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UNIVERSAL BOUNDS FOR GLOBAL SOLUTIONS OF A DIFFUSION EQUATION WITH A MIXED LOCAL-NONLOCAL REACTION TERM

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Abstract. In this paper we prove the existence of a universal, i.e. independent of the initial data, bound for global positive solutions of a diffusion equation with a mixed local-nonlocal reaction term. Such results are already known in some cases of local or nonlocal reaction terms.

1. Introduction and main result

In this paper, we are interested in the global solutions of the following problem:

$$(1.1)\quad (u_t - \Delta u)(t,x) = \left(1 + \frac{1}{p+1}\int_\Omega u^{p+1}(t,y)\,dy\right)^k u^p(t,x), \qquad t>0,\ x\in\Omega,$$

$$(1.2)\qquad u(t,x) = 0, \qquad t>0,\ x\in\partial\Omega,$$

$$(1.3)\qquad u(0,x) = u_0(x) \geq 0, \qquad x\in\Omega,$$

with $p > 1$, $k > \frac{1-p}{p+1}$, Ω is a smoothly bounded domain of $\mathbb{R}^d$ and $u_0 \in L^\infty(\Omega)$. More precisely, we will prove the existence of universal bounds for the global nonnegative solutions of (1.1)–(1.3) in the case of $k > 0$.

In order to illustrate our results and our motivations, we will recall some known results concerning the problem (1.1), (1.2), (1.3) with

$$(1.4)\qquad u_t(t,x) - \Delta u(t,x) = u^p(t,x), \qquad t>0,\ x\in\Omega,$$

and the problem (1.5), (1.2), (1.3) with

$$(1.5)\qquad u_t(t,x) - \Delta u(t,x) = \int_\Omega u^p(t,y)\,dy, \qquad t>0,\ x\in\Omega.$$

First, notice that the only difference between (1.1), (1.4) and (1.5) is the right hand side of those equalities, namely, the reaction term. In the case of (1.4), it is said to be *local* because the reaction is given at each point of the domain. In the

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case of (1.5), it is said to be *nonlocal* in opposition of the precedent definition. For (1.1), we called it a *mixed* local-nonlocal reaction term since one part is (1.4) and the other one corresponds to problems like (1.5). Let us recall now some known results.

The proof of local existence and uniqueness of the solution for the problem (1.2)–(1.4) is well-known; for (1.2), (1.3), (1.5), we refer to the articles of Ph. Souplet (see [**16**], [**17**]), where this problem and more general nonlocal terms were studied from the point of view of blow-up in finite time; for (1.1)–(1.3) in the articles of M. Fila and the same author and H. A. Levine (see [**2**] and [**3**]), where they studied the boundedness of global solutions.

We call $T^\star$ the maximal time of existence of the solutions of problems (1.1)–(1.5). We will be here interested in the case of $T^\star = \infty$, which means that the solution u is global. Let

$$p_s = \begin{cases} \infty & \text{if } d \leq 2, \\ \dfrac{d+2}{d-2} & \text{if } d \geq 3. \end{cases}$$

It is known that all global solutions of (1.2)–(1.4) are bounded if $p < p_s$, whereas if $p \geq p_s$, there exist unbounded global weak solutions, i.e. $\sup_{t\geq 0} |u(t)|_\infty = \infty$ (see [**9**], [**1**], [**5**]), where $|\cdot|_q$, $1 \leq q \leq \infty$ denotes the $L^q(\Omega)$-norm. Moreover, some unbounded global classical solutions even exist when $p = p_s$ and Ω is a ball. In the case of (1.5), (1.2), (1.3) it was proved in a precedent paper (see [**14**]) that all global solutions are uniformly bounded. For (1.1)–(1.3), the boundedness of global solutions was proved in [**2**] and [**3**], assuming $p < p_s$ and $k > \frac{1-p}{p+1}$. Note that the restriction on the values of k is not technical: the authors showed the existence of global unbounded solutions in the case of $k = \frac{1-p}{p+1}$.

