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ON THE STATIONARY MOTION OF A STOKES FLUID IN A THICK ELASTIC TUBE: A 3D/3D INTERACTION PROBLEM

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Abstract. We study the problem of a steady-state fluid-flexible structure interaction in 3D: a Stokes flow moving in an elastic tube. We consider periodic conditions (in the direction parallel to the tube's axis) and assume the exterior lateral surface of the flexible wall clamped. We prove the existence of a solution of the coupled problem.

1. Introduction

In this paper we study the problem of a steady-state fluid-flexible structure interaction in 3D. A stationary fluid-structure interaction problem in space dimension three was treated also in [**10**], in the case where the fluid was completely enclosed by the elastic structure, in 2D/1D in [**9**] when the equations of the fluid were coupled with those of an elastic beam. In the 2D/1D case, in [**4**] is analyzed a non-homogeneous Stokes-rod coupled problem. We take here the Stokes equations for describing the behavior of the fluid moving inside a flexible tube with thickness. The equations of linearized elasticity are used for the elastic structure. The stress acting on the structure is supposed to come from the fluid, thus this would be the stress on the structure at the interface between the two media. Fluid and solid mechanics are coupled through the wall position and the traction exerted by the fluid on the tube wall. Assuming that periodic boundary conditions are prescribed at the ends of the tube and that the exterior lateral surface of the elastic cylinder is clamped, we prove the existence of a solution for the coupled problem, for small enough data.

2. The mathematical model

We denote by $\tilde{C}_f := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ x_1^2 + x_2^2 < r_1^2\}$ the infinite undeformed cylindrical pipe occupied by the viscous, incompressible fluid, with viscosity $\nu > 0$ and by $\tilde{C}_s := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ r_1^2 < x_1^2 + x_2^2 < r_2^2\}$ the initial configuration of

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the elastic structure. Let $\widetilde{Cyl} \subset \mathbf{R}^3$ be the union of these two infinite cylinders. Thus we have $\widetilde{Cyl} = \tilde{C}_f \cup \tilde{C}_s, \quad \tilde{C}_f \cap \tilde{C}_s = \emptyset$. The fluid-structure interface is $\tilde{\Gamma}_{fs} := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ x_1^2 + x_2^2 = r_1^2\}$.

We consider for the fluid the Stokes equations and for the elastic structure the Lamé equations and we denote by $\tilde{\boldsymbol{\phi}}(\tilde{\boldsymbol{u}})$ the deformation of the interface between the two media, for which we have:

$$\tilde{\boldsymbol{\phi}}\ (\tilde{\mathbf{u}})(\mathbf{x}) = \mathbf{x} + \tilde{\mathbf{u}}(\mathbf{x}). \tag{1}$$

Observe that this mapping depends on the displacement $\tilde{\mathbf{u}}$ of the elastic structure.

The following equations describe the behavior of the elastic structure – a St. Venant-Kirchhoff material – in the small deformations regime (linearized elasticity):

$$-\mathbf{div}(\lambda \text{trace } \mathbf{e}(\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u})) = \mathbf{g} \qquad \text{in } \tilde{C}_s, \tag{2}$$

$$(\lambda \text{trace } \mathbf{e}(\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u})) \cdot \mathbf{n} = \mathbf{G} \qquad \text{on } \tilde{\Gamma}_{fs}, \tag{3}$$

$$\mathbf{u} = 0 \qquad \text{on } \tilde{\Gamma}_0, \tag{4}$$

$$\mathbf{u}(x_1, x_2, x_3) = \mathbf{u}(x_1, x_2, x_3 + \frac{2\pi}{a}) \qquad \text{in } \tilde{C}_s \tag{5}$$

where $\mathbf{n}$ is the unit outer normal vector along $\partial\tilde{C}_s \cap \partial\tilde{C}_f =: \tilde{\Gamma}_{fs}$ and $\mathbf{g}$ denotes the exterior volumic force applied to the structure part. $\tilde{\Gamma}_0$ is the exterior boundary of the elastic tube, while the interior one is obviously $\tilde{\Gamma}_{fs}$. $a \in \mathbf{R}_+^*$ is a constant such that $a << 2\pi$.

$\mathbf{G} := -\boldsymbol{\sigma}_\mathbf{f} \cdot \mathbf{n}$ is the surfacic force, which the fluid applies on the interface, with $\boldsymbol{\sigma}_\mathbf{f}^\phi := -p^\phi \cdot \mathbf{I} + 2\nu\mathbf{e}(\mathbf{v}^\phi)$ the fluid stress on the deformed interface, where $\mathbf{e}(\mathbf{v}^\phi) := \frac{1}{2}(\nabla\mathbf{v}^\phi + (\nabla\mathbf{v}^\phi)^t)$ denotes the fluid strain tensor; they are to be written on the reference (undeformed) interface. The outer normal $\mathbf{n}^\phi$ on the deformed interface $\tilde{\boldsymbol{\phi}}(\tilde{\Gamma}_{fs})$ transforms to the outer normal $\mathbf{n}$ on the reference interface Γ_{fs}.

