{ab : a \in A,\ b \in B\}$. Then AB is again a compact subset of G and thus $AB \in C(G)$. This shows that $C(G)$ becomes a semigroup under the aforesaid binary operation. Also $\{e\} \in C(G)$, where 'e' denotes the identity of the topological group G. We note that, $\{e\}$ also acts as an identity in $C(G)$.

We now show that $C(G)$ cannot be a group unless $G = \{e\}$.

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Proposition 2.1. *$C(G)$ is a group iff $|G| = 1$, where $|A|$ denotes the cardinality of $A(\subseteq G)$.*

Proof. Let $A, B \in C(G)$ and $a \in A$, $b_1, b_2 \in B$. Then $ab_1 = ab_2$ iff $b_1 = b_2$. Thus $|aB| = |B|$, for any $a \in A$. Therefore

$$(*) \qquad |AB| \geq |B|.$$

Let $A^\star$ denotes the inverse of A in $C(G)$. Then $1 = |\{e\}| = |A^\star A| \geq |A| \geq 1$ [by $(*)$]$\Rightarrow |A| = 1 \Rightarrow A$ is a singleton set. Thus all invertible elements of $C(G)$ are singleton. Therefore it follows that $C(G)$ cannot be a group if $|G| > 1$. □

Note 2.2. Although $C(G)$ itself cannot be a group if $|G| > 1$, $C(G)$ contains the subgroup $\{\{a\} : a \in G\} = \mathcal{G}$ (say). We claim that $\mathcal{G}$ is a unique maximal subgroup of $C(G)$. It follows from the fact that, singletons of G are the only invertible elements of $C(G)$ [as seen in Proposition 2.1]. Clearly this subgroup $\mathcal{G}$ of $C(G)$ is isomorphic to G.

Definition 2.3. Let $\mathcal{H} \subseteq C(G)$. We define,

$$\uparrow\mathcal{H} = \{L \in C(G) : H \subseteq L \text{ for some } H \in \mathcal{H}\}$$
$$\downarrow\mathcal{H} = \{L \in C(G) : L \subseteq H \text{ for some } H \in \mathcal{H}\}$$

Then clearly,

$$\text{(i)} \qquad \mathcal{H} \subseteq \uparrow\mathcal{H}, \quad \mathcal{H} \subseteq \downarrow\mathcal{H};$$
$$\text{(ii)} \qquad \uparrow(\mathcal{H}_1 \cup \mathcal{H}_2) = \uparrow\mathcal{H}_1 \cup \uparrow\mathcal{H}_2, \quad \downarrow(\mathcal{H}_1 \cup \mathcal{H}_2) = \downarrow\mathcal{H}_1 \cup \downarrow\mathcal{H}_2;$$
$$\text{(iii)} \qquad \uparrow(\mathcal{H}_1 \cap \mathcal{H}_2) \subseteq \uparrow\mathcal{H}_1 \cap \uparrow\mathcal{H}_2, \quad \downarrow(\mathcal{H}_1 \cap \mathcal{H}_2) \subseteq \downarrow\mathcal{H}_1 \cap \downarrow\mathcal{H}_2;$$
$$\text{(iv)} \qquad \uparrow(\uparrow\mathcal{H}) = \uparrow\mathcal{H}, \quad \downarrow(\downarrow\mathcal{H}) = \downarrow\mathcal{H};$$
$$\text{(v)} \qquad \mathcal{H}_1 \subseteq \mathcal{H}_2 \Rightarrow \uparrow\mathcal{H}_1 \subseteq \uparrow\mathcal{H}_2, \quad \downarrow\mathcal{H}_1 \subseteq \downarrow\mathcal{H}_2$$

Proposition 2.4. *$(\downarrow\mathcal{H})A \subseteq \downarrow(\mathcal{H}A)$, for any $A \in C(G)$ and $\mathcal{H} \subseteq C(G)$.*

Proof. Let $Z \in (\downarrow\mathcal{H})A \Rightarrow Z = XA$, for some $X \in \downarrow\mathcal{H}$. So $\exists\ H \in \mathcal{H}$ such that $X \subseteq H \Rightarrow Z = XA \subseteq HA \Rightarrow Z \in \downarrow(\mathcal{H}A)$. □

Proposition 2.5. *Let $\mathcal{H}$ be a subsemigroup of $C(G)$ and $H \in \mathcal{H}$. Then $(\downarrow\mathcal{H})H \subseteq \downarrow\mathcal{H}$.*

Proof. Since $\mathcal{H}$ is a subsemigroup and $H \in \mathcal{H}$, so $\mathcal{H}H \subseteq \mathcal{H} \Rightarrow (\downarrow\mathcal{H})H \subseteq \downarrow(\mathcal{H}H) \subseteq \downarrow(\mathcal{H})$ [by Proposition 2.4]. □

Proposition 2.6. *Let $\mathcal{H} \subseteq C(G)$ and $S \in C(G)$. Then $\downarrow(\mathcal{H}S) = \downarrow((\downarrow\mathcal{H})S)$.*

Proof. $\downarrow((\downarrow\mathcal{H})S) \subseteq \downarrow\downarrow(\mathcal{H}S) = \downarrow(\mathcal{H}S)$. Conversely, $\mathcal{H} \subseteq \downarrow\mathcal{H} \Rightarrow \downarrow(\mathcal{H}S) \subseteq \downarrow((\downarrow\mathcal{H})S)$. □

Proposition 2.7. *Let $\mathcal{H}$ be a subsemigroup of $C(G)$ such that $H \in \mathcal{H} \Rightarrow H^{-1} \in \mathcal{H}$ where, $H^{-1} = \{a^{-1} : a \in H\}$. Then $\downarrow\mathcal{H}$ is also a subsemigroup such that $H \in \downarrow\mathcal{H} \Rightarrow H^{-1} \in \downarrow\mathcal{H}$.*

Proof. Let $X, Y \in \downarrow\mathcal{H}$. Then $\exists\ H_1, H_2 \in \mathcal{H}$ such that $X \subseteq H_1$ and $Y \subseteq H_2 \Rightarrow XY \subseteq H_1H_2 \in \mathcal{H}$ [since $\mathcal{H}$ is a subsemigroup] $\Rightarrow XY \in \downarrow\mathcal{H}$. So $\downarrow\mathcal{H}$ is a subsemigroup of $C(G)$. Now, let $X \in \downarrow\mathcal{H}$. So $\exists\ H \in \mathcal{H}$ such that $X \subseteq H \Rightarrow X^{-1} \subseteq H^{-1} \in \mathcal{H} \Rightarrow X^{-1} \in \downarrow\mathcal{H}$. This completes the proof. □

We can get same type of results if we replace $\downarrow\mathcal{H}$ by $\uparrow\mathcal{H}$ in the above propositions.