Later, Giga (see [**6**]) gave a more precise picture of the set of global bounded positive solutions for the problem (1.2)–(1.4). Assuming $p < p_s$ and u is global, he gives *a priori* estimates:

$$|u(t)|_\infty \leq C\big(|u_0|_\infty\big), \quad t > 0. \tag{1.6}$$

Such estimates were obtained for (1.1)–(1.3) in a work of P. Quittner (see [**10**]) with $p < p_s$ and $\frac{1-p}{p+1} < k \leq 0$. Using the same arguments as in this paper, it is in fact possible to prove (1.6) for all global solutions of (1.1)–(1.3) with $k > 0$ and $p < p_s$. For the problem (1.5), (1.2), (1.3), (1.6) was obtained by completely different methods in [**15**].

The last two years have seen some new results in the study of the global bounded positive solutions of (1.2), (1.4), namely the proof of existence of *universal bounds* for this problem. Those results were initiated by M. Fila, Ph. Souplet and F. Weissler (see [**4**]). Let us state the result of [**4**]:

Theorem A. *Assume*

$$1 < p < \frac{d+1}{d-1} \tag{1.7}$$

and let $\tau > 0$. There exists a constant $C(\Omega, p, \tau) > 0$, independent of u, such that for all nonnegative global solutions of (1.2)–(1.4), *it holds*

$$\sup_{\Omega} u(t,\cdot) \le C(\Omega, p, \tau), \quad t \ge \tau. \tag{1.8}$$

Note that the bound in (1.8) is independent of the initial data u_0, that is why it is called universal. It is easy to show that (1.8) implies (1.6) for positive solutions. In [**11**] and [**13**], P. Quittner, Ph. Souplet and M. Winkler proved that (1.7) can be improved, more than this, they showed that one can reach $p < p_s$ for $d \le 4$. For the problem (1.2), (1.3), (1.5), in [**15**] we obtained exactly Theorem A, without the assumption (1.7) which means for all $p > 1$.

The question is: is it possible to obtain a similar theorem for the global nonnegative solutions of (1.1)–(1.3)? The answer is:

Theorem 1.1. *Assume that $k > 0$ and that $1 < p < p_s$. Let*

$$\frac{1}{(k+1)p+k} > \frac{d-2}{2} \qquad \text{and} \qquad \frac{(k+1)(p+1)}{k+2} < \frac{d}{d-1}. \tag{1.9}$$

Let $\tau > 0$. There exists a constant $C(\Omega, \tau, p, k) > 0$, independent of u, such that for all nonnegative global solutions of (1.1)–(1.3), *it holds*

$$\sup_{\Omega} u(t,\cdot) \le C(\Omega, \tau, p, k), \quad t \ge \tau. \tag{1.10}$$

Remark 1.1. *Note that the condition* (1.9) *implies $p < p_s$.*

The proof of this theorem relies on three tools:

1. Universal bound of a weighted L^1-norm, obtained by Kaplan's type arguments.
2. $L^r - L^\infty$ estimates.
3. *A priori* estimates of Giga's type.

Tools one and two will be developped in Sections 2 and 3 respectively. Theorem is proved in Section 4.

2. Kaplan's argument

We begin with a first estimate, which is obtained by Kaplan's classical eigenfunction method, see [**7**].

Lemma 2.1. *Assume that $k \ge 0$ and let $\lambda_1 > 0$ be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and φ_1 the associated eigenfunction such that*

$$\varphi_1 = \varphi_1(x) > 0 \quad \text{and} \quad \int_{\Omega} \varphi_1(x)dx = 1. \tag{2.1}$$

Let

$$y(t) = \int_{\Omega} u(t,x)\varphi_1(x)dx, \qquad 0 < t < T^\star \tag{2.2}$$

and

$$\mathcal{C} = \mathcal{C}(\Omega, p, k) = \Big(\lambda_1 (p+1)^k \Big(\int_\Omega \varphi_1^{\frac{p+1}{p}}\Big)^{kp}\Big)^{\frac{1}{(k+1)p+k-1}} > 0. \tag{2.3}$$

