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REGULAR ADDITIVELY INVERSE SEMIRINGS

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ABSTRACT. In this paper we show that in a regular additively inverse semiring $(S, +, \cdot)$ with 1 satisfying the conditions
(A) $a(a + a') = a + a'$;
(B) $a(b + b') = (b + b')a$
and (C) $a + a(b + b') = a$, for all $a, b \in S$, the sum of two principal left ideals is again a principal left ideal. Also, we decompose S as a direct sum of two mutually inverse ideals.

1. INTRODUCTION

A semiring is a nonempty set S on which operations of addition, $+$, and multiplication, $\cdot$, have been defined such that the following conditions are satisfied:

(1) $(S, +)$ is a semigroup.
(2) $(S, \cdot)$ is a semigroup.
(3) Multiplication distributes over addition from either side.

A semiring $(S, +, \cdot)$ is called an additive inverse semiring if $(S, +)$ is an inverse semigroup, that is, for each $a \in S$ there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. In 1974, Karvellas [**3**] studied additive inverse semiring and he proved the following:

(Karvellas (1974), Theorem 3(ii) and Theorem 7) Take any additive inverse semiring $(S, +, \cdot)$.

(i) For all $x, y \in S, (x \cdot y)' = x' \cdot y = x \cdot y'$ and $x' \cdot y' = x \cdot y$
(ii) If $a \in aS \cap Sa$ for all $a \in S$ then S is additively commutative.

We say that an additive inverse semiring S satisfies conditions (A), (B) and (C) if for all $a, b \in S$,

(A) $a(a + a') = a + a'$;
(B) $a(b + b') = (b + b')a$;
(C) $a + a(b + b') = a$.

Clearly, rings, distributive lattices and direct products of a distributive lattice and ring are natural examples of these types of semirings. Semirings satisfying conditions (A), (B) and (C) were first introduced and studied by Bandelt & Petrich [**1**].

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We consider the set $S = \{0, a, b\}$. On S we define addition and multiplication by the following Cayley tables:

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

It is easy see that $(S, +, \cdot)$ is an additive inverse semiring satisfying conditions (A), (B) and (C).

In the remaining part of this paper we assume that S denotes an additive inverse semiring with 1 satisfying conditions (A), (B) and (C). Also we assume that $E^+(S) = \{a \in S : a + a = a\}$ and $E^\bullet(S) = \{e \in S : e \cdot e = e\}$. Note that $E^+(S)$ is an ideal of S. For notations and terminologies not given in this paper, the reader is referred to the monograph of Golan [**2**] and Neumann [**4**].

2. Mutually inverse ideals in S

In this section we define the notion of mutually inverse ideals in S. Then we establish the actual form of two ideals such that these two ideals become mutually inverses.

Lemma 2.1. *If $e \in E^\bullet(S)$ then $1 + e' \in E^\bullet(S)$.*

Proof. Now, $(1+e')^2 = (1+e')+e'(1+e') = 1+e'+e'+e'e' = 1+e'+e'+e = 1+e'$. Hence $(1 + e') \in E^\bullet(S)$. $\square$

Definition 2.2. For every right ideal A of S we define

$$A^l = \{y \in S : \text{ for every } z \in A, yz \in E^+(S)\}$$

and for every left ideal B of S we define

$$B^r = \{z \in S : \text{ for every } y \in B, yz \in E^+(S)\}.$$

Notation 2.3. The sets of all left ideals and right ideals of S are denoted by L_S and R_S respectively.

From the Definition 2.2, we have the following result.

Corollary 2.4. *A^l is a left ideal and B^r is a right ideal. The transformation $A \longrightarrow A^l$ maps R_S on a part of L_S and the transformation $B \longrightarrow B^r$ maps L_S on a part of R_S.*

Lemma 2.5. *Let A, B be two left ideals. Then*

(i) *$A \subseteq B$ implies $B^r \subseteq A^r$,*
(ii) *$A \subseteq A^{rl} (\equiv (A^r)^l)$,*
(iii) *$A^r = A^{rlr}$.*

(The left-right symmetric results will be denoted by (i)′, (ii)′, (iii)′)

Proof. (i) If $y \in B^r$ then for every $z \in B$ we have $zy \in E^+(S)$. Then in particular for every $z \in A$ we have $zy \in E^+(S)$ and hence $y \in A^r$. Thus $B^r \subseteq A^r$.