3. Partial Equivalence on $C(G)$

Definition 3.1. [**1**] Let S be a set and $T \subseteq S$. Let ρ be an equivalence relation on T. Then ρ will also be called a partial equivalence on S, T will be called the domain of ρ.

Note 3.2. [**1**] It is easily verified that a binary relation ρ on S is a partial equivalence on S iff ρ is symmetric and transitive.

Definition 3.3. [**1**] The domain of a partial equivalence ρ on S is the set $S_\rho = \{x \in S : s\rho x \text{ for some } s \in S\}$

Lemma 3.4. *Let ρ be a partial equivalence on S and S_ρ denotes the domain of ρ. Then $s \in S_\rho$ iff $s\rho s$ holds.*

Proof. If $s\rho s$ holds then by definition of S_ρ, $s \in S_\rho$. If $s \in S_\rho$ then $\exists\ x \in S$ such that $x\rho s$ holds. Since ρ is an equivalence relation on S_ρ, it is symmetric and transitive. Consequently, $(x, s) \in \rho \Rightarrow (s, x) \in \rho \Rightarrow (s, s) \in \rho$. □

Definition 3.5. [**1**] Let S be a semigroup and ρ be a partial equivalence on S. Then ρ is said to be right [left] compatible on S if, for each $s \in S, (a, b) \in \rho \Rightarrow$ either $(as, bs) \in \rho$ [$(sa, sb) \in \rho$] or $as, bs \notin S_\rho$ [$sa, sb \notin S_\rho$]

Definition 3.6. [**1**] A partial equivalence on S which is right [left] compatible on S is called a partial right [left] congruence on S.

Partial equivalence on $C(G)$:

Let $\mathcal{K}$ be a subsemigroup of $C(G)$ with the property : $K \in \mathcal{K} \Rightarrow K^{-1} \in \mathcal{K}$.[1]

Let, $\mathcal{H} = \downarrow\mathcal{K}$. Then $\mathcal{H}$ is also a subsemigroup of $C(G)$ with the same property [by Proposition 2.7].

We define, $\Pi_{\mathcal{K}} = \{(S, T) \in C(G) \times C(G) : ST^{-1} \in \mathcal{H}\}$. We claim that, $\Pi_{\mathcal{K}}$ is a partial right congruence on $C(G)$.

(i) Let, $(S, T) \in \Pi_{\mathcal{K}}$. Then $ST^{-1} \in \mathcal{H}$. So $TS^{-1} = (ST^{-1})^{-1} \in \mathcal{H}$ [by hypothesis] $\Rightarrow (T, S) \in \Pi_{\mathcal{K}}$. Thus $\Pi_{\mathcal{K}}$ is symmetric.

(ii) Let (S, T) and $(T, V) \in \Pi_{\mathcal{K}}$. Then $ST^{-1} \in \mathcal{H}$ and $TV^{-1} \in \mathcal{H}$. Now, $SV^{-1} \subseteq ST^{-1}TV^{-1} \in \mathcal{H}$ [since $\mathcal{H}$ is a subsemigroup] $\Rightarrow SV^{-1} \in \downarrow\mathcal{H} = \mathcal{H}$ $\Rightarrow (S, V) \in \Pi_{\mathcal{K}}$. Therefore $\Pi_{\mathcal{K}}$ is transitive.

[1] That such a subsemigroup of $C(G)$ exists follows from the following fact: Let M be a subgroup of G. Then $\mathcal{K} = \{\{m\} : m \in M\}$ is a subsemigroup (in fact a subgroup) of $C(G)$ with the desired property.

Thus $\Pi_{\mathcal{K}}$ is a partial equivalence on $C(G)$ [by Note 3.2].

Now let, $(A,B) \in \Pi_{\mathcal{K}}$ and $S \in C(G)$. It now suffices to prove that, if $AS \in D_{\mathcal{K}}$ [the domain of $\Pi_{\mathcal{K}}$] then $(AS, BS) \in \Pi_{\mathcal{K}}$.

$(A,B) \in \Pi_{\mathcal{K}} \Rightarrow AB^{-1} \in \mathcal{H}$. Again by Lemma 3.4, $AS \in D_{\mathcal{K}} \Rightarrow (AS, AS) \in \Pi_{\mathcal{K}} \Rightarrow ASS^{-1}A^{-1} \in \mathcal{H}$. Now, $ASS^{-1}B^{-1} \subseteq ASS^{-1}A^{-1}AB^{-1} \in \mathcal{H}AB^{-1} \subseteq \mathcal{H}$ [since $AB^{-1} \in \mathcal{H}$ and $\mathcal{H}$ is a semigroup] $\Rightarrow ASS^{-1}B^{-1} \in \downarrow \mathcal{H} = \mathcal{H}$ $\Rightarrow (AS, BS) \in \Pi_{\mathcal{K}}$.

This shows that $\Pi_{\mathcal{K}}$ is right compatible on $C(G)$.

