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ON EXPONENTIAL DICHOTOMY OF SEMIGROUPS

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Abstract. The aim of this paper is to analyze the connections between the exponential dichotomy of a semigroup on a Banach space X and the admissibility of the pair $(\ell^p(\mathbf{N}, X), \ell^q(\mathbf{N}, X))$. We obtain necessary and sufficient conditions for exponential dichotomy of exponentially bounded semigroups using discrete time techniques.

1. Introduction

Asymptotic behaviour of semigroups in Banach spaces is a classical and well-studied subject (see [**2**], [**5**], [**11**], [**12**], [**15**], [**18**]). In the last decades an impressive progress has been made in the qualitative theory of evolution equations, by associating to an evolution family or to a linear skew-product flow an evolution semigroup on different function spaces and by expressing their asymptotic properties in terms of the characteristic particularities of this semigroup (see [**2**], [**10**]). In fact, this new approach allowed, in certain situations, the treatment of the non-autonomous case in the unified setting of the autonomous one. One of the main results in [**2**] states that a linear skew-product flow is exponentially dichotomic if and only if the associated evolution semigroup is hyperbolic. In [**10**] exponential stability and exponential dichotomy of an evolution family are related to the properties of the infinitesimal generator of the evolution semigroup associated to it. An extensive study concerning the applicability of the theory of evolution semigroups in dynamical systems has been presented in [**2**].

Discrete methods in the study of the exponential dichotomy of evolution equations have proved to be the starting points for important results in this field (see [**1**]–[**3**], [**6**], [**8**], [**9**], [**16**], [**18**]). These techniques have the origin in the work of Henry (see [**6**]). In [**2**] and in [**9**], it is proved by different methods the equivalence between the exponential dichotomy of a linear skew-product flow and the exponential dichotomy of the discrete linear skew-product flow associated to it. The main advantage of the discrete characterizations of diverse asymptotic properties is that they are applicable for a large class of systems without imposing continuity

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or measurability conditions. Therefore in what follows our central concern will be the study of a very general class of semigroups, without requiring continuity or measurability properties – the class of exponentially bounded semigroups.

The purpose of the present paper is to obtain characterizations for the exponential dichotomy of exponentially bounded semigroups in terms of the solvability of discrete-times equations on $l^p(\mathbf{N}, X)$-spaces and to point out the special properties of the autonomous case. We associate to a semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ the subspace

$$X_1 = \{x \in X : \sum_{n=0}^{\infty} ||T(n)x||^p < \infty\}$$

and we discuss the properties implied by the solvability of a discrete-time equation associated to the semigroup, under the assumption that X_1 is closed and complemented. We study when the admissibility of the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is a sufficient condition for exponential dichotomy of a semigroup. For $p, q \in [1, \infty), p \geq q$ we prove that an exponentially bounded semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is exponentially dichotomic if and only if the pair $(l^p(\mathbf{N}, X), l^q(\mathbf{N}, X))$ is admissible for $\mathbf{T}$ and X_1 is closed and it has a $\mathbf{T}$-invariant complement.

2. Exponential dichotomy of semigroups

Let X be a real or a complex Banach space and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X. In what follows we denote by $||\cdot||$ the norm on X and on $\mathcal{L}(X)$, respectively. An operator $P \in \mathcal{L}(X)$ will be called *projection* if $P^2 = P$.

Definition 2.1. A family $\mathbf{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{L}(X)$ is called a *semigroup* on X if $T(0) = I$ and $T(t + s) = T(t)T(s)$, for all $t, s \geq 0$.

A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be *exponentially bounded* if there are $M \geq 1$ and $\omega > 0$ such that $||T(t)|| \leq Me^{\omega t}$, for all $t \geq 0$.