$\lambda > 0$ and $\mu > 0$ are the Lamé constants of the St. Venant-Kirchhoff material considered and

$$\mathbf{e}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$$

is Green's strain tensor for the elastic material.

Denoting the deformed fluid domain by $\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f)$, we can write the equations for the fluid flow ($\mathbf{v}^{\tilde{\phi}}$, $p^{\tilde{\phi}}$ are the velocity, respectively the pressure of the fluid in the deformed configuration):

$$-\nu\Delta\mathbf{v}^{\tilde{\phi}} + \nabla p^{\tilde{\phi}} = \mathbf{f}^{\tilde{\phi}} \text{ in } \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f) \tag{6}$$

$$\text{div } \mathbf{v}^{\tilde{\phi}} = 0 \text{ in } \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f) \tag{7}$$

$$\mathbf{v}^{\tilde{\phi}} = 0 \text{ on } \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{\Gamma}_{fs}), \tag{8}$$

to which we add a periodicity condition for the velocity:

$$\mathbf{v}^{\tilde{\phi}}(\mathbf{x}) = \mathbf{v}^{\tilde{\phi}}(x_1, x_2, x_3 + \frac{2\pi}{a}), \ \mathbf{x} = (x_1, x_2, x_3) \in \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f). \tag{9}$$

Now, since the fluid equations are written in Eulerian coordinates in the unknown deformed domain and the structure equations are expressed in Lagrange (material) coordinates in the reference configuration, in order to study the problem in the known reference configuration we have to do some tranformations on the equations for the fluid.

We thus want to transform the unknown domain $\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f)$ into the fixed one $\tilde{C}_f$. We therefore define $\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})$ in $\tilde{C}_f$ as:

$$\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}) := \mathrm{Id} + \mathcal{L}(\mathrm{trace}_{fs}(\tilde{\mathbf{u}})),$$

where Id is the identity, trace_{fs} is the trace operator over $\tilde{\Gamma}_{fs}$ and $\mathcal{L} : \tilde{\Gamma}_{fs} \to \tilde{C}_f$ is a linear, continuous lifting. Denote $\mathbf{x}^{\tilde{\phi}} := \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x})$, $\mathbf{x} \in \tilde{C}_f$.

With the following transformations:

$$p^{\tilde{\phi}}(\mathbf{x}^{\tilde{\phi}}) = p^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x})) =: p(\tilde{\mathbf{u}}(\mathbf{x})), \qquad \mathbf{v}^{\tilde{\phi}}(\mathbf{x}^{\tilde{\phi}}) = \mathbf{v}^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x})) =: \mathbf{v}(\tilde{\mathbf{u}}(\mathbf{x}))$$

$$\mathbf{f}^{\tilde{\phi}}(\mathbf{x}^{\tilde{\phi}}) = \mathbf{f}^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x})) =: \mathbf{f}(\tilde{\mathbf{u}}(\mathbf{x})), \qquad J(\tilde{\mathbf{u}}) := \det\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}) \tag{10}$$

and

$$\mathbf{n}^{\tilde{\phi}} = \frac{\mathrm{cof}\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}) \cdot \mathbf{n}}{||\mathrm{cof}\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}) \cdot \mathbf{n}||}, \qquad d\sigma^{\tilde{\phi}} = ||\mathrm{cof}\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})||d\sigma$$

$$\mathbf{M} := \mathrm{cof}\nabla\tilde{\boldsymbol{\phi}} \text{ (cofactor matrix)}, \qquad \mathbf{N} := (\nabla\tilde{\boldsymbol{\phi}})^{-1} \cdot \mathrm{cof}\nabla\tilde{\boldsymbol{\phi}}, \tag{11}$$

the system (6)–(9) becomes (when written in the reference configuration):

$$-\nu \mathrm{div}\ ((\mathbf{N}\nabla)\mathbf{v}) + (\mathbf{M}\nabla)p = \mathbf{f}^J \qquad \text{in } \tilde{C}_f, \tag{12}$$

$$\mathrm{div}\ (\mathbf{M}^t\mathbf{v}) = 0 \qquad \text{in } \tilde{C}_f, \tag{13}$$

$$\mathbf{v} = 0 \qquad \text{on } \tilde{\Gamma}_{fs} \tag{14}$$

where $\mathbf{f}^J(\tilde{\mathbf{u}}) := \mathbf{f}(\tilde{\mathbf{u}})J(\tilde{\mathbf{u}})$ and the periodicity condition

$$\mathbf{v}(\mathbf{x}) = \mathbf{v}(x_1, x_2, x_3 + \frac{2\pi}{a}) \text{ in } \tilde{C}_f. \tag{15}$$

We shall keep in mind that the functions above are related to the (initial) displacement $\tilde{\mathbf{u}}$, but we omit it in the writing.