Then the following property holds:

$$\text{if } T^\star = \infty \text{ then } y(t) \le \mathcal{C}, \text{ for all } t > 0. \tag{2.4}$$

Proof. Multiplying (1.1) by φ_1, it follows that

$$\int_\Omega u_t \varphi_1 - \int_\Omega \Delta u \varphi_1 = \Big(1 + \frac{1}{p+1} \int_\Omega u^{p+1}(t, y) dy\Big)^k \int_\Omega u^p \varphi_1.$$

Integrating by parts, we obtain

$$\frac{d}{dt} \int_\Omega u\varphi_1 + \lambda_1 \int_\Omega u\varphi_1 = \Big(1 + \frac{1}{p+1} \int_\Omega u^{p+1}(t, y) dy\Big)^k \int_\Omega u^p \varphi_1.$$

Using (2.2), the last equality becomes:

$$y' + \lambda_1 y \ge (p+1)^{-k} \Big(\int_\Omega u^{p+1}\Big)^k \int_\Omega u^p \varphi_1.$$

By Hölder's and Jensen's inequalities, knowing (2.1), the inequality above implies

$$y' + \lambda_1 y \ge (p+1)^{-k} \Big(\int_\Omega \varphi_1^{(p+1)/p}\Big)^{-kp} y^{k(p+1)+p}.$$

That is

$$y' \ge y\Big((p+1)^{-k} \Big(\int_\Omega \varphi_1^{(p+1)/p}\Big)^{-kp} y^{(k+1)p+k-1} - \lambda_1\Big). \tag{2.5}$$

By well-known arguments, (2.5) implies finite time blow-up of y whenever $y > \mathcal{C}$ (see (2.3)), for some $t > 0$. Since $T^\star = \infty$, we conclude that $y \le \mathcal{C}(\Omega, p, k)$, $t \ge 0$. Hence (2.4) is verified, which ends the proof of Lemma 2.1. □

3. L^q–L^r estimates

In this section we are going to prove the following theorem:

Theorem 3.1. *Let $p < p_s$ and assume that*

$$q > \max\Big\{\frac{d(p+1)\big((k+1)p + k\big)}{\big(d(k+1)+2\big)p + kd + 2}, \frac{d(k+1)p + d(k-1)}{kd+2}\Big\}. \tag{3.1}$$

For all $M > 0$, there exist $T = T(M) > 0$ and $K = K(M) > 0$ such that if $u_0 \in L^\infty(\Omega)$ with $|u_0|_q \le M$, then the maximal solution $u \in L^\infty_{loc}\big([0, T^\star), L^\infty\big)$ of (1.1)–(1.3) satisfies $T^\star > T(M)$ and

$$t^{\frac{d}{2}(\frac{1}{q} - \frac{1}{r})} |u(t)|_r \le K, \quad 0 < t < T, \quad q \le r \le \infty, \quad p+1 \le r.$$

Before beginning the proof of Theorem 1.3, note that u solves the integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t \left(1 + \frac{1}{p+1}\int_\Omega u^{p+1}(s,y)dy\right)^k e^{(t-s)\Delta}\big(u^p(s,\cdot)\big)ds, \tag{3.2}$$

where here, and in what follows, $e^{t\Delta}$ denotes the Dirichlet heat semi-group.

Proof. We proceed in three steps. C and C' denote various constants which may change from line to line, and which will depend only on Ω, p, k and q.