(ii) Let $u \in A$ and consider $y \in A^r$. Now $z \in A$ implies $zy \in E^+(S)$. Hence in particular $uy \in E^+(S)$ and $u \in A^{rl}$. Thus $A \subseteq A^{rl}$.

(iii) Since $A \subseteq A^{rl}$, $A^{rlr} \subseteq A^r$, by (i). But by (ii)$'$ with A replaced by A^r, $A^r \subseteq A^{rlr}$. Hence $A^r = A^{rlr}$. $\square$

Definition 2.6. An ideal I of a semiring S is called full if $E^+(S) \subseteq I$. The principal left ideal of S of the form $E^+(S) + (a)_l$ is called a full principal left ideal of S.

Lemma 2.7. *$E^+(S)$ is a full principal left ideal.*

Proof. Let $e \in E^+(S)$. Since S is an additive inverse semiring so we have $E^+(S)+(e)_l \subseteq E^+(S)$. Let $f \in E^+(S)$. Now by condition (C), $f = f+f(e+e') = f + fe \in E^+(S) + (e)_l$. Hence $E^+(S) = E^+(S) + (e)_l$ and consequently $E^+(S)$ is a full principal left ideal. $\square$

Remark 2.8. We note that for any $g \in E^+(S)$, we have $g = g+ga = g(1+a) \in (1+a)_l$. Thus any principal left (right) ideal of the form $(1+a)_l$ (resp. $(1+a)_r$) is a full ideal of S. Hence in particular for any $e \in E^\bullet(S)$, $(1+e)_l$ is also a full left ideal of S. In this connection we have the following result.

Theorem 2.9. *Let $e \in E^\bullet(S)$. Then the principal left ideal $(e)_l$ is full if and only if $(e)_l = (1+f')_l$ for some $f \in E^\bullet(S)$.*

Proof. First suppose that $(e)_l$ is a full ideal. Since $e \in E^\bullet(S)$ we have $(1+e') \in E^\bullet(S)$. Let $f = 1+e'$. Now, $1+f' = 1+1'+e \in (e)_l$. This leads to $(1+f')_l \subseteq (e)_l$. Also, $e = e+e'+e = e+e'+e^2 = e(1+1'+e) = e(1+f') \in (1+f')_l$. Thus $(e)_l \subseteq (1+f')_l$. Consequently, $(e)_l = (1+f')_l$. $\square$

Converse part follows from Remark 2.8.

Lemma 2.10. *Let $a, b \in S$ be such that $a+b' \in E^+(S)$ and $a+a' = b+b'$. Then $a = b$.*

Proof. Since $a+b' \in E^+(S)$ so we have

$$a+b' = (a+b') + (a+b')' = a+b'+b+a' = a+a'+b+b' = b+b'.$$

This leads to,

$$a+b'+b = b+b'+b, \qquad \text{i.e.,} \qquad a+a'+a = b.$$

Hence $a = b$. $\square$

We now give the following definition.

Definition 2.11. Two left ideals A and B of a semiring S are said to be mutually inverses if $A+B = S$ and $A \cap B = E^+(S)$. A left ideal B of S is said to be an inverse of a left ideal A of S if A and B are mutually inverses.

Lemma 2.12. *In S the principal left ideals $(1+e')_l$ and $(1+1'+e)_l$ where $e \in E^\bullet(S)$ are mutually inverses.*

Proof. First, $(1+e')_l+(1+1'+e)_l$ contains $1+e'+1+1'+e = 1+e+e'+1+1' = 1$, whence $(1+e')_l + (1+1'+e)_l = S$. Now, if $x \in (1+e')_l \cap (1+1'+e)_l$ then $x = x(1+e') = x(1+1'+e)$. Now, $x = x(1+1'+e) = x + x(1'+e) = x + x'$ (since $x = x(1+e')) \in E^+(S)$. Thus, $(1+e')_l \cap (1+1'+e)_l = E^+(S)$ and hence $(1+e')_l$ is inverse to $(1+1'+e)_l$. □

We now prove the following theorem.