Note 3.7. Clearly $\mathcal{H} \subseteq D_{\mathcal{K}}$. Actually, $\mathcal{H}$ is a $\Pi_{\mathcal{K}}$-class. In fact: $H_1, H_2 \in \mathcal{H}$ $\Rightarrow (H_1, H_2) \in \Pi_{\mathcal{K}}$. Again, $S \in D_{\mathcal{K}}$ and $(S, H) \in \Pi_{\mathcal{K}}$ for some $H \in \mathcal{H} \Rightarrow SH^{-1} \in \mathcal{H} \Rightarrow S \subseteq SH^{-1}H \in \mathcal{H}H \subseteq \mathcal{H}$ [since $H \in \mathcal{H}$ and $\mathcal{H}$ is a semigroup] $\Rightarrow S \in \downarrow\mathcal{H} = \mathcal{H}$.

Proposition 3.8. *Let $S \in C(G)$. Then $\mathcal{H}S \subseteq D_{\mathcal{K}}$ iff $S \in D_{\mathcal{K}}$.*

Proof. If $S \in D_{\mathcal{K}}$ then $SS^{-1} \in \mathcal{H}$ [by Lemma 3.4]. Now, $HSS^{-1}H^{-1} \in H\mathcal{H}H^{-1} \subseteq \mathcal{H}$ [since $H \in \mathcal{H}$ and $\mathcal{H}$ is a semigroup] $\Rightarrow HS \in D_{\mathcal{K}}$ for any $H \in \mathcal{H}$. Thus, $\mathcal{H}S \subseteq D_{\mathcal{K}}$.

Conversely let, $\mathcal{H}S \subseteq D_{\mathcal{K}}$. Then, $HSS^{-1}H^{-1} \in \mathcal{H}$, for any $H \in \mathcal{H}$. Now, $SS^{-1} \subseteq H^{-1}HSS^{-1}H^{-1}H \in H^{-1}\mathcal{H}H \subseteq \mathcal{H} \Rightarrow SS^{-1} \in \downarrow\mathcal{H} = \mathcal{H} \Rightarrow S \in D_{\mathcal{K}}$. □

Proposition 3.9. *Let $S \in D_{\mathcal{K}}$ and $H_1, H_2 \in \mathcal{H}$. Then $(H_1S, H_2) \in \Pi_{\mathcal{K}}$ iff $S \in \mathcal{H}$.*

Proof. $(H_1S, H_2) \in \Pi_{\mathcal{K}} \Rightarrow H_1SH_2^{-1} \in \mathcal{H} \Rightarrow S \subseteq H_1^{-1}H_1SH_2^{-1}H_2 \in H_1^{-1}\mathcal{H}H_2 \subseteq \mathcal{H} \Rightarrow S \in \downarrow\mathcal{H} = \mathcal{H}$.

Conversely let, $S \in \mathcal{H}$. Then $H_1SH_2^{-1} \in H_1\mathcal{H}H_2^{-1} \subseteq \mathcal{H}$. Thus $(H_1S, H_2) \in \Pi_{\mathcal{K}}$. □

Corollary 3.10. *If $S \notin \mathcal{H}$ then $\mathcal{H}S$ does not belong to the class $\mathcal{H}$.*

Proof. It follows from Proposition 3.9. □

Proposition 3.11. *$D_{\mathcal{K}}$ is a decreasing set i.e. $\downarrow D_{\mathcal{K}} = D_{\mathcal{K}}$.*

Proof. Let $K \in C(G)$ be such that $K \subseteq S$ for some $S \in D_{\mathcal{K}}$. $S \in D_{\mathcal{K}} \Rightarrow SS^{-1} \in \mathcal{H}$. Again $K \subseteq S \Rightarrow K^{-1} \subseteq S^{-1} \Rightarrow KK^{-1} \subseteq SS^{-1} \in \mathcal{H} \Rightarrow KK^{-1} \in \mathcal{H}$ $\Rightarrow K \in D_{\mathcal{K}}$. Thus $\downarrow D_{\mathcal{K}} = D_{\mathcal{K}}$. □

Note 3.12. From Proposition 3.8, if $S \in D_{\mathcal{K}}$ then $\mathcal{H}S \subseteq D_{\mathcal{K}} \Rightarrow \downarrow(\mathcal{H}S) \subseteq \downarrow D_{\mathcal{K}} = D_{\mathcal{K}}$ [by Proposition 3.11].

Proposition 3.13. *If $S \in D_{\mathcal{K}}$ then $\downarrow(\mathcal{H}S)$ is a $\Pi_{\mathcal{K}}$-class.*

Proof. From Note 3.12, $\downarrow(\mathcal{H}S) \subseteq D_{\mathcal{K}}$. We first prove that, any two member of $\downarrow(\mathcal{H}S)$ are comparable.

Let, $K_1, K_2 \in \downarrow(\mathcal{H}S)$. Then $\exists\, H_1, H_2 \in \mathcal{H}$ such that $K_1 \subseteq H_1S$ and $K_2 \subseteq H_2S \Rightarrow K_2^{-1} \subseteq S^{-1}H_2^{-1} \Rightarrow K_1K_2^{-1} \subseteq H_1SS^{-1}H_2^{-1}$.

Since $S \in D_{\mathcal{K}}$ so $SS^{-1} \in \mathcal{H} \Rightarrow H_1SS^{-1}H_2^{-1} \in \mathcal{H}$ [since $\mathcal{H}$ is a semigroup] $\Rightarrow K_1K_2^{-1} \in \mathcal{H} \Rightarrow (K_1, K_2) \in \Pi_{\mathcal{K}}$.