Definition 2.2. A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be *exponentially dichotomic* if there exist a projection $P \in \mathcal{L}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that:

(i) $T(t)P = PT(t)$, for all $t \geq 0$;
(ii) $||T(t)x|| \leq Ke^{-\nu t}||x||$, for all $x \in \operatorname{Im} P$ and all $t \geq 0$;
(iii) $||T(t)x|| \geq \frac{1}{K}\, e^{\nu t}||x||$, for all $x \in \operatorname{Ker} P$ and all $t \geq 0$;
(iv) $T(t)_{|} : \operatorname{Ker} P \to \operatorname{Ker} P$ is an isomorphism, for all $t \geq 0$.

Definition 2.3. If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a semigroup on X and $U \subset X$ is a linear subspace, U is said to be $\mathbf{T}$-*invariant* if $T(t)U \subset U$, for all $t \geq 0$.

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be an exponentially bounded semigroup on X and let $p \in [1, \infty)$. We define the linear subspace

$$X_1 = \{x \in X : \sum_{n=0}^{\infty} ||T(n)x||^p < \infty\}.$$

In what follows we suppose that X_1 is closed and it has a closed $\mathbf{T}$-invariant complement X_2 such that $X = X_1 \oplus X_2$.

For $p \in [1,\infty)$ let $\ell^p(\mathbf{N}, X) = \{s : \mathbf{N} \to X : \sum_{n=0}^{\infty} ||s(n)||^p < \infty\}$ which is a Banach space with respect to the norm

$$||s||_p = (\sum_{n=0}^{\infty} ||s(n)||^p)^{1/p}.$$

Definition 2.4. Let $p, q \in [1,\infty)$. The pair $(\ell^p(\mathbf{N}, X), \ell^q(\mathbf{N}, X))$ is said to be *admissible* for $\mathbf{T}$ if for every $s \in \ell^q(\mathbf{N}, X)$ there is $\gamma \in \ell^p(\mathbf{N}, X)$ such that

$$(E_d) \qquad \gamma(n+1) = T(1)\gamma(n) + s(n), \quad \forall n \in \mathbf{N}.$$

Remark 2.1. If the pair $(\ell^p(\mathbf{N}, X), \ell^q(\mathbf{N}, X))$ is admissible for $\mathbf{T}$, then:

(i) for every $s \in \ell^q(\mathbf{N}, X)$ there is a unique $\gamma_s \in \ell^p(\mathbf{N}, X)$ such that $\gamma_s(0) \in X_2$ and (γ_s, s) satisfies the equation (E_d);

(ii) there is $\alpha > 0$ such that $||\gamma_s||_p \leq \alpha ||s||_q$, for every pair (γ_s, s) which verifies the equation (E_d) and $\gamma_s(0) \in X_2$.

Indeed, if D is the linear subspace of all $\gamma \in \ell^p(\mathbf{N}, X)$ with $\gamma(0) \in X_2$ and there is $s \in \ell^q(\mathbf{N}, X)$ such that the pair (γ, s) satisfies (E_d), we deduce that the operator $W : D \to \ell^q(\mathbf{N}, X), W\gamma = s$ is invertible. Considering the graph norm on D, i.e. $||\gamma||_D = ||\gamma||_p + ||W\gamma||_q$, by the Banach principle there is $\alpha > 0$ such that $||\gamma||_p \leq ||\gamma||_D \leq \alpha ||W\gamma||_q$, for all $\gamma \in D$.

Theorem 2.1. *There are two constants $K, \nu > 0$ such that*

$$||T(t)x|| \leq K e^{-\nu t} ||x||, \quad \forall t \geq 0, \forall x \in X_1.$$

Proof. Let $T_1(t) := T(t)_{|X_1}$, for all $t \geq 0$. Then we have that $\mathbf{T_1} = \{T_1(t)\}_{t \geq 0}$ is a semigroup on X_1.