In order to analyze the above three-dimensional linear problems with mixed boundary conditions, we proceed like in [**11**], treating equivalent problems with homogeneous Dirichlet boundary conditions on tori.

Let T be a torus in $\mathbf{R}^3$. We transform the cylinders

$$C_f := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 0 < x_3 < \frac{2\pi}{a},\ x_1^2 + x_2^2 < r_1^2\}$$

and

$$C_s := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 0 < x_3 < \frac{2\pi}{a},\ r_1^2 < x_1^2 + x_2^2 < r_2^2\}$$

(having the interface $\Gamma_{fs} := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 0 < x_3 < \frac{2\pi}{a},\ x_1^2 + x_2^2 = r_1^2\}$) into the tori T_f, respectively T_s by identifying the top and the bottom parts of the corresponding cylinders: the disk $\{x_3 = 0,\ x_1^2 + x_2^2 \le r_1^2\}$ with the disk

$\{x_3 = \frac{2\pi}{a},\ x_1^2 + x_2^2 \le r_1^2\}$, respectively $\{x_3 = 0,\ r_1^2 \le x_1^2 + x_2^2 \le r_2^2\}$ with $\{x_3 = \frac{2\pi}{a},\ r_1^2 \le x_1^2 + x_2^2 \le r_2^2\}$.

We consider the mapping $[0, L] \ni s \mapsto \delta(s) = \varphi,\ \delta(s) = \frac{2\pi s}{L},\ \forall s \in [0, L]$, where we take $L = \frac{2\pi}{a}$.

Then the mapping transforming the cylinder $Cyl = C_f \cup C_s$ into the torus is of the form:

$$\mathbf{t} : Cyl \subset \mathbf{R}^3 \to T \subset \mathbf{R}^3 \tag{16}$$

$$t_1(\mathbf{x}) = (\frac{1}{a} + x_1)\cos(ax_3); \quad t_2(\mathbf{x}) = (\frac{1}{a} + x_1)\sin(ax_3), \quad t_3(\mathbf{x}) = x_2. \tag{17}$$

We denote by T_f and T_s the fluid domain, respectively the domain of the elastic structure, both transformed by (16), (17).

Thus, the system (12)–(15) is equivalent to

$$\begin{aligned} -\nu\text{div }((\mathcal{N}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f\cdot\nabla)\mathbf{v}(\tilde{\mathbf{u}})\cdot\boldsymbol{\gamma}^f) + \nu(\mathcal{N}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f\cdot\nabla)\mathbf{v}(\tilde{\mathbf{u}})\cdot\text{div }\boldsymbol{\gamma}^f & \\ +(\mathcal{M}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f\cdot\nabla)p(\tilde{\mathbf{u}}) = \widetilde{\mathbf{f}^J}(\tilde{\mathbf{u}}) & \quad \text{in } T_f \\ \mathcal{M}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f : \nabla\mathbf{v}(\tilde{\mathbf{u}}) = 0 & \quad \text{in } T_f \\ \mathbf{v}(\tilde{\mathbf{u}}) = 0 & \quad \text{on } \partial T_f, \end{aligned} \tag{18}$$

where $\mathcal{M}(\tilde{\mathbf{u}})(\mathbf{X}) := \text{cof }(\boldsymbol{\gamma}^f\cdot\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}(\mathbf{X})))$,
$\mathcal{N}(\tilde{\mathbf{u}})(\mathbf{X}) := (\boldsymbol{\gamma}^f\cdot\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}))^{-1}\cdot\text{cof }(\boldsymbol{\gamma}^f\cdot\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}))$
and $\gamma_{ij}^f(\mathbf{x}) := \frac{\partial t_j}{\partial x_i}(\mathbf{x}),\ i, j = 1, 2, 3$
(we make the notation $\mathbf{t}(\mathbf{x}) = \mathbf{X}$ and $\gamma_{ij}^f(\mathbf{X}) := \gamma_{ij}^f \circ \mathbf{t}^{-1}(\mathbf{X})$). We also have

$$\begin{aligned} \widetilde{\mathbf{f}^J}(\tilde{\mathbf{u}})(\mathbf{X}) &:= (\mathbf{f}(\tilde{\mathbf{u}})\circ\mathbf{t}^{-1})(\mathbf{X})J((\boldsymbol{\gamma}^f\nabla)\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}(\mathbf{X}))); \\ \mathbf{v}(\tilde{\mathbf{u}})(\mathbf{X}) &:= (\mathbf{v}(\tilde{\mathbf{u}})\circ\mathbf{t}^{-1})(\mathbf{X}); \\ p(\tilde{\mathbf{u}})(\mathbf{X}) &:= (p(\tilde{\mathbf{u}})\circ\mathbf{t}^{-1})(\mathbf{X}). \end{aligned}$$