Now let, $S_1 \in D_{\mathcal{K}}$ and $(S_1, K) \in \Pi_{\mathcal{K}}$ for some $K \in \downarrow(\mathcal{H}S)$. So $\exists\ H \in \mathcal{H}$ such that $K \subseteq HS \Rightarrow \mathcal{H}K \subseteq \mathcal{H}HS \subseteq \mathcal{H}S$. Now, $(S_1, K) \in \Pi_{\mathcal{K}} \Rightarrow S_1K^{-1} \in \mathcal{H} \Rightarrow S_1 \subseteq S_1K^{-1}K \in \mathcal{H}K \subseteq \mathcal{H}S \Rightarrow S_1 \in \downarrow(\mathcal{H}S)$.
This shows that $\downarrow(\mathcal{H}S)$ is a $\Pi_{\mathcal{K}}$-class when $S \in D_{\mathcal{K}}$. We also note that, $\mathcal{H}S \subseteq \downarrow(\mathcal{H}S)$. □

Proposition 3.14. *$\mathcal{H}$ must contain $\{e\}$ as an element.*

Proof. Since $H \in \mathcal{H} \Rightarrow H^{-1} \in \mathcal{H}$ by hypothesis and $HH^{-1} \in \mathcal{H}$ [since $\mathcal{H}$ is a semigroup] we have $\{e\} \subseteq HH^{-1} \in \mathcal{H} \Rightarrow \{e\} \in \downarrow\mathcal{H} = \mathcal{H}$. □

Corollary 3.15. *For any $a \in G, \{a\} \in D_{\mathcal{K}}$.*

Proof. Since $\{a\}\{a^{-1}\} = \{e\} \in \mathcal{H}$ the corollary follows immediately. □

Corollary 3.16. *For each $S \in D_{\mathcal{K}}, S \in \mathcal{H}S$.*

Since $\{e\} \cdot S = S$ and $\{e\} \in \mathcal{H}$, the corollary follows.

Proposition 3.17. *If $S \notin \mathcal{H}$ then $\mathcal{H}$ and $\downarrow(\mathcal{H}S)$ [assume $S \in D_{\mathcal{K}}$] are two distinct $\Pi_{\mathcal{K}}$-classes.*

Proof. If not, $\exists\ H \in \mathcal{H}$ and $K \in \downarrow(\mathcal{H}S)$ such that $(H, K) \in \Pi_{\mathcal{K}}$. Since, $S \in \downarrow(\mathcal{H}S)$ so it follows that $(S, H) \in \Pi_{\mathcal{K}}$. Then by Note 3.7, $S \in \mathcal{H}$ □

Proposition 3.18. (i) $D_{\mathcal{K}} = \bigcup_{S \in D_{\mathcal{K}}} \downarrow(\mathcal{H}S)$,
(ii) $\downarrow(\mathcal{H}S_1) = \downarrow(\mathcal{H}S_2)$ *iff* $(S_1, S_2) \in \Pi_{\mathcal{K}}$.

Proof. Obvious. □

Proposition 3.19. *Let $M = \bigcup_{H \in \mathcal{H}} H$. Then M is a subgroup of G.*

Proof. Let, $a, b \in M$. Then $\exists\ H_1, H_2 \in \mathcal{H}$ such that $a \in H_1$ and $b \in H_2$. So $b^{-1} \in H_2^{-1}$. Therefore, $ab^{-1} \in H_1H_2^{-1} \in \mathcal{H}$ [since $\mathcal{H}$ is a semigroup] $\Rightarrow ab^{-1} \in M$. So, M is a subgroup of G. □

Proposition 3.20. *$S \notin \mathcal{H}$ iff $S \cap M = \Phi$.*

Proof. Let $S \notin \mathcal{H}$ and if possible let, $S \cap M \neq \Phi$. Then $\exists\ a \in G$ such that $a \in S \cap M \Rightarrow a \in H$ for some $H \in \mathcal{H}$. Now $H \in \mathcal{H} \Rightarrow H^{-1} \in \mathcal{H}$. So, $e \in H^{-1}S \in \mathcal{H}S \Rightarrow \{e\} \in \downarrow(\mathcal{H}S)$. Thus $\downarrow(\mathcal{H}S) = \mathcal{H}$ [since $\{e\} \in \mathcal{H}$] $\Rightarrow S \in \mathcal{H}$ a contradiction.
Converse part is obvious. □

Proposition 3.21. *Let E be an idempotent element of $C(G)$ such that $E \in D_{\mathcal{K}}$. Then $E \in \mathcal{H}$.*

Proof. E being idempotent we have $E^2 = E \Rightarrow E \subseteq E^2E^{-1} = EE^{-1} \in \mathcal{H}$ [since $E \in D_{\mathcal{K}}$] $\Rightarrow E \in \downarrow\mathcal{H} = \mathcal{H}$.
Thus, if E be an idempotent element of $C(G)$ such that $E \notin \mathcal{H}$ then $E \notin D_{\mathcal{K}}$. □

Proposition 3.22. *If $\mathcal{K}_1 \subseteq \mathcal{K}_2$ then, $D_{\mathcal{K}_1} \subseteq D_{\mathcal{K}_2}$.*

Proof. $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \mathcal{H}_1 \subseteq \mathcal{H}_2$. So $S \in D_{\mathcal{K}_1} \Rightarrow SS^{-1} \in \mathcal{H}_1 \Rightarrow SS^{-1} \in \mathcal{H}_2 \Rightarrow S \in D_{\mathcal{K}_2}$. □

Proposition 3.23. *If $\mathcal{H} \subseteq \mathcal{G}$ then $D_{\mathcal{K}} = \mathcal{G}$, where $\mathcal{H} = \downarrow\mathcal{K}$.*

Proof. From Corollary 3.15, we know $\mathcal{G} \subseteq D_{\mathcal{K}}$. Now let $S \in D_{\mathcal{K}} \Rightarrow SS^{-1} \in \mathcal{H} \subseteq \mathcal{G} \Rightarrow S$ must be a singleton set. Consequently, $S \in \mathcal{G}$. Therefore $D_{\mathcal{K}} = \mathcal{G}$. □

4. Topologization of $C(G)$

Here $C(G)$ is topologized by Vietoris topology [**3**] which will be compatible with its algebraic structure.

The Vietoris topology is defined as follows: for each subset S of G we define, $S^+ = \{A \in C(G) : A \subseteq S\}$ and $S^- = \{A \in C(G) : A \cap S \neq \Phi\}$. A subbase for the Vietoris topology on $C(G)$ is given by $\{W^+ : W \text{ is open in } G\} \bigcup \{W^- : W \text{ is open in } G\}$. It is easy to see that, $V_1^+ \cap \cdots \cap V_n^+ = (V_1 \cap \cdots \cap V_n)^+$ and hence a basic open set in Vietoris topology is of the form $V_1^- \cap \cdots \cap V_n^- \cap V_0^+$, where $V_0, V_1, \ldots, V_n$ are open in G; We may also choose each $V_i \subseteq V_0, i = 1, 2, \ldots, n$ in such a basic open set.