For every $x \in X_1$ we consider the mapping $\varphi_x : \mathbf{N} \to X,\ \varphi_x(n) = T(n)x$. We define the operator

$$\Gamma : X_1 \to \ell^p(\mathbf{N}, X_1), \quad \Gamma x = \varphi_x.$$

Then it is easy to verify that Γ is a closed linear operator, so it is bounded. It results that

$$(2.1) \qquad \sum_{n=0}^{\infty} ||T_1(n)x||^p \leq ||\Gamma||^p \ ||x||^p, \quad \forall x \in X_1.$$

In particular, from relation (2.1) we have that $||T_1(n)|| \leq ||\Gamma||$, for all $n \in \mathbf{N}$. Let $m \in \mathbf{N}^*$ be such that $m \geq 2^p ||\Gamma||^{2p}$. Then from relation (2.1) we deduce that

$$m||T_1(m)x||^p \leq ||\Gamma||^p \ \sum_{j=1}^{m} ||T_1(j)x||^p \leq ||\Gamma||^{2p} \ ||x||^p, \quad \forall x \in X_1.$$

This implies that

$$||T_1(m)x|| \leq \frac{1}{2} \ ||x||, \quad \forall x \in X_1.$$

If M, ω are the constants given by Definition 2.1, setting $\nu = (\ln 2/m)$ and $K = Me^{(\omega+\nu)m}$ we obtain the conclusion. $\square$

If $A \subset \mathbf{N}$ we denote by χ_A the characteristic function of the set A.

Theorem 2.2. *If the pair $(\ell^p(\mathbf{N}, X), \ell^q(\mathbf{N}, X))$ is admissible for $\mathbf{T}$, then:*

(i) *for every $t \geq 0$, the restriction $T(t)_| : X_2 \to X_2$ is an isomorphism;*

(ii) *there are $K, \nu > 0$ such that*

$$||T(t)x|| \geq \frac{1}{K}\, e^{\nu t}||x||, \quad \forall x \in X_2.$$

Proof. Let $\alpha > 0$ be given by Remark 2.1 (ii).

(i) It is sufficient to prove that for every $h \in \mathbf{N}^*$ the operator $T(h)_| : X_2 \to X_2$ is an isomorphism. Indeed, let $h \in \mathbf{N}^*$.

Injectivity. Let $x \in X_2$, with $T(h)x = 0$. We consider the sequence

$$\gamma : \mathbf{N} \to X, \qquad \gamma(n) = \chi_{\{0,\ldots,h-1\}}(n)T(n)x.$$

It is easy to see that the pair $(\gamma, 0)$ satisfies the equation (E_d). Since $\gamma(0) = x \in X_2$ from Remark 2.1 (i) it follows that $\gamma = 0$. In particular this implies that $x = \gamma(0) = 0$, so $T(h)_|$ is injective.

Surjectivity. Let $x \in X_2$. We define

$$s : \mathbf{N} \to X, \qquad s(n) = -x\chi_{\{h-1\}}(n).$$

Let $\gamma \in \ell^p(\mathbf{N}, X)$ be such that (γ, s) verifies the equation (E_d). From our assumption $X = X_1 \oplus X_2$, so there are $x_1 \in X_1$ and $x_2 \in X_2$ with $\gamma(0) = x_1 + x_2$. Taking into account that $\gamma(h) = T(h)\gamma(0) - x$, it follows that $T(h)x_2 - x = \gamma(h) - T(h)x_1$. Since $\gamma(n) = T(n-h)\gamma(h)$, for all $n \geq h$, it immediately follows that $\gamma(h) \in X_1$. Hence $x = T(h)x_2$, so $T(h)_|$ is also surjective.

(ii) Let $M, \omega > 0$ be the constants given by Definition 2.1. Let $x \in X_2$ and let $n \in \mathbf{N}^*$. We consider the sequences

$$s, \gamma : \mathbf{N} \to X, \qquad s(k) = -\chi_{\{n\}}(k)\ T(1)x \quad \gamma(k) = \chi_{\{0,\ldots,n\}}(k)T(n-k)_|^{-1}x$$

where $T(j)_|^{-1}$ is the inverse of the operator $T(j)_| : X_2 \to X_2$. It is easy to see that the pair (γ, s) satisfies the equation (E_d). Since $\gamma(0) \in X_2$, from Remark 2.1 it follows that $||\gamma||_p \leq \alpha ||s||_q$. This shows that

$$\sum_{j=0}^{n} ||T(j)_|^{-1}x||^p \leq \lambda^p\ ||x||^p$$

where $\lambda = \alpha ||T(1)||$. Since $n \in \mathbf{N}^*$ and $x \in X_2$ were arbitrary we obtain that

$$\sum_{j=0}^{\infty} ||T(j)_|^{-1}x||^p \leq \lambda^p\ ||x||^p, \quad \forall x \in X_2.$$