Analogously, the system (2)–(5) is equivalent to:

$$-\text{div }(\lambda\text{tr }\mathbf{E}(\boldsymbol{\gamma}^s\cdot(\nabla\mathbf{U}(\tilde{\mathbf{u}}))^t)\cdot\mathbf{I} + 2\mu\mathbf{E}(\boldsymbol{\gamma}^s\cdot(\nabla\mathbf{U}(\tilde{\mathbf{u}}))^t)) = K(\boldsymbol{\gamma}^s)^{-1}\cdot\mathbf{g} \quad \text{in } T_s \tag{19}$$

$$(\lambda\text{tr }\mathbf{E}(\boldsymbol{\gamma}^s\cdot(\nabla\mathbf{U}(\tilde{\mathbf{u}}))^t)\cdot\mathbf{I} + 2\mu\mathbf{E}(\boldsymbol{\gamma}^s\cdot(\nabla\mathbf{U}(\tilde{\mathbf{u}}))^t)\cdot\mathbf{n} = K\mathbf{G} \quad \text{on } \Gamma_{fs} \tag{20}$$

$$\mathbf{U}(\tilde{\mathbf{u}}) = 0 \quad \text{on } \Gamma_0, \tag{21}$$

where $K := \lambda + \mu$ and $\mathbf{E}(\boldsymbol{\gamma}^s\cdot\nabla\mathbf{U}^t) := \frac{1}{2}(\boldsymbol{\gamma}^s\cdot\nabla\mathbf{U}^t + (\boldsymbol{\gamma}^s\cdot\nabla\mathbf{U}^t)^t)$.
Γ_0 denotes the exterior boundary surface of the elastic torus. Observe that

$$\mathbf{G}(\tilde{\mathbf{u}}) = p(\tilde{\mathbf{u}})\mathcal{M}(\tilde{\mathbf{u}})\cdot\mathbf{n} - \nu(\mathcal{N}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f\nabla)\mathbf{v}(\tilde{\mathbf{u}})\cdot\mathbf{n}.$$

Here we make the same convention of notation, where

$$\gamma_{mj}^s(\mathbf{x}) = (\lambda + \mu)\frac{\partial t_j}{\partial x_m}(\mathbf{x}), \qquad j, m = 1, 2, 3$$

and

$$\mathbf{g}(\mathbf{X}) = (\mathbf{g}\circ\mathbf{t}^{-1})(\mathbf{X}), \quad \mathbf{G}(\mathbf{X}) = (\mathbf{G}\circ\mathbf{t}^{-1})(\mathbf{X}).$$

Now, having all equations set in a known configuration, we want to prove the existence of a solution to the coupled problem. This will be done in the following way: for a given displacement $\tilde{\mathbf{u}}$, we split the equations for the fluid and the equations for the structure, then prove for each of these systems the existence of a unique solution. This means that we prove the existence and uniqueness for the solution of the fluid equations (for $\tilde{\mathbf{u}}$ known), then solve the problem for the elastic structure as a mixed Dirichlet-Neumann boundary value problem (with the right-hand side known, since we would have solved the fluid problem). Then the existence of the solution for the coupled problem will be done by a fixed point theorem.

3. The fluid problem

Let $p \in \mathbf{R}$ with $3 < p < \infty$. We consider the following system

$$\begin{aligned} &-\nu \mathrm{div}\, ((\mathcal{N}\boldsymbol{\gamma}^f \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\gamma}^f) + \nu(\mathcal{N}\boldsymbol{\gamma}^f \cdot \nabla)\mathbf{v} \cdot \mathrm{div}\, \boldsymbol{\gamma}^f \\ &\qquad\qquad\qquad\qquad + (\mathcal{M}\boldsymbol{\gamma}^f \cdot \nabla)p = \hat{\mathbf{f}} \quad \text{in } T_f, \\ &\qquad\qquad\qquad\qquad\quad \mathcal{M}\boldsymbol{\gamma}^f : \nabla\mathbf{v} = 0 \quad \text{in } T_f, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \mathbf{v} = 0 \quad \text{on } \partial T_f, \end{aligned} \tag{22}$$

which is of the same type as (18). We keep here the notations for the matrices in (18), but for the matrices involved in (22) we forget about the dependence on some displacement $\tilde{\mathbf{u}}$ and only assume that the following hypotheses are satisfied:

(H_1) $\mathcal{N}$ is a symmetric and positive definite matrix such that coeff $(\mathcal{N}) \in \mathbf{W}^{1,p}(T_f)$, $\boldsymbol{\gamma}^f$ is a regular enough matrix; also assume that $\exists c > 0$ a constant such that $\mathcal{N}\boldsymbol{\gamma}^f \geq c\mathbf{I}$;

(H_2) $\mathcal{M}$ is invertible in $\mathbf{W}^{1,p}(T_f)$ and $\exists\, \boldsymbol{\Theta}$ with $\mathcal{M} =$ cof $\nabla\boldsymbol{\Theta}$;

(H_3) $\exists\, C > 0$ a constant with $\|\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f\|_{\mathbf{W}^{1,p}(T_f)} \leq C$, $\|\mathbf{I} - (\mathcal{M}\boldsymbol{\gamma}^f)^t\|_{\mathbf{W}^{1,p}(T_f)} \leq C$ and $\|\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f\|_{\mathbf{W}^{1,p}(T_f)} \leq C$.