We note the following properties [**3**]:

(i) $A = B \Leftrightarrow A^+ = B^+$;

(ii) $(\bar{A})^+ = \overline{A^+}$;

(iii) $(A^+)^o = (A^o)^+$ [A^o being interior of A];

(iv) $\overline{A_1^- \cap \cdots \cap A_n^- \cap A_0^+} = (\overline{A_1})^- \cap \cdots \cap (\overline{A_n})^- \cap (\overline{A_0})^+$

Now our first attempt is to show that $C(G)$ with this Vietoris topology is a topological semigroup [**4**]. For this, we require the following lemma.

Lemma 4.1. *Let A, B be two compact subsets of G and $AB \subseteq V$, where V is open in G. Then $\exists$ an open neighbourhood (nbd. in short) L of 'e' in G such that $LABL \subseteq V$.*

Proof. Let $a \in A, b \in B$. Then $\exists$ open nbds. W'_{ab}, W''_{ab} of e such that $W'_{ab} \cdot ab \cdot W''_{ab} \subseteq V$. Let $W_{ab} = W'_{ab} \cap W''_{ab}$. Then $W_{ab} \cdot ab \cdot W_{ab} \subseteq V$. Let L_{ab} be an open nbd. of e such that $L_{ab}^2 \subseteq W_{ab}$. Now $\{L_{ab} \cdot ab \cdot L_{ab} : a \in A, b \in B\}$ is an open cover of AB and hence has a finite subcover such that $AB \subseteq \bigcup_{i=1}^n L_{a_ib_i} \cdot a_ib_i \cdot L_{a_ib_i}$. Put, $\bigcap_{i=1}^n L_{a_ib_i} = L$; Then L is an open nbd. of e. Now, $LABL \subseteq L(\bigcup_{i=1}^n L_{a_ib_i} \cdot a_ib_i \cdot L_{a_ib_i})L \subseteq \bigcup_{i=1}^n L_{a_ib_i}^2 \cdot a_ib_i \cdot L_{a_ib_i}^2 \subseteq \bigcup_{i=1}^n W_{a_ib_i} \cdot a_ib_i \cdot W_{a_ib_i} \subseteq V$. This completes the proof. □

Definition 4.2. [**4**] A semigroup $(S, \cdot)$ having a topological structure is said to be a topological semigroup or mob if the binary operation '$\cdot$ ' is continuous in this topology.

Theorem 4.3. *The semigroup $C(G)$ together with the Vietoris topology is a mob.*

Proof. We define a map $f : C(G) \times C(G) \longrightarrow C(G)$ by, $f(A, B) = AB$. To show that $C(G)$ is a mob we are only to prove that f is continuous. Let, $A, B \in$

$C(G)$ and $V_0, V_1, \ldots, V_n$ be open in G such that $V_i \subseteq V_0, i = 1, 2, \ldots, n$ and $f(A, B) = AB \in V_1^- \cap \ldots \cap V_n^- \cap V_0^+$. We have to find two open nbds. $\mathcal{L}, \mathcal{M}$ of A, B respectively in $C(G)$ such that $f(\mathcal{L} \times \mathcal{M}) \subseteq V_1^- \cap \ldots \cap V_n^- \cap V_0^+$ i.e. $\mathcal{L} \cdot \mathcal{M} \subseteq V_1^- \cap \ldots \cap V_n^- \cap V_0^+$. $AB \subseteq V_0 \Rightarrow \exists$ open nbd. $\widehat{V}_0$ of e in G such that $\widehat{V}_0 \cdot AB \cdot \widehat{V}_0 \subseteq V_0$ [by Lemma 4.1(i)]. Now, $AB \cap V_i \neq \Phi, i = 1, \ldots, n$. So $\exists\ a_i \in A, b_i \in B$ such that $a_i b_i \in V_i$ and this is true for all $i = 1, \ldots, n$. Since G is a topological group, $\exists$ an open nbd. $\widehat{V}_i$ of e in G such that $a_i \cdot \widehat{V}_i \widehat{V}_i \cdot b_i \subseteq V_i$ i.e. $a_i \cdot \widehat{V}_i^2 \cdot b_i \subseteq V_i$ for $i = 1, \ldots, n$ [by(ii)]

We take,

$$\mathcal{L} = (a_1 \cdot \widehat{V}_1)^- \cap \ldots \cap (a_n . \widehat{V}_n)^- \cap (\widehat{V}_0 \cdot A)^+$$

$$\text{and} \qquad \mathcal{M} = (\widehat{V}_1 \cdot b_1)^- \cap \ldots \cap (\widehat{V}_n \cdot b_n)^- \cap (B \cdot \widehat{V}_0)^+$$

Clearly, $\mathcal{L}, \mathcal{M}$ are open sets in $C(G)$. We claim that $A \in \mathcal{L}$, $B \in \mathcal{M}$. In fact: $A \subseteq \widehat{V}_0 \cdot A$ [since $\widehat{V}_0$ contains e] and $A \cap a_i \cdot \widehat{V}_i \neq \Phi$, $i = 1, \ldots, n$ [since $e \in \widehat{V}_i$, $i = 1, \ldots, n$]. Similarly, $B \in \mathcal{M}$.

Now let, $T_1 \in \mathcal{L}, T_2 \in \mathcal{M}$. So, $T_1 \subseteq \widehat{V}_0 \cdot A$ and $T_2 \subseteq B \cdot \widehat{V}_0$. So, $T_1 T_2 \subseteq \widehat{V}_0 \cdot AB \cdot \widehat{V}_0 \subseteq V_0$ [by (i)].

Now, $T_1 \cap a_i \cdot \widehat{V}_i \neq \Phi$, $i = 1, \ldots, n \Rightarrow \exists\ t_i \in T_1$ and $v_i \in \widehat{V}_i$ such that $t_i = a_i \cdot v_i$, $i = 1, \ldots, n$.