Using similar arguments as in the proof of Theorem 2.1 it follows that there is $h \in \mathbf{N}^*$ such that

$$||T(h)_|^{-1}x|| \leq \frac{1}{2}\ ||x||, \quad \forall x \in X_2.$$

This implies that
$$||T(h)x|| \geq 2||x||, \quad \forall x \in X_2.$$
Taking $K = 1/(Me^{\omega h})$ and $\nu = (ln\ 2)/h$ we deduce that
$$||T(t)x|| \geq \frac{1}{K}\, e^{\nu t}||x||, \quad \forall t \geq 0, \forall x \in X_2.$$
□

The first main result of this paper is

Theorem 2.3. *If the pair $(\ell^p(\mathbf{N}, X), \ell^q(\mathbf{N}, X))$ is admissible for the exponentially bounded semigroup $\mathbf{T}$ and the subspace X_1 is closed and it has a $\mathbf{T}$-invariant complement, then $\mathbf{T}$ is exponentially dichotomic.*

Proof. From our assumption $X = X_1 \oplus X_2$. We denote by P the projection corresponding to the above decomposition, i.e. $\operatorname{Im} P = X_1$ and $\operatorname{Ker} P = X_2$. Then it is easy to see that
$$T(t)P = PT(t), \quad \forall t \geq 0.$$
Applying Theorem 2.1 and Theorem 2.2 we obtain the conclusion. □

In what follows we study when the admissibility of the pair $(\ell^p(\mathbf{N}, X), \ell^q(\mathbf{N}, X))$ is a sufficient condition for exponential dichotomy.

Lemma 2.1. *Let $p, q \in [1, \infty)$ with $p \geq q$ and let $\nu > 0$. If $s \in \ell^q(\mathbf{N}, \mathbf{R}_+)$ and*
$$\delta_s, \beta_s : \mathbf{N} \to \mathbf{R}_+, \qquad \delta_s(n) = \sum_{k=0}^{n} e^{-\nu(n-k)}s(k) \quad \beta_s(n) = \sum_{k=n+1}^{\infty} e^{-\nu(k-n)}s(k)$$
then $\delta_s, \beta_s \in \ell^p(\mathbf{N}, \mathbf{R}_+)$.

Proof. It immediately follows applying Hölder's inequality. □

The second main result of this paper is

Theorem 2.4. *Let $p, q \in [1, \infty)$ with $p \geq q$. Then $\mathbf{T}$ is exponentially dichotomic if and only if the pair $(\ell^p(\mathbf{N}, X), \ell^q(\mathbf{N}, X))$ is admissible for $\mathbf{T}$ and the subspace X_1 is closed and it has a $\mathbf{T}$-invariant complement.*

Proof. Necessity. Let P be the projection and let $K, \nu > 0$ be the constants given by Definition 2.2. If $s \in \ell^q(\mathbf{N}, X)$ setting $s(-1) = 0$ we consider the sequence
$$\gamma : \mathbf{N} \to X, \qquad \gamma(n) = \sum_{k=0}^{n} T(n-k)Ps(k-1) - \sum_{k=n+1}^{\infty} T(k-n)_{|}^{-1}(I-P)s(k-1)$$
where $T(k)_{|}^{-1}$ is the inverse of the operator $T(k)_{|} : \operatorname{Ker} P \to \operatorname{Ker} P$. Using Lemma 2.1 we deduce that $\gamma \in \ell^p(\mathbf{N}, X)$ and an immediate computation shows that the pair (γ, s) verifies the equation (E_d). It follows that the pair $(\ell^p(\mathbf{N}, X),$ $\ell^q(\mathbf{N}, X))$ is admissible for $\mathbf{T}$.