Theorem 3.1. *Let $\hat{\mathbf{f}} \in \mathbf{L}^p(T_f)$. There exists a unique solution $(\mathbf{v}, p)$ of the system* (22) *in $(\mathbf{W}^{2,p}(T_f) \cap \mathbf{W}^{1,p}_{0,\partial T_f}(T_f)) \times W^{1,p}(T_f)$, with:*

$$||\mathbf{v}||_{\mathbf{W}^{2,p}(T_f)} + ||p||_{W^{1,p}(T_f)} \leq C_1||\hat{\mathbf{f}}||_{\mathbf{L}^p(T_f)} \tag{23}$$

(C_1 is a positive constant).

Proof. The existence of a (unique) solution $(\mathbf{v}, p) \in \mathbf{H}^1_{0,\Gamma_{fs}}(T_f) \times L^2_0(T_f)$ of (22) can be shown e.g., like in [**6**]. Concerning the existence of a unique pressure, we verify a corresponding inf-sup condition:

$$\begin{aligned} &\exists k > 0 \ \ \text{(constant) s.t.} \\ &\sup_{\boldsymbol{\psi} \in \mathbf{H}^1_0(T_f)} \frac{\int_{T_f} \tau \mathcal{M}\boldsymbol{\gamma}^f : \nabla\boldsymbol{\psi}}{||\boldsymbol{\psi}||_{\mathbf{H}^1_0(T_f)}} \geq k\|\tau\|_{L^2(T_f)}, \quad \forall \tau \in L^2_0(T_f) \end{aligned} \tag{24}$$

Indeed, it is known see [**7**, ch. III, s. 3] that $\forall\tau\in L^2_0(T_f)$ $\exists\widehat{\boldsymbol{\psi}}\in\mathbf{H}^1_0(T_f)$ such that

$$\operatorname{div}\widehat{\boldsymbol{\psi}}=\tau \text{ and } \|\widehat{\boldsymbol{\psi}}\|_{\mathbf{H}^1(T_f)}\le C\|\tau\|_{L^2(T_f)}.$$

Thus, to any given $\tau\in L^2_0(T_f)$ we associate a $\widehat{\boldsymbol{\psi}}$ and we take $\boldsymbol{\psi}$ such that $\nabla\boldsymbol{\psi}=(\mathcal{M}\boldsymbol{\gamma}^f)^{-t}\nabla\widehat{\boldsymbol{\psi}}$. It follows that $\boldsymbol{\psi}\in\mathbf{H}^1_0(T_f)$ and (using the above estimate for $\|\widehat{\boldsymbol{\psi}}\|$) that condition (24) above is satisfied, with the constant k depending on $\|(\mathcal{M}\boldsymbol{\gamma}^f)^{-t}\|_{\mathbf{L}^\infty(T_f)}$; the rest is classical.

The regularity stated in the theorem and the estimate (23) will be proved in what follows.

Thus, let us consider the sequence $S(n)$:

$$\begin{aligned}
-\nu\operatorname{div}(\nabla\mathbf{v}^n\cdot\boldsymbol{\gamma}^f)+\nu\nabla\mathbf{v}^n\cdot\operatorname{div}\boldsymbol{\gamma}^f+\nabla p^n&=\hat{\mathbf{f}}-\nu\operatorname{div}(((\mathbf{I}-\mathcal{N}\boldsymbol{\gamma}^f)\nabla)\mathbf{v}^{n-1})\cdot\boldsymbol{\gamma}^f)\\
&\quad+\nu((\mathbf{I}-\mathcal{N}\boldsymbol{\gamma}^f)\nabla)\mathbf{v}^{n-1}\cdot\operatorname{div}\boldsymbol{\gamma}^f+(\mathbf{I}-\mathcal{M}\boldsymbol{\gamma}^f)\nabla p^{n-1} \quad \text{in } T_f,\\
\mathbf{I}:\nabla\mathbf{v}^n&=(\mathbf{I}-\mathbf{M}\boldsymbol{\gamma}^f):\nabla\mathbf{v}^{n-1} \quad \text{in } T_f,\\
\mathbf{v}^n&=0 \quad \text{on } \Gamma_{fs},
\end{aligned}\tag{25}$$

having the first term $S(0)$:

$$\begin{aligned}
-\nu\operatorname{div}(\nabla\mathbf{v}^0\cdot\boldsymbol{\gamma}^f)+\nu\nabla\mathbf{v}^0\cdot\operatorname{div}\boldsymbol{\gamma}^f+\nabla p^0&=\hat{\mathbf{f}} \quad \text{in } T_f\\
\operatorname{div}\mathbf{v}^0&=0 \quad \text{in } T_f\\
\mathbf{v}^0&=0 \quad \text{on } \Gamma_{fs}.
\end{aligned}\tag{26}$$