Step 1. First, note that we have, using (3.2)

$$u(t) \le e^{t\Delta}u_0 + C\int_0^t \big(1 + |u(s)|_{p+1}^{k(p+1)}\big)e^{(t-s)\Delta}\big(u^p(s,\cdot)\big)ds. \tag{3.3}$$

Let us introduce the following quantities

$$N_0 = \max\{p+1,\ q\}, \qquad \alpha = \frac{d}{2}\left(\frac{1}{q} - \frac{1}{N_0}\right), \qquad \beta = \frac{d}{2}\,\frac{(p-1)}{N_0} \tag{3.4}$$

and put

$$K(t) = \sup_{s\in(0,t)} s^\alpha |u(s)|_{N_0} < \infty, \quad 0 < t \le T, \tag{3.5}$$

where $T < T^\star$ will be specified later. Using (3.3), we have

$$|u(t)|_{N_0} \le |e^{t\Delta}u_0|_{N_0} + C\int_0^t \big(1 + |u(s)|_{p+1}^{k(p+1)}\big)\big|e^{(t-s)\Delta}\big(u^p(s,\cdot)\big)\big|_{N_0}ds.$$

Using L^q-L^{N_0} and $L^{N_0/p}$-L^{N_0} estimates for the heat semi-group, the last inequality becomes

$$|u(t)|_{N_0} \le C't^{-\alpha}|u_0|_q + C'\int_0^t \big(1 + |u(s)|_{p+1}^{k(p+1)}\big)(t-s)^{-\beta}|u^p(s)|_{N_0/p}ds.$$

Knowing (3.4) eq. 1, we can apply the Hölder inequality to the L^{p+1} and $L^{N_0/p}$-norm terms and we obtain

$$|u(t)|_{N_0} \le Ct^{-\alpha}|u_0|_q + C\int_0^t \big(1 + |u(s)|_{N_0}^{k(p+1)}\big)(t-s)^{-\beta}|u(s)|_{N_0}^p ds.$$

Now, using (3.5) we obtain

$$|u(t)|_{N_0} \le Ct^{-\alpha}|u_0|_q + CK^p(T)\int_0^t (t-s)^{-\beta}s^{-p\alpha}ds$$
$$+ CK^{(k+1)p+k}(T)\int_0^t s^{-k(p+1)\alpha}(t-s)^{-\beta}s^{-p\alpha}ds,$$

in other words, we have

$$|u(t)|_{N_0} \le Ct^{-\alpha}|u_0|_q + CK^p(T)t^{1-p\alpha-\beta}\int_0^1 (1-\sigma)^{-\beta}\sigma^{-p\alpha}d\sigma$$
$$+ CK^{(k+1)p+k}(T)t^{1-[(k+1)p+k]\alpha-\beta}\int_0^1 (1-\sigma)^{-\beta}\sigma^{-[(k+1)p+k]\alpha}d\sigma.$$

Thanks to $p < p_s$ and (3.1), the integrals above are convergent. Multiplying by t^α and taking the sup in time on $(0,T)$ in the left side of our last inequality we obtain

$$K(T) \le C|u_0|_q + C\big(K^p(T) + K^{(k+1)p+k}(T)\big)T^{1-[(k+1)p+k-1]\alpha-\beta}$$

provided $T \le 1$. (Note that (3.1) implies that $1 - \big((k+1)p+k-1\big)\alpha - \beta > 0$.)

For any T such that

$$\begin{aligned} &T < T^\star, \\ &T \le T_0 = \min\Big(1, c_2\big(M^{p-1}(1+M^{k(p+1)})\big)^{-1/(1-[(k+1)p+k-1]\alpha-\beta)}\Big), \end{aligned} \tag{3.6}$$

with $c_2 = c_2(d,p,q,k) > 0$ sufficiently small, we easily get:

$$K(T) \le 2CM. \tag{3.7}$$

Note in particular that (3.6), (3.7) imply

$$K^p(T)\big(1 + K^{k(p+1)}(T)\big)T^{1-[(k+1)p+k-1]\alpha-\beta} \le C'M. \tag{3.8}$$

<u>Step 2.</u> Assume that m, r satisfy

$$N_0 \le m < r \le \infty \quad \text{and} \quad \frac{p}{m} - \frac{2}{d} < \frac{1}{r}. \tag{3.9}$$

Put $\gamma = \dfrac{d}{2}\left(\dfrac{1}{q} - \dfrac{1}{m}\right)$ and suppose we know that

$$H(m) = \sup_{t\in(0,T)} t^\gamma |u(t)|_m < \infty, \quad 0 < t < T, \tag{3.10}$$

where T is given by (3.6).