It is easy to see that $\text{Im } P \subset X_1$. Conversely, let $x \in X_1$. Then

$$||x - Px|| \le Ke^{-\nu n}||T(n)(I-P)x|| \le K(1+||P||)\; e^{-\nu n}||T(n)x||,\ \forall n \in \mathbf{N}$$

so $x - Px = 0$, which yields that $x \in \text{Im } P$. It results that $X_1 = \text{Im } P$, so it is closed and it has a complement – Ker P – which is $\mathbf{T}$-invariant.

Sufficiency. It follows from Theorem 2.3. □

We present now an example in order to illustrate that for $p < q$ the exponential dichotomy of a semigroup does not imply the admissibility of the pair $(\ell^p(\mathbf{N}, X), \ell^q(\mathbf{N}, X))$.

Example 2.1. On $X = \mathbf{R}^2$ endowed with the norm $||(x_1, x_2)|| = |x_1| + |x_2|$ we define $T(t) : X \to X$ by

$$T(t)(x_1, x_2) = (e^{-t}x_1, e^t x_2), \quad \forall x = (x_1, x_2) \in \mathbf{R}^2, \forall t \ge 0.$$

Then the semigroup $\mathbf{T} = \{T(t)\}_{t\ge 0}$ is exponentially dichotomic.

Let $p, q \in [1, \infty)$ with $p < q$. Suppose by contrary that the pair $(\ell^p(\mathbf{N}, \mathbf{R}^2), \ell^q(\mathbf{N}, \mathbf{R}^2))$ is admissible for $\mathbf{T}$.

Let $r \in (p, q)$. We define $s : \mathbf{N} \to \mathbf{R}^2$, $s(n) = (0, \tilde{s}(n))$, where $\tilde{s}(n) = (n+1)^{-1/r}$. Then $s \in \ell^q(\mathbf{N}, \mathbf{R}^2) \setminus \ell^p(\mathbf{N}, \mathbf{R}^2)$. From the supposed admissibility there is $\tilde{\gamma} \in \ell^p(\mathbf{N}, \mathbf{R})$ such that

$$\frac{\tilde{\gamma}(n+1)}{e} = \tilde{\gamma}(n) + \frac{\tilde{s}(n)}{e}, \quad \forall n \in \mathbf{N}. \tag{2.2}$$

Using relation (2.2) and the fact that $\lim\limits_{n\to\infty} \tilde{\gamma}(n) = 0$ we deduce that

$$\tilde{\gamma}(n) = -e^n \sum_{k=n}^{\infty} \frac{\tilde{s}(k)}{e^{k+1}}, \quad \forall n \in \mathbf{N}. \tag{2.3}$$

Using Stolz-Cesaro theorem and relation (2.3) we have that

$$\lim_{n\to\infty} \frac{|\tilde{\gamma}(n)|}{\tilde{s}(n)} = \lim_{n\to\infty} \frac{-e^{-(n+1)}\tilde{s}(n)}{e^{-(n+1)}\tilde{s}(n+1) - e^{-n}\tilde{s}(n)} = \lim_{n\to\infty} \frac{1}{e - \ (\frac{n+1}{n+2})^{1/r}} = \frac{1}{e-1}\ .$$

Since $\tilde{s} \notin \ell^p(\mathbf{N}, \mathbf{R})$ it follows that $\tilde{\gamma} \notin \ell^p(\mathbf{N}, \mathbf{R})$, which is absurd.

In conclusion the pair $(\ell^p(\mathbf{N}, \mathbf{R}^2), \ell^q(\mathbf{N}, \mathbf{R}^2))$ is not admissible for $\mathbf{T}$.

Remark 2.2. The above example points out the fact that in Theorem 2.4 the condition $p \ge q$ is essential.

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