Problems of this type are treated in [**6**]. Existence, uniqueness and regularity of a solution $\mathbf{v}^0\in\mathbf{H}^2(T_f)$, $p^0\in H^1(T_f)$ can be proved similarly (see also [**3**]). Since the ellipticity condition in [**2**] is satisfied for the system (26), it follows (see [**8**]) that $\mathbf{v}^0\in\mathbf{W}^{2,p}(T_f)$, $p^0\in W^{1,p}(T_f)$.

Then for any positive integer n, $(\mathbf{v}^n,p^n)\in(\mathbf{W}^{2,p}(T_f)\cap\mathbf{W}^{1,p}_0(T_f))\times W^{1,p}(T_f)$ and it converges to the unique solution of the system (18). Indeed, we argument here by mathematical induction on n.

Assuming that $(\mathbf{v}^n,p^n)\in(\mathbf{W}^{2,p}(T_f)\cap\mathbf{W}^{1,p}_0(T_f))\times W^{1,p}(T_f)$, it follows that $\hat{\mathbf{f}}-\nu\operatorname{div}(((\mathbf{I}-\mathcal{N}\boldsymbol{\gamma}^f)\nabla)\mathbf{v}^n)\cdot\boldsymbol{\gamma}^f)+\nu((\mathbf{I}-\mathcal{N}\boldsymbol{\gamma}^f)\nabla)\mathbf{v}^n\cdot\operatorname{div}\boldsymbol{\gamma}^f+(\mathbf{I}-\mathcal{M}\boldsymbol{\gamma}^f)\nabla p^n\in L^p(T_f)$, by the hypotheses we have made and the fact that $W^{1,p}$ is a Banach algebra for $p>3$.

Also by (H_1), (H_2), it follows that $(\mathbf{I}-\mathbf{M}\boldsymbol{\gamma}^f):\nabla\mathbf{v}^n\in\mathbf{W}^{1,p}(T_f)$ and it has zero mean over T_f, since $\mathbf{v}^n$ satisfies the boundary condition in (25). It follows then that $S(n+1)$ has a unique solution $(\mathbf{v}^{n+1},p^{n+1})\in(\mathbf{W}^{2,p}(T_f)\cap\mathbf{W}^{1,p}_0(T_f))\times W^{1,p}(T_f)$ and thus the induction on the regularity of the solutions for the fluid system is complete.

Let us now prove that the solution of $S(n)$ converges to the unique solution of (22). This is done by showing that (v^n,p^n) is a Cauchy sequence in $\mathbf{W}^{2,p}(T_f)\times W^{1,p}(T_f)$ and by passing to the limit for $n\to\infty$ in $S(n)$.

$S(n+1) - S(n)$:

$$\begin{aligned} &-\nu \mathrm{div}(\nabla(\mathbf{v}^{n+1} - \mathbf{v}^n) \cdot \boldsymbol{\gamma}^f) + \nu \nabla(\mathbf{v}^{n+1} - \mathbf{v}^n) \mathrm{div}\, \boldsymbol{\gamma}^f + \nabla(p^{n+1} - p^n) \\ &\quad = -\nu \mathrm{div}(((\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f)\nabla)(\mathbf{v}^n - \mathbf{v}^{n-1}) \cdot \boldsymbol{\gamma}^f) \\ &\qquad + \nu((\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f)\nabla)(\mathbf{v}^n - \mathbf{v}^{n-1}) \mathrm{div}\, \boldsymbol{\gamma}^f \\ &\qquad + ((\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f)\nabla)(p^n - p^{n-1}) \qquad \text{in } T_f \end{aligned} \tag{27}$$

$$\mathrm{div}\, (\mathbf{v}^{n+1} - \mathbf{v}^n) = (\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f) : \nabla(\mathbf{v}^n - \mathbf{v}^{n-1}) \qquad \text{in } T_f \tag{28}$$

$$\mathbf{v}^{n+1} - \mathbf{v}^n = 0 \qquad \text{on } \Gamma_{fs}. \tag{29}$$

Upon using again for this Stokes system estimates of the kind of those in [**6**], one gets:

$$\begin{aligned} ||\mathbf{v}^{n+1} - \mathbf{v}^n||_{\mathbf{W}^{2,p}(T_f)} &+ ||p^{n+1} - p^n||_{W^{1,p}(T_f)} \\ &\le \text{const } \{||\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f||_{\mathbf{W}^{1,p}(T_f)} ||\mathbf{v}^n - \mathbf{v}^{n-1}||_{\mathbf{W}^{2,p}(T_f)} \\ &\quad + ||\mathbf{I} - (\mathcal{M}\boldsymbol{\gamma}^f)^t||_{\mathbf{W}^{1,p}(T_f)} ||\mathbf{v}^n - \mathbf{v}^{n-1}||_{\mathbf{W}^{2,p}(T_f)} \\ &\quad + ||\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f||_{\mathbf{W}^{1,p}(T_f)} ||p^n - p^{n-1}||_{W^{1,p}(T_f)}\}, \end{aligned}$$

const > 0 being a constant independent of n, $\mathcal{N}$, $\mathcal{M}$, but depending on the bound of $\boldsymbol{\gamma}^f$.

Now we choose C in the hypotheses we made in order to satisfy const $\cdot\, C < 1$ and it follows that:

$$\begin{aligned} \|\mathbf{v}^{n+1} - \mathbf{v}^n\|_{\mathbf{W}^{2,p}(T_f)} &+ \|p^{n+1} - p^n\|_{W^{1,p}(T_f)} \\ &\le C_{prod}\{\|\mathbf{v}^n - \mathbf{v}^{n-1}\|_{\mathbf{W}^{2,p}(T_f)} + \|p^n - p^{n-1}\|_{W^{1,p}(T_f)}\}, \end{aligned}$$

with $0 < C_{prod} < 1$.

Consequently, the sequence $(\mathbf{v}^n, p^n)$ converges in $\mathbf{W}^{2,p}(T_f) \times W^{1,p}(T_f)$. Thus, there exists the limit $(\mathbf{v}_C, p_C) \in (\mathbf{W}^{2,p}(T_f) \cap \mathbf{W}_0^{1,p}(T_f)) \times W^{1,p}(T_f)$ such that

$$\mathbf{v}^n \to \mathbf{v}_C \text{ in } \mathbf{W}^{2,p}(T_f) \text{ as } n \to \infty$$

and

$$p^n \to p_C \text{ in } W^{1,p}(T_f) \text{ as } n \to \infty.$$

Passing now to the limit in $S(n)$, we conclude that $(\mathbf{v}_C, p_C)$ is the unique solution of (22) and thus $\mathbf{v}_C = \mathbf{v}$ and $p_C = p$.

We still have to prove the inequality (23). Using the above estimations, observe that we can write

$$\begin{aligned} \|\mathbf{v}^n\|_{\mathbf{W}^{2,p}(T_f)} &+ \|p^n\|_{W^{1,p}(T_f)} \\ &\le C\|\hat{\mathbf{f}}\|_{\mathbf{L}^p(T_f)} + C_{prod}\{\|\mathbf{v}^{n-1}\|_{\mathbf{W}^{2,p}(T_f)} + \|p^{n-1}\|_{W^{1,p}(T_f)}\}, \end{aligned}$$

where $C > 0$ is a constant. For $n \to \infty$ this estimation becomes

$$||\mathbf{v}||_{\mathbf{W}^{2,p}(T_f)} + ||p||_{W^{1,p}(T_f)} \le C(1 - C_{prod})^{-1}||\hat{\mathbf{f}}||_{\mathbf{L}^p(T_f)}.$$

Thus we obtain the inequality (23), with $C_1 = C(1 - C_{prod})^{-1}$. □

4. The flexible structure

Consider now the following system for the flexible structure:

$$-\text{div }(\lambda \text{tr } \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla U^t) \cdot \mathbf{I} + 2\mu \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla U^t)) = K(\boldsymbol{\gamma}^s)^{-1}\mathbf{g} \qquad \text{in } T_s \tag{30}$$

$$(\lambda \text{tr } \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla U^t) \cdot \mathbf{I} + 2\mu \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla U^t)) \cdot \mathbf{n} = K\mathbf{G} \qquad \text{on } \Gamma_{fs} \tag{31}$$

$$\mathbf{U} = 0 \qquad \text{on } \Gamma_0, \tag{32}$$

where $\mathbf{g}$ is a given volumic force, while $\mathbf{G}$ is a given surfacic force (which is actually related to the fluid stress tensor). Then we have:

Theorem 4.1. *For $p \in \mathbf{R}$, $3 < p < \infty$ let $\mathbf{g} \in \mathbf{L}^p(T_s)$ and $\mathbf{G} \in \mathbf{W}^{1-1/p,p}(\Gamma_{fs})$. Then there exists a unique solution $\mathbf{U} \in \mathbf{W}^{2,p}(T_s) \cap \mathbf{W}^{1,p}_{0,\Gamma_0}(T_s)$ of the system (30)–(32) and it satisfies:*

$$||\mathbf{U}||_{\mathbf{W}^{2,p}(T_s)} \leq \text{const } (||\mathbf{g}||_{\mathbf{L}^p(T_s)} + ||\mathbf{G}||_{\mathbf{W}^{1-1/p,p}(\Gamma_{fs})}). \tag{33}$$