Theorem 2.13. *Two left ideals A and B of S are mutually inverses if and only if there exists $e \in E^\bullet(S)$ such that $A = (1+1'+e)_l$ and $B = (1+e')_l$.*

Proof. The reverse implication follows from Lemma 2.12. Let A and B be two mutually inverse left ideals. Then there exist elements x, y with $x + y = 1$, $x \in A, y \in B$. Now, $x + y = 1$ implies $x = x^2 + xy$. This leads to $(x^2)' + x = (x^2)' + x^2 + xy \in A \cap B = E^+(S)$. Also, $x^2 + (x^2)' = x + x'$ (by condition (A)). Hence by Lemma 2.10, we have $x^2 = x$ and hence $x \in E^\bullet(S)$. Now, $1 + 1' + x \in A$. This gives $(1+1'+x)_l \subseteq A$. Let $z \in A$. Now, $x + y = 1$ implies that $z = zx + zy$. This leads to, $zx' + z = zx' + zx + zy \in A \cap B = E^+(S)$. Hence $(zx'+z)' = (zx'+z) + (zx'+z)' = zx' + zx + z + z' = z + z'$ (by condition (C)) i.e., $z + z' = z' + zx$. This gives

$$z = z + z' + z = z + z' + zx = z(1+1'+x) \in (1+1'+x)_l.$$

Thus $A \subseteq (1+1'+x)_l$. Consequently, $A = (1+1'+x)_l$. Similarly, we can show that $B = (1+x')_l$. Thus $e = x$ is effective in this theorem. □

3. Principal ideals in S

In this section we study the principal ideals in S. We generalize some results of regular ring to regular semiring. Finally, we prove that the set of all full principal left ideals of S is a complemented modular lattice. This is the main theorem of this section.

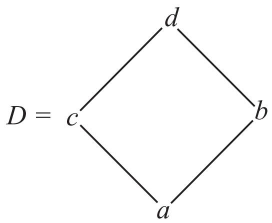
Definition 3.1. A semiring S is called a regular semiring if for each $a \in S$ there exists an element $x \in S$ such that $a = axa$.

A regular semiring S contains element e such that $e \cdot e = e$.

Note that every regular ring and every distributive lattice is regular semiring. So the direct product of a regular ring and a distributive lattice is also regular semiring.

We now consider the following example.

Example 3.2. Let D denote the distributive lattice D given by



Let $R = \mathbb{R} \times \mathbb{R}$ and $I = \mathbb{R} \times \{0\}$, where $\mathbb{R}$ is the ring of all real numbers. Then $S = (I \times \{a, c\}) \cup (R \times \{b, d\})$ is a regular semiring which is not the whole direct product of a regular ring and a distributive lattice.

Theorem 3.3. *The following statements are equivalent in S:*

(1) *Every principal left ideal of the form $E^+(S) + (a)_l$ has an inverse.*
(2) *For every $a \in S$ there exists $e \in E^\bullet(S)$ such that $E^+(S) + (a)_l = (1 + e')_l$.*
(3) *S is regular.*
(4) *For every $a \in S$ there exists $e \in E^\bullet(S)$ such that $E^+(S) + (a)_r = (1 + e')_r$.*
(5) *Every principal right ideal of the form $E^+(S) + (a)_r$ has an inverse.*

Proof. (1) $\Longrightarrow$ (2): This follows from Theorem 2.13.

(2) $\Longrightarrow$ (1): This follows from Theorem 2.13.

(2) $\Longrightarrow$ (3): Now, $a \in E^+(S) + (a)_l = (1 + e')_l$. This implies that $a = a(1 + e')$. Again $(1 + e') \in (1 + e')_l = E^+(S) + (a)_l$. This leads to, $1 + e' = g + za$ for some $g \in E^+(S)$ and $z \in S$. Then $a = ag + aza$. This implies that

$$\begin{aligned} a &= a + a' + a = a + a' + ag + aza = a + a' + aza \quad \text{(by condition (C))} \\ &= a(a + a' + z)a = axa \quad \text{where} \quad x = a + a' + z \in S. \end{aligned}$$

Thus, for each $a \in S$ there exists an element $x \in S$ such that $a = axa$. Hence S is regular.