We see that for $t \in (0,T)$

$$|u(t)|_r \le \left|e^{\frac{t}{2}\Delta} u\big(\tfrac{t}{2}\big)\right|_r + C\int_{t/2}^{t} \big(1+|u(s)|_{p+1}^{k(p+1)}\big)(t-s)^{-\frac{d}{2}\left(\frac{p}{m}-\frac{1}{r}\right)} |u^p(s)|_{\frac{m}{p}}\, ds.$$

Using Hölder's inequality, and L^m-L^r, $L^{m/p}$-L^r estimates for the heat semi-group, the last expression becomes

$$|u(t)|_r \le Ct^{-\frac{d}{2}\left(\frac{1}{m}-\frac{1}{r}\right)} \left|u\big(\tfrac{t}{2}\big)\right|_m + C'\int_{t/2}^{t} \big(1+|u(s)|_{N_0}^{k(p+1)}\big)(t-s)^{-\frac{d}{2}\left(\frac{p}{m}-\frac{1}{r}\right)} |u(s)|_m^p\, ds$$

and using (3.10) we obtain

$$\begin{aligned}
|u(t)|_r &\le CHt^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} + CH^p \int_{t/2}^{t} (t-s)^{-\frac{d}{2}\left(\frac{p}{m}-\frac{1}{r}\right)} s^{-p\gamma} ds \\
&\quad + CK^{k(p+1)}(T)H^p \int_{t/2}^{t} (t-s)^{-\frac{d}{2}\left(\frac{p}{m}-\frac{1}{r}\right)} s^{-k(p+1)\alpha - p\gamma} ds \\
&= CHt^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} + CH^p t^{1-\frac{d}{2}\left(\frac{p}{m}-\frac{1}{r}\right)-p\gamma} \int_{1/2}^{1} (1-\sigma)^{-\frac{d}{2}\left(\frac{p}{m}-\frac{1}{r}\right)} \sigma^{-p\gamma} d\sigma \\
&\quad + CK^{k(p+1)}(T)H^p t^{1-\frac{d}{2}\left(\frac{p}{m}-\frac{1}{r}\right)-k(p+1)\alpha-p\gamma} \\
&\qquad\qquad \cdot \int_{1/2}^{1} (1-\sigma)^{-\frac{d}{2}\left(\frac{p}{m}-\frac{1}{r}\right)} \sigma^{-k(p+1)\alpha-p\gamma} d\sigma.
\end{aligned}$$

The finiteness of the integrals is guaranteed by (3.9), then we have

$$t^{\frac{d}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} |u(t)|_r \le CH + CH^p\big(1+K^{k(p+1)}(T)\big) t^{1-\frac{d}{2}\frac{p-1}{q}-k(p+1)\alpha}.$$

As

$$T^{1-\frac{d}{2}\frac{p-1}{q}-k(p+1)\alpha} \le T^{1-[(k+1)p+k-1]\alpha-\beta}$$

knowing (3.8) we have

$$1 + H^{p-1}(m)\big(1+K^{k(p+1)}(T)\big)T^{1-[(k+1)p+k-1]\alpha-\beta} \le C(d,p,q,r,H),$$

we easily obtain

$$\sup_{t\in(0,T)} t^{\frac{d}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} |u(t)|_r \le C'(d,p,q,r,H)H < \infty.$$

Step 3. By standard arguments (see [**8**, proof of Theorem 2.2] one shows that it is possible to define a nondecreasing sequence (r_k), with $r_0 = N_0$, such that $r_{k+1} = r$ and $r_k = m$ satisfy (3.9) for every k; and $r_k = \infty$ is reached in a finite number of iterations. Hence the conclusion of Theorem 3.1 follows by finite iterations of step 2. $\square$