$T_2 \cap \widehat{V}_i \cdot b_i \neq \Phi$, $i = 1, \ldots, n \Rightarrow \exists\ t'_i \in T_2$ and $w_i \in \widehat{V}_i$ such that $t'_i = w_i \cdot b_i$, $i = 1, \ldots, n$. So, $t_i \cdot t'_i = a_i \cdot v_i \cdot w_i \cdot b_i \in a_i \cdot \widehat{V}_i^2 \cdot b_i \subseteq V_i$ for $i = 1, \ldots, n$ [by (ii)] $\Rightarrow T_1 T_2 \cap V_i \neq \Phi$, $i = 1, \ldots, n$.

Therefore $T_1 T_2 \in V_1^- \cap \ldots \cap V_n^- \cap V_0^+$. Thus, $\mathcal{L}\mathcal{M} \subseteq V_1^- \cap \ldots \cap V_n^- \cap V_0^+$. This completes the proof. □

Now we study the topological status of $C(G)$; for this we require the following definition and theorems:

Definition 4.4. [**2**] Let '$\leq$' be a preorder in a topological space X; the preorder is said to be closed iff its graph $\{(x, y) \in X \times X : x \leq y\}$ is closed in $X \times X$ (endowed with the product topology).

Theorem 4.5. [**2**] *The preorder '$\leq$' of X is closed iff for every two points $a, b \in X$ with $a \not\leq b, \exists$ nbds. V, W of a, b respectively in X such that $\uparrow V \cap \downarrow W = \Phi$ [where $\uparrow V$ and $\downarrow W$ are defined with the help of the preorder '$\leq$'].*

Theorem 4.6. *The inclusion order '$\subseteq$' which is obviously a partial order in $C(G)$ is closed as well.*

Proof. Let $A \not\subseteq B$ where $A, B \in C(G)$. Then $\exists\ x \in A$ such that $x \notin B$. Since G is T_2 and $\{x\}$, B are two disjoint compact subsets of G, $\exists$ two open sets U, V in G such that $x \in U$, $B \subseteq V$ and $U \cap V = \Phi \ \Rightarrow A \in U^-$ and $B \in V^+$. Now $U^- \cap V^+ = \Phi$ since $U \cap V = \Phi$ and $\uparrow(U^-) \cap \downarrow(V^+) = U^- \cap V^+ = \Phi$ [**2**]. Then by Theorem 4.5, it follows that '$\subseteq$' is a closed order. □

Theorem 4.7. *$C(G)$ is a T_2-space.*

Proof. Let $A, B \in C(G)$ with $A \neq B$. Then either $A \not\subseteq B$ or $B \not\subseteq A$. We assume $A \not\subseteq B$. Since '$\subseteq$' is a closed order by Theorem 4.6, $\exists$ two open sets U, V in G such that $A \in U^-$, $B \in V^+$ and $U^- \cap V^+ = \Phi$. This shows that $C(G)$ is a T_2-space. □

Theorem 4.8. *The family of all finite subsets of G is dense in $C(G)$.*

Proof. Let $\mathcal{F} = V_1^- \cap \ldots \cap V_n^- \cap V_0^+$ be any nonempty basic open set in $C(G)$ where $V_i \subseteq V_0, i = 1, \ldots, n$. Since $\mathcal{F}$ is nonempty, $\exists\ F \in \mathcal{F}$. So $\exists\ p_i \in F \cap V_i$, $i = 1, \ldots, n$ and $p_0 \in F \subseteq V_0$. We take $K = \{p_0, p_1, \ldots, p_n\}$. Clearly $K \in \mathcal{F}$ and K is in the aforesaid family. This completes the proof. □

We have seen in the first article that although $C(G)$ itself cannot be a group, it contains a unique maximal subgroup $\mathcal{G}$. This $\mathcal{G}$ inherits a subspace topology from $C(G)$. It is now a pertinent question to ask whether $\mathcal{G}$ is a topological group. The answer is in the affirmative. This follows from the following theorem.

Theorem 4.9. *The mapping*

$$\left.\begin{array}{rcl} F : C(G) & \longrightarrow & C(G) \\ A & \longmapsto & A^{-1} \end{array}\right\}$$

is continuous.

Proof. Let $A \in C(G)$ and $V_0, V_1, \ldots, V_n$ be open in G such that $V_i \subseteq V_0$, $i = 1, \ldots, n$ and $F(A) = A^{-1} \in V_1^- \cap \ldots \cap V_n^- \cap V_0^+ = \mathcal{H}$. Then, $A^{-1} \subseteq V_0 \Rightarrow A \subseteq V_0^{-1}$ and $A^{-1} \cap V_i \neq \Phi$, $i = 1, \ldots, n \Rightarrow A \cap V_i^{-1} \neq \Phi$, $i = 1, \ldots, n$. Since G is a topological group and inversion of an element in G is a homeomorphism, it follows that V_i^{-1}, $i = 0, 1, \ldots, n$ are open in G. Clearly, $A \in (V_1^{-1})^- \cap \ldots \cap (V_n^{-1})^- \cap (V_0^{-1})^+ = \mathcal{L}$.
Now, $\mathcal{L}$ is open in $C(G)$. Also, $F(\mathcal{L}) = \mathcal{L}^{-1} \subseteq \mathcal{H}$, where $\mathcal{L}^{-1} = \{B^{-1} : B \in \mathcal{L}\}$. This shows that F is continuous. □

Corollary 4.10. *$\mathcal{G}$ is a topological group.*

Proof. Since $C(G)$ is a topological semigroup [by Theorem 4.3] and restriction of a continuous function is again continuous, it follows that the group operation on $\mathcal{G}$ is continuous. Also by above Theorem 4.9, the mapping $\{x\} \longrightarrow \{x^{-1}\}$ is continuous. Consequently, $\mathcal{G}$ is a topological group. □

We have seen in first article that $\mathcal{G}$ and G are group isomorphic. Now it is very natural to ask whether they are topologically same or not. Answer to this question follows from the next theorem.