Proof. The problem corresponding to the system (30)–(32) is equivalent to the problem of finding a solution $\mathbf{U}$ of the following equation:

$$\mathbf{A}(\mathbf{U}, \boldsymbol{\psi}) = \mathbf{L}(\boldsymbol{\psi}), \ \forall \boldsymbol{\psi} \in \mathbf{V}, \tag{34}$$

where

$$\mathbf{A}(\mathbf{U}, \boldsymbol{\psi}) := \int_{T_s} \mathbf{S}(\mathbf{U}) : \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla \boldsymbol{\psi}^t),$$

with $\mathbf{S}(\mathbf{U}) := \lambda \text{tr } \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla \mathbf{U}^t)\mathbf{I} + 2\mu \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla \mathbf{U}^t)$ and

$$\mathbf{L}(\boldsymbol{\psi}) := \int_{T_s} K\mathbf{g} \cdot \boldsymbol{\psi} d\mathbf{y} + \int_{\Gamma_{fs}} K\mathbf{G} \cdot \boldsymbol{\psi} d\sigma.$$

$\mathbf{V}$ denotes a space of smooth enough vector-valued functions $\boldsymbol{\psi} : \bar{T}_s \to \mathbf{R}^3$ that vanish on Γ_0. We take here $\mathbf{V} := \{\boldsymbol{\psi} \in \mathbf{H}^1(T_s) \ : \ \boldsymbol{\psi} = 0 \text{ on } \Gamma_0\} = \mathbf{H}^1_{0,\Gamma_0}(T_s)$.

Now, $\mathbf{A}$ is a continuous, bilinear form that is also $\mathbf{V}$-elliptic (via Korn's inequality) and $\mathbf{L}$ is a continuous linear form defined on $\mathbf{V}$. By the Lax-Milgram lemma it follows that there is one and only one function $\mathbf{U}$ in the space $\mathbf{V}$, solution of (34). Moreover, using the regularity of the data and the regularity properties of the mixed Neumann-Dirichlet boundary value problem see [**5**, Th. 6.3.6 and the remarks after it], it follows that $\mathbf{U} \in \mathbf{W}^{2,p}(T_s)$ and the estimate (33) holds. □

5. The coupling

We now come to the coupled problem. The following is the main result for the fluid-structure interaction problem (on the torus):

Theorem 5.1. *Let $p \in \mathbf{R}$ with $3 < p < \infty$, $\mathbf{f}^{\tilde{\phi}} \in \mathbf{L}^p(\mathbf{R}^3)$ and $\mathbf{g} \in \mathbf{L}^p(T_s)$. Assume there exists a constant $\chi > 0$ with:*

$$C_{coupl}(\|\mathbf{f}^{\tilde{\phi}}\|_{\mathbf{L}^p(\mathbf{R}^3)} + \|\mathbf{g}\|_{\mathbf{L}^p(T_s)}) \leq \chi. \tag{35}$$

Then there exists a solution $(\mathbf{v}, p, \mathbf{U})$ of the equations (18), (19)–(21), with $\mathbf{v} \in \mathbf{W}^{2,p}(T_f) \cap \mathbf{W}^{1,p}_0(T_f)$, $p \in W^{1,p}(T_f)$ and $\mathbf{U}$ sufficiently small in $\mathbf{W}^{2,p}(T_s)$.

Proof. The idea of the proof is the following: let

$$\mathcal{U}_\chi := \{\tilde{\mathbf{u}} \in \mathbf{W}^{2,p}(T_s) \ : \ \|\tilde{\mathbf{u}}\|_{\mathbf{W}^{2,p}(T_s)} \leq \chi\}.$$

The mapping

$$\mathcal{U}_\chi \ni \tilde{\mathbf{u}} \overset{\mathcal{A}}{\mapsto} \mathbf{U}(\tilde{\mathbf{u}}) \in \mathbf{W}^{2,p}(T_s)$$

has at least one fixed point.

Let $\tilde{\mathbf{u}} \in \mathcal{U}_\chi$. Then $\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})$ is an invertible matrix in $\mathbf{W}^{1,p}(T_f)$ $(p > 3)$ and (for χ sufficiently small) we have $\det\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x}) > 0$, thus the deformation $\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}) = \mathrm{Id} + \tilde{\mathbf{u}}$ is orientation preserving [**5**, Theorem 5.5] and injective. Indeed, by the mean value theorem:

$$\begin{aligned} \|\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}(\mathbf{x