4. Proof of Theorem 1.1

Denote $\delta(x) = \mathrm{dist}\big(x, \partial\Omega\big)$, let us recall that L^q_δ spaces are defined by

$$L^q_\delta = L^q_\delta(\Omega) = L^q\big(\Omega;\ \delta(x)dx\big) \quad 1 \le q < \infty.$$

We will denote by $|\cdot|_{q,\delta}$ the associated norm. For more details concerning the L^q_δ theory, see [**4**]. First, we are going to prove the following lemma:

Lemma 4.1. *Assume that the solution u of the problem* (1.1)–(1.3) *is global and let $\tau > 0$ and $\varepsilon > 0$ be sufficiently small. Assume that*

$$r = p + 1 - \frac{(p-1)(p+1+2\varepsilon)}{(p-1)(k+2) - 2k\varepsilon} \tag{4.1}$$

and that the following inequality holds

$$\int_0^{\tau/2} \Big(1 + |u(t)|^{p+1}_{p+1}\Big)^k |u(t)|^p_{p,\delta} \le C(\Omega, p, k, \tau). \tag{4.2}$$

Then there exists a $\tau_1 \in (0, \tau/2)$ such that

$$|u(\tau_1)|_r \le C'(\Omega, p, k, \tau). \tag{4.3}$$

Proof. As $T^\star = \infty$, by Lemma 2.1 we know that

$$\forall t \ge 0, \quad \int_\Omega u(t,x)\varphi_1(x) \le \mathcal{C}(\Omega, p, k). \tag{4.4}$$

Using the test function χ introduced in [**13**], which solves the following problem

$$\begin{cases} -\Delta\chi(x) = \varphi_1^{-\alpha}(x), & x \in \Omega, \\ \chi \equiv 0, & x \in \partial\Omega, \end{cases}$$

where $\alpha \in (0,1)$, we have, multiplying (1.1) by χ

$$\int_\Omega u_t\chi - \int_\Omega \chi\Delta u = \int_\Omega u^p\chi\Big(1 + \tfrac{1}{p+1}|u|^{p+1}_{p+1}\Big)^k.$$

Integrating this inequality by parts and in time and knowing that $\chi(x) \le c\delta(x)$, we obtain

$$\int_0^{\tau/2}\int_\Omega u\varphi_1^{-\alpha} \le C\Big[\int u\chi\Big]^0_{\tau/2} + C\int_0^{\tau/2} |u|^p_{p,\delta}\big(1 + |u|^{p+1}_{p+1}\big)^k.$$

Using (4.4) and (4.2), we deduce from the last inequality that

$$\int_0^{\tau/2}\int_\Omega u\varphi_1^{-\alpha} \le C(\Omega, \tau, p, k). \tag{4.5}$$

Thanks to the Hölder inequality with

$$N_1 = \frac{2(p-1)}{p-1-2\varepsilon}, \qquad 1 = \frac{1}{N_1} + \frac{1}{N_2}, \qquad \alpha = 1 - \frac{4\varepsilon}{p-1+2\varepsilon},$$

we have

$$\int_\Omega u^{\frac{p+1}{2}-\varepsilon} \le \left(\int_\Omega u^p \varphi_1\right)^{1/N_1} \left(\int_\Omega u\varphi_1^{-\alpha}\right)^{1/N_2}. \tag{4.6}$$

Setting

$$g(t) = \left(1 + \frac{1}{p+1}\int_\Omega u^{p+1}\right)^k, \tag{4.7}$$

multiplying (4.6) by $\big(g(t)\big)^{1/N_1}$ and integrating in time over $(0, \tau/2)$, we obtain

$$\int_0^{\tau/2} \left(\big(g(t)\big)^{1/N_1} \int_\Omega u^{\frac{p+1}{2}-\varepsilon}\right) \le \int_0^{\tau/2} \left(g(t)\int_\Omega u^p\varphi_1\right)^{1/N_1} \Big(\int_\Omega u\varphi_1^{-\alpha}\Big)^{1/N_2}.$$