(3) $\Longrightarrow$ (2): Let $a \in S$. Then $a = axa$ for some $x \in S$. Let $c = xa$. Then $c \in E^\bullet(S)$. Let $e = 1 + c'$. Now

$$a = a + a' + a = a + a' + axa = a(1 + 1' + c) = a(1 + e') \in (1 + e')_l.$$

Thus, $E^+(S) + (a)_l \subseteq (1 + e')_l$. Again, let $y \in (1 + e')_l$. Then $y = b(1 + e')$ for some $b \in S$. Then

$$y = b(1 + e') = b(1 + 1' + xa) = b + b' + bxa \in E^+(S) + (a)_l.$$

Thus,

$$(1 + e')_l \subseteq E^+(S) + (a)_l \quad \text{and hence} \quad E^+(S) + (a)_l = (1 + e')_l.$$

The equivalence of (3), (4), (5) is right-left symmetric to that of (1), (2), (3). Hence the proof is completed. □

Lemma 3.4. *A semiring S is regular if and only if for any $a \in S$ there exists an element $e \in E^\bullet(S)$ such that $Sa = Se$.*

Proof. The proof is similar to ring theory and we omit the proof.

In the remaining part of the section we assume that S is regular and an additive inverse semiring with 1 satisfying conditions (A), (B) and (C). □

Lemma 3.5. (i) *If $A = (1 + e')_l$ ($e \in E^\bullet(S)$) is a full principal left ideal then $A = C^l$ wherel $C = (1 + 1' + e)_r$.*
(ii) *If A is a full principal left ideal then $A = A^{rl}$.*
(iii) *If A is a full principal left ideal, A^r is a full principal right ideal.*

Proof.

(i)
$$\begin{aligned} A = (1+e')_l &= \{x : x = x(1+e')\} \\ &= \{x : x + x(1+e')' = x(1+e') + x(1+e')'\} \\ &\subseteq \{x : x(1+1'+e) \in E^+(S)\} \\ &= \{x : \text{ for all } u \in S, x(1+1'+e)u \in E^+(S)\} \\ &= \{x : \text{ for all } y \in (1+1'+e)_r, xy \in E^+(S)\} \\ &= C^l \qquad \text{where } C = (1+1'+e)_r. \end{aligned}$$

Again, let $x \in c^l$. Then for all $y \in C = (1+1'+e)_r$, we have $xy \in E^+(S)$, i.e., for all $u \in S$, $x(1+1'+e)u \in E^+(S)$. This implies $x(1+1'+e) \in E^+(S)$. Hence,

$$x(1+1'+e) = x(1+1'+e) + x(1+1'+e)' = x + x'.$$

This leads to

$$x' = x' + x + x' = x' + xe = x'(1+e'),$$

i.e.,

$$x = x(1+e') \in (1+e')_l = A.$$

Thus, $C^l \subseteq A$ and hence $A = C^l$.

(ii) Since S is regular and A is a full principal left ideal so $A = (1+e')_l = (f)_l$ where $f = 1+e'$. Then $A = (f)_l = C^l$ where $C = (1+f')_r$ [by (i)]. Then $A^{rl} = C^{lrl} = C^l = A$.

(iii) $A = C^l$ where C is a full principal right ideal, whence $A^r = C^{lr} = C$ [by (ii)$'$]. Hence A^r is a full principal right ideal. □

Theorem 3.6. *In S, the sum of two principal left ideals of S is again a principal left ideal.*

Proof. Let Sa and Sb be two principal ideals in a regular semiring S. Then there exists an idempotent $e \in S$ such that $Sa = Se$. Now,

$$\begin{aligned} Sa + Sb &= Se + Sb \\ &= \{re + tb : r, t \in S\} \\ &= \{re + tb + tbe + tbe' : r, t \in S\} \qquad \text{(by condition (C))} \\ &= \{(r + tb)e + tb(1+e') : r, t \in S\} \\ &\subseteq Se + Sb(1+e'). \end{aligned}$$

Let $re + tb(1+e') \in Se + Sb(1+e')$. Then

$$\begin{aligned} re + tb(1+e') &= re + tb + tbe' \\ &= re + tb + tb(e+e') + tbe' \\ &= (r + tb' + tb)e + tb(1+e') \\ &= (r_1 + tb)e + tb(1+e') \qquad \text{where } r_1 = r + tb' \in S. \end{aligned}$$

Hence, $Sa + Sb = Se + Sb(1+e')$. Let $c = b(1+e')$. Then $ce \in E^+(S)$. Now by Lemma 3.4., $Sc = Sf$ for some idempotent $f \in S$. Also, $f \in Sf = Sc$ This implies