Theorem 4.11. *The map*

$$\left.\begin{array}{rcl} f : G & \longrightarrow & C(G) \\ x & \longmapsto & \{x\} \end{array}\right\}$$

is a homeomorphism.

Proof. Obviously, f is a bijective function between G and $\mathcal{G}$. Let U be open in G. Now $f(U) = \{\{x\} : x \in U\} = U^- \cap \mathcal{G}$. Thus $f(U)$ is open in $\mathcal{G}$ (with the relative topology). Consequently, f is an open map.
Again let, $V_1^- \cap \ldots \cap V_n^- \cap V_0^+ \cap \mathcal{G}$ be any open set in $\mathcal{G}$ containing $\{x\}$. Since $\mathcal{G}$ consists of singletons only, $\{y\} \in V_0^+ \cap \mathcal{G} \Longleftrightarrow \{y\} \in V_0^- \cap \mathcal{G}$ and thus, $V_1^- \cap \ldots \cap V_n^- \cap V_0^+ \cap \mathcal{G} = V_1^+ \cap \ldots \cap V_n^+ \cap V_0^+ \cap \mathcal{G} = (V_1 \cap \ldots \cap V_n \cap V_0)^+ \cap \mathcal{G}$. Obviously, $V_1 \cap \ldots \cap V_n \cap V_0$ is an open nbd. of x in G such that, $f(V_1 \cap \ldots \cap V_n \cap V_0) = (V_1 \cap \ldots \cap V_n \cap V_0)^+ \cap \mathcal{G} = V_1^- \cap \ldots \cap V_n^- \cap V_0^+ \cap \mathcal{G}$. Thus f is continuous. Hence f is a homeomorphism and $f(G) = \mathcal{G}$.

Also f is a group isomorphism. Thus, $\mathcal{G}$ and G are isomorphic and homeomorphic. □

Note 4.12. It is very easy to check that, if G_1 and G_2 be two isomorphic and homeomorphic topological groups then so are the corresponding semigroups $C(G_1)$ and $C(G_2)$. Our question is: "is the converse true?". The answer is in the affirmative.

Theorem 4.13. *If G_1 and G_2 are topological groups such that $C(G_1)$ and $C(G_2)$ are homeomorphic and isomorphic then G_1 and G_2 are also homeomorphic and isomorphic.*

Proof. Let $F : C(G_1) \longrightarrow C(G_2)$ be the homeomorphism and isomorphism. $\mathcal{G}_1$ is a unique maximal subgroup of $C(G_1)$. So $F(\mathcal{G}_1)$ is also a subgroup of $C(G_2)$. We claim that $F(\mathcal{G}_1)$ is also a maximal subgroup of $C(G_2)$. If not, $\exists$ a subgroup $\mathcal{H}$ of $C(G_2)$ such that $F(\mathcal{G}_1) \subset \mathcal{H} \Rightarrow \mathcal{G}_1 \subset F^{-1}(\mathcal{H})$. F being an isomorphism, $F^{-1}(\mathcal{H})$ is also a subgroup of $C(G_1)$. Since $\mathcal{G}_1$ is a maximal subgroup, $\mathcal{G}_1 = F^{-1}(\mathcal{H}) \Rightarrow F(\mathcal{G}_1) = \mathcal{H}$. Again, since $\mathcal{G}_2$ is the unique maximal subgroup of $C(G_2)$ it follows that $F(\mathcal{G}_1) = \mathcal{G}_2$. Thus $\mathcal{G}_1$ and $\mathcal{G}_2$ are homeomorphic [restriction of a homeomorphism being an homeomorphism] and isomorphic. The rest follows from Theorem 4.11. □

5. Uniform Structure on $C(G)$

Michael [**3**] proved that $C(G)$ is completely regular iff G is so. G being a topological group it is completely regular. Also we know that, a topological space is uniformizable iff it is completely regular. Thus we find that, $C(G)$ is uniformizable. Here we construct an uniformity on $C(G)$ giving its topology. Let,

$$\begin{aligned} V_s = \big\{(M,N) \in C(G) \times C(G) :\ & \text{for each } m \in M, \exists\ n \in N \text{ such that } n^{-1}m \in V \\ & \&\ \text{for each } n \in N, \exists\ m \in M \text{ such that } m^{-1}n \in V\big\} \end{aligned}$$

where, $V \in \eta_e, \eta_e$ being the nbd. system of e in G.

We claim that $\mathcal{B} = \{V_s : V \in \eta_e\}$ forms a base for some uniformity on $C(G)$.

(i) $\{(M,M) : M \in C(G)\} \subseteq V_s$ for each $V \in \eta_e$. In fact: for each $m \in M$ $\exists\ m \in M$ such that $m^{-1}m = e \in V$.
(ii) Obviously, $V_s^{-1} = V_s, \forall\ V \in \eta_e$ where, $V_s^{-1} = \{(M,N) : (N,M) \in V_s\}$.

(iii) Let $V_s \in \mathcal{B}$. Then $V \in \eta_e$. So $\exists\ W \in \eta_e$ such that $W^2 \subseteq V$. We claim that $W_s \circ W_s \subseteq V_s$ where, $W_s \circ W_s = \{(M, N) \in C(G) \times C(G) : (M, P), (P, N) \in W_s$, for some $P \in C(G)\}$. In fact: $(M, N) \in W_s \circ W_s \Rightarrow \exists\ P \in C(G)$ such that $(M, P), (P, N) \in W_s \Rightarrow$ for each $m \in M, \exists\ p \in P$ such that $p^{-1}m \in W$ and for $p \in P \exists\ n \in N$ such that $n^{-1}p \in W$. So, $n^{-1}m = n^{-1}p \cdot p^{-1}m \in W^2 \subseteq V$. Similarly, for each $n \in N \exists\ q \in P$ such that $q^{-1}n \in W$ and for $q \in P, \exists\ m \in M$ such that $m^{-1}q \in W \Rightarrow m^{-1}n = m^{-1}q \cdot q^{-1}n \in W^2 \subseteq V$. This shows that $(M, N) \in V_s$. i.e. $W_s \circ W_s \subseteq V_s$.