Now applying Hölder's inequality in time to the right hand side of the last inequality, we have

$$\int_0^{\tau/2} \left(\big(g(t)\big)^{1/N_1} \int_\Omega u^{\frac{p+1}{2}-\varepsilon}\right) \le \left(\int_0^{\tau/2} g(t)\int_\Omega u^p\varphi_1\right)^{1/N_1} \left(\int_0^{\tau/2}\int_\Omega u\varphi_1^{-\alpha}\right)^{1/N_2}.$$

Using (4.7) and knowing (4.2) and (4.5) we easily deduce that

$$\int_0^{\tau/2} \left(\int_\Omega u^{\frac{p+1}{2}-\varepsilon}\Big(1 + \frac{1}{p+1}\int_\Omega u^{p+1}\Big)^{k/N_1}\right) \le C(\Omega, \tau, p, k).$$

There exists some $\tau_1 \in (0, \tau/2)$ such that, using the last inequality,

$$\begin{aligned} &\int_\Omega u^{\frac{p+1}{2}-\varepsilon}(\tau_1)\Big(\int_\Omega u^{p+1}(\tau_1)\Big)^{k/N_1} \\ &\le \frac{2}{\tau}\int_0^{\tau/2}\left(\int_\Omega u^{\frac{p+1}{2}-\varepsilon}\Big(\int_\Omega u^{p+1}\Big)^{k/N_1}\right) \\ &\le C(\Omega, \tau, p, k). \end{aligned} \tag{4.8}$$

Knowing (4.1) and that $N_1 = \frac{2(p-1)}{p-1-2\varepsilon} > 1$; using the Hölder inequality, we have

$$\Big(\int_\Omega u^r\Big)^{(N_1+k)/N_1} \le \int_\Omega u^{\frac{p+1}{2}-\varepsilon}\Big(\int_\Omega u^{p+1}\Big)^{k/N_1}.$$

From the last inequality and (4.8), we easily deduce (4.3), which ends the proof of Lemma 4.1. □

Proof of Theorem 1.1. Multiplying (1.1) by φ_1 and integrating by parts we have

$$\int_\Omega u_t(t,x)\varphi_1(x)dx + \lambda_1 \int_\Omega u(t,x)\varphi_1(x)dx = \int_\Omega u^p(t,x)\varphi_1(x)dx \left(1 + \frac{1}{p+1}|u(t)|_{p+1}^{p+1}\right)^k.$$

Integrating in time the last equality and using Lemma 2.1, we deduce that

$$\begin{aligned}\int_0^{\tau/2} \left(1 + \frac{1}{p+1}|u(t)|_{p+1}^{p+1}\right)^k |u(t)|_{p,\delta}^p \, dt &\le \int_\Omega u\left(\tfrac{\tau}{2}\right)\varphi_1 + \lambda_1 \int_0^{\tau/2}\int_\Omega u\varphi_1 \\ &\le C + \lambda_1 \tau C \\ &\le C'(\Omega,\tau,p,k).\end{aligned} \tag{4.9}$$

By Lemma 4.1, we know that (4.9) implies

$$|u(\tau_1)|_r \le C'(\Omega,\tau,p,k),$$

for some $\tau_1 \in (0,\tau)$. Using Theorem 3.1 and the last inequality (note that (1.9) ensures that $q = r$ satisfies (3.1), we obtain

$$|u(\tau_2)|_\infty \le C(\Omega,\tau,p,k),$$

for some $\tau_2 \in (\tau_1,\tau)$. Thanks to the proof of Theorem 4.3 in [**10**] and the appendix below, knowing *a priori* estimates, the result follows. □