$f = yc = yb(1+e')$. This leads to, $fe = yb(1+e')e = yb(e+e') \in E^+(S)$. Hence, $Sa + Sb = Se + Sf$, where $e^2 = e, f^2 = f$ and $fe \in E^+(S)$. Let $g = (1+e')f$. Then $eg = e(1+e')f = (e+e')f$ and $ge = (1+e')fe$. Thus, $eg,\ ge \in E^+(S)$. Also,

$$g^2 = g(1+e')f = gf + ge'f = gf = g.$$

Now,

$$\begin{aligned} Sg = S(1+e')f &\subseteq Sf = Sff \\ &= Scf = Sb(1+e')f \\ &= Sbg \subseteq Sg. \end{aligned}$$

Hence,

$$Sf = Sg.$$

Then $Sa + Sb = Se + Sg$, where $e^2 = e, g^2 = g$ and $eg,\ ge \in E^+(S)$.

We show that $Se + Sg = S(e+g)$. Clearly, $S(e+g) \subseteq Se + Sg$.

Now,

$$e = e^2 + eg = e(e+g) \in S(e+g).$$

Then $Se \subseteq S(e+g)$. Similarly, $Sf \subseteq S(e+g)$. Hence, $Se + Sg \subseteq S(e+g)$. Consequently,

$$Sa + Sb = S(e+g).$$

Thus, the proof is completed. □

Corollary 3.7. *The sum of two full principal left ideals of S is again a full principal left ideal.*

Lemma 3.8. *If C, D are left ideals of S then $(C+D)^r = C^r \cap D^r$.*

Proof.

$$\begin{aligned} C^r \cap D^r &= \{y : \text{ for all } z \in C,\ zy \in E^+(S) \text{ and for all } z \in D,\ zy \in E^+(S)\} \\ &= \{y : \text{ for all } t \in C+D, ty \in E^+(S)\} \\ &\subseteq (C+D)^r. \end{aligned}$$

Let $y \in (C+D)^r$. Then for all $t \in C+D$, $ty \in E^+(S)$. Let $c \in C$ and $d \in D$. Now by condition (C), $c = c + c(d+d')$. Hence

$$cy = \Big((c + c(d+d')\Big)y = cy + c(d+d')y \in E^+(S).$$

Thus $y \in C^r$. Similarly, $y \in D^r$. Hence $y \in C^r \cap D^r$. Consequently,

$$C^r \cap D^r = (C+D)^r.$$

□

Lemma 3.9. *Let A, B be two full principal left ideals of S. Then $A \cap B$ is again a full principal left ideal.*

Proof.

$$\begin{aligned} A \cap B &= A^{rl} \cap B^{rl} \\ &= (A^r + B^r)^l. \end{aligned}$$

But A^r, B^r are full principal right ideals by Lemma 3.5(iii). Hence again by Corollary 3.7, $(A^r + B^r)$ is a full principal right ideal. Thus, $A \cap B$ is a full principal left ideal by 3.5 (iii)$'$. □

Theorem 3.10. *The set $\overline{L}_S$ of all full principal left ideals of S is a complemented modular lattice, partially ordered by set inclusion relation, the meet being $\cap$ and the join is the sum of two ideals, its least element is $E^+(S)$ and its greatest element is S.*

Proof. The fact that $\overline{L}_S$ is a lattice follows from Corollary 3.7 and Lemma 3.9 The regularity of S and Theorem 2.13, yields that $\overline{L}_S$ is complemented. The modularity is established as follows. Let A, B, C be full right ideals with $A \subseteq C$. Clearly, $A + (B \cap C) \subseteq (A + B) \cap C$. Let $x \in (A + B) \cap C$. Then $x = a + b$ for some $a \in A$, $b \in B$ and $x \in C$. Then

$$a' + x = a' + a + b \in B \cap C.$$

Now, $a' + x = a' + a + b$ implies that $a + a' + x = a + b = x$. Hence

$$x = a + a' + x \in A + (B \cap C).$$

Thus, $(A + B) \cap C \subseteq A + (B \cap C)$. Therefore,

$$(A + B) \cap C = A + (B \cap C)$$

and the proof is completed. □

4. Decomposition of S

In this section we decompose S as direct sum of two mutually inverse ideals.

We now define the center of a semiring.

Definition 4.1. Let S be a semiring. The center $Z(S)$ of S is the set $Z(S) = \{a \in S : ax = xa \text{ for every } x \in S\}$.