(iv) Let, $(V_1)_s, (V_2)_s \in \mathcal{B}$. Then clearly, $(V_1 \cap V_2)_s \subseteq (V_1)_s \cap (V_2)_s$.

Thus our assertion is proved.

Let the topology that this uniformity generates be $\tau(\mathcal{B})$. Obviously $\{V_s[M] : V \in \eta_e\}$ forms the nbd. system of $M \in C(G)$ in $\tau(\mathcal{B})$ where, $V_s[M] = \{N \in C(G) : (M, N) \in V_s\}$. We now show that this uniformity actually gives the Vietoris topology on $C(G)$.

Theorem 5.1. *$\tau(\mathcal{B})$ is the Vietoris topology on $C(G)$.*

Proof. Let $\mathcal{H}$ be open in $\tau(\mathcal{B})$ and $M \in \mathcal{H}$. Then $\exists$ an open set $V \in \eta_e$ such that $M \in V_s[M] \subseteq \mathcal{H}$. Let $\widehat{V}$ be a symmetric open nbd. of 'e' in G such that $\widehat{V}^2 \subseteq V$. Since M is compact and $\{m\widehat{V} : m \in M\}$ is an open cover of M, it has a finite subcover $\{m_i\widehat{V} : i = 1, \ldots, t\}$ (say). Also MV is open in G. Let, $\mathcal{L} = (m_1\widehat{V})^- \cap \ldots \cap (m_t\widehat{V})^- \cap (MV)^+$. Then $\mathcal{L}$ is an open set in $C(G)$ in the Vietoris topology and clearly $M \in \mathcal{L}$. We claim that, $\mathcal{L} \subseteq V_s[M]$.
Let, $N \in \mathcal{L}$. Then $N \cap m_i\widehat{V} \neq \Phi, i = 1, \ldots, t$. Let $m \in M$. Then $m \in m_i\widehat{V}$, for some $i \Rightarrow m = m_iv$, for some $v \in \widehat{V}$. Now, $m_i\widehat{V} \cap N \neq \Phi \Rightarrow \exists\ n \in N$ such that $n = m_iv_1$ for some $v_1 \in \widehat{V}$. Therefore, $n^{-1}m = v_1^{-1}m_i^{-1}m_iv = v_1^{-1}v \in \widehat{V}^2 \subseteq V$ [since $\widehat{V}$ is symmetric]. Again, $N \subseteq MV \Rightarrow$ for each $n \in N, \exists\ m \in M$ and $v \in V$ such that $n = mv \Rightarrow m^{-1}n = v \in V$. Thus, $(M, N) \in V_s$. Consequently, $N \in V_s[M]$. Hence $M \in \mathcal{L} \subseteq V_s[M] \subseteq \mathcal{H}$.

Conversely, let $M \in V_1^- \cap \ldots \cap V_t^- \cap V_0^+ = \mathcal{L}$ where, V_i, $i = 0, 1, \ldots, t$ are open in G with $V_i \subseteq V_0$, $i = 1, \ldots, t$. Now, $M \subseteq V_0 \Rightarrow$ for $m \in M, \exists$ an open set $V_0^m \in \eta_e$ such that $m \in mV_0^m \subseteq V_0$. Let W_0^m be an open symmetric nbd. of 'e' in G such that $(W_0^m)^2 \subseteq V_0^m$.
Now $\{mW_0^m : m \in M\}$ is an open cover of M and M is compact. So it has a finite subcover $\{m_jW_0^{m_j} : j = 1, \ldots, p\}$. We put, $W_0 = \bigcap_{j=1}^{p} W_0^{m_j}$. Then W_0 is an open symmetric nbd. of 'e' in G.
Again, $M \cap V_i \neq \Phi$, $i = 1, \ldots, t \Rightarrow \exists\ m_i' \in M \cap V_i$, $i = 1, \ldots, t$. Let W_i be an open symmetric nbd. of 'e' in G such that $m_i' \in m_i'W_i \subseteq V_i$, $i = 1, \ldots, t$. We put $\widehat{W} = \bigcap_{i=1}^{t} W_i$. Then $\widehat{W}$ is an open symmetric nbd. of 'e' in G. Let, $W = W_0 \cap \widehat{W}$. Then W is an open symmetric nbd. of 'e' in G. We claim that, $M \in W_s[M] \subseteq V_1^- \cap \ldots \cap V_t^- \cap V_0^+ = \mathcal{L}$.
Let, $N \in W_s[M]$. Then $(M, N) \in W_s$. Let, $n \in N$. Then $\exists\ m \in M$ such that $m^{-1}n \in W \Rightarrow n \in mW$. Now, $m \in m_jW_0^{m_j}$ for some $j \in \{1, \ldots, p\}$. So $n \in mW \subseteq m_jW_0^{m_j}W_0^{m_j} = m_j(W_0^{m_j})^2 \subseteq m_jV_0^{m_j} \subseteq V_0$ [since $W \subseteq W_0^{m_j}, \forall\ j$]. Thus, $N \subseteq V_0$.

Now, for $m'_i \in M, \exists\ n_i \in N$ such that $n_i^{-1} m'_i \in W \Rightarrow n_i^{-1} \in W(m'_i)^{-1} \Rightarrow n_i \in m'_i W^{-1} = m'_i W$ [since W is symmetric]
$\Rightarrow n_i \in m'_i W \subseteq m'_i W_i \subseteq V_i, i = 1, \ldots, t$ [since $W \subseteq W_i, \forall\ i$]
$\Rightarrow N \cap V_i \neq \Phi$, $i = 1, \ldots, t$. Consequently, $N \in V_1^- \cap \ldots \cap V_t^- \cap V_0^+ = \mathcal{L}$. Hence, $M \in W_s[M] \subseteq \mathcal{L}$.
This shows that, $\tau(\mathcal{B})$ is actually the Vietoris topology on $C(G)$. $\square$

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