5. Appendix

In this Section, we want to give for reader's convenience, some indications for the proof of *a priori* estimates for global solutions to (1.1)–(1.3). Such results were already shown for (1.1)–(1.3) by P. Quittner, in the case of $k < 0$ (see [**10**]). It is possible to readapt Quittner's proof in the same article, to obtain those estimates for (1.1)–(1.3) for $k > 0$. A straightforward modification of [**10**, Lemma 2.2 and Remark 2.5] yields

$$|u(t)|_{p+1-\varepsilon} \le c(\varepsilon), \quad \forall \varepsilon > 0. \tag{5.1}$$

Then, one obtains *a priori* estimates in the same way as [**10**]. To be complete, we have to prove (2.34) in [**10**, Theorem 2.6]. In order to do this, we shall show that

$$|u(t)|_{p+1} \le C. \tag{5.2}$$

By proving that

$$|u(t)|_{p+1+\alpha} \le C,$$

for some $\alpha > 0$, we will be done. Fix $\alpha > 0$ and $\beta > 0$ such that

$$L^{(p+1)/p} \hookrightarrow W^{-2+2\beta}_{p+1+\alpha} \quad (p < p_S). \tag{5.3}$$

We denote by $||\cdot||_{p+1+\alpha}$ the norm in $W^{-2+2\beta}_{p+1+\alpha}$. Then, using (3.2) we obtain

$$\begin{aligned} |u(t)|_{p+1+\alpha} &\le C + C\int_0^t \big(1+|u(s)|_{p+1}^{p+1}\big)^k \big|e^{(t-s)\Delta}\big(u^p(s,\cdot)\big)\big|_{p+1+\alpha}\, ds \\ &\le C + C\int_0^t (t-s)^{-(1-\beta)}\big(1+|u(s)|_{p+1}^{p+1}\big)^k ||u^p(s)||_{p+1+\alpha}\, ds. \end{aligned}$$

Now, using the embedding (5.3), the last inequality becomes

$$|u(t)|_{p+1+\alpha} \le C + C\int_0^t \big(1+|u(s)|_{p+1}^{p+1}\big)^k (t-s)^{-(1-\beta)}|u^p(s)|_{(p+1)/p}\, ds,$$

which gives, using (3.3),

$$\begin{aligned} |u(t)|_{p+1+\alpha} \le C + C&\int_0^t (t-s)^{-(1-\beta)}|u(s)|_{p+1}^{p}\, ds \\ + C&\int_0^t (t-s)^{-(1-\beta)}|u(s)|_{p+1}^{k(p+1)+p}\, ds. \end{aligned}$$

By interpolation we have

$$\begin{aligned} \big|u(s)\big|_{p+1}^{p} &\le \big|u(s)\big|_{p+1-\varepsilon}^{p-1+\theta}\big|u(s)\big|_{p+1+\alpha}^{1-\theta}, \\ \big|u(s)\big|_{p+1}^{k(p+1)+p} &\le \big|u(s)\big|_{p+1-\varepsilon}^{k(p+1)+p-1+\theta'}\ \big|u(s)\big|_{p+1+\alpha}^{1-\theta'} \end{aligned}$$

with

$$\begin{aligned} \frac{p}{p+1} &= \frac{p-1+\theta}{p+1-\varepsilon} + \frac{1-\theta}{p+1+\alpha}, \\ \frac{k(p+1)+p}{p+1} &= \frac{k(p+1)+p-1+\theta'}{p+1-\varepsilon} + \frac{1-\theta'}{p+1+\alpha}. \end{aligned}$$

By choosing $\varepsilon > 0$ sufficiently small, we ensure that $\theta,\ \theta' > 0$. Denoting $\varphi(t) = |u(t)|_{p+1+\alpha}$ and using (5.1), we finally obtain the following inequality

$$\varphi(t) \le C + c\int_0^t (t-s)^{-(1-\beta)}\varphi^{1-\theta}(s)\, ds + c'\int_0^t (t-s)^{-(1-\beta)}\varphi^{1-\theta'}(s)\, ds.$$

The inequality above implies the boundedness of φ, hence (5.2) follows, which ends this section. $\square$

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