Definition 4.2. A subsemiring A is called a full subsemiring if $E^+(S) \subseteq A$.

Lemma 4.3. *The center $Z(S)$ of S is a multiplicative commutative, regular and additive inverse full subsemiring of S with 1.*

Proof. The proof is similar to ring theory and we omit the proof. □

Lemma 4.4. *For every $a \in Z(S)$ the principal left ideal $(a)_l$ and the principal right ideal $(a)_r$ are the same. They will be denoted by $(a)_*$. Moreover, if $a \in Z(S)$, $(a)^l_* = (a)^r_*$ and the common value will be denoted by $(a)^*_*$.*

Proof. Follows from [**4**, Lemma 2.5]. □

Lemma 4.5. (i) *If A is both left and right ideal of the form $(e)_l$ (or $(e)_r$) with $e \in E^\bullet(S)$, then e is unique, $e \in Z(S)$ and $A = (e)_*$.*

(ii) *A principal left ideal A is a right ideal if and only if there exists a unique $e \in E^{\bullet}(S)$ such that $e \in Z(S)$ and $A = (e)_*$.*

Proof. (i) Let $A = (e)_l$ be a right ideal, $e \in E^{\bullet}(S)$. For every $y \in S$, $ey \in (e)_l$ whence $eye = ey$, i.e., $ey(1 + e') \in E^+(S)$. Now for every $x \in S$, there is some $y \in S$ with

$$(1 + e')xey(1 + e')xe = (1 + e')xe.$$

Since $ey(1 + e') \in E^+(S)$, it follows that

$$(1 + e')xe = xe + e'xe \in E^+(S).$$

Also

$$\begin{aligned} exe + exe' &= (ex + ex')e = e(ex + ex') \quad \text{(by condition (C))} \\ &= ex + ex'. \end{aligned}$$

Hence by Lemma 2.10, we have $ex = exe$ and hence $ex = xe$. This shows that $e \in Z(S)$ and $a = (e)_*$. The uniqueness of e follows from [**4**, Lemma 2.6(i)].

(ii) The reverse implication is trivial. suppose A is a principal left ideal and also a right ideal. Then there exists an element $e \in E^{\bullet}(S)$ such that $A = (e)_l$, since S is regular. Then by part (i), e is unique, $e \in Z(S)$ and $A = (e)_*$. □

Definition 4.6. A semiring S is said is said to be the direct sum of two full subsemirings S_1 and S_2 if every element $x \in S$ is expressible in the form $y + z, y \in S_1, z \in S_2$ and $yz, zy \in E^+(S)$ for every $y \in S_1, z \in S_2$.

Theorem 4.7. *If S is the direct sum of S_1 and S_2, then S_1 and S_2 are mutually inverse ideals (both left or right ideals). Conversely, any two mutually inverse left ideals yields a direct sum decomposition of S.*

Proof. If $y \in S_1$, $x \in S$, then $x = y' + z'$, $y' \in S_1, z' \in S_2$, and $yx = y(y' + z') = yy' + yz' \in S_1$, whence S_1 is a right ideal. Likewise $xy = (y' + z')y = y'y + z'y \in S_1$, whence S_1 is a left ideal. Thus S_1 is an ideal. Similarly, S_2 is an ideal. Let $x \in S_1 \cap S_2$. Now, $1 = y_1 + z_1, y_1 \in S_1$, $z_1 \in S_2$. Then

$$x = x \cdot 1 = x(y_1 + z_1) = xy_1 + xz_1 \in E^+(S).$$

Moreover, $S_1 + S_2 = S$. Hence S_1 and S_2 are inverses to each other.

Converse part is obvious. □

Theorem 4.8. *The only direct sum decompositions of S are those of the form*

$$S = (1 + e')_* + (1 + 1' + e)_*, \tag{1}$$

where $e \in E^{\bullet}(S)$ and in $Z(S)$.

Proof. Clearly, any decomposition of the form (1) a direct sum decomposition. Let $S = A + B$, $A \cap B = E^+(S)$, with A, B ideals. Then by Theorem 2.13, there exists an element $e \in E^{\bullet}(S)$ such that $A = (1 + e')_r, B = (1 + 1' + e)_r$. But since A, B are also left ideals, Lemma 4.5(i), yields that $e \in Z(S), A = (1 + e')_*$, $B = (1 + 1' + e)_*$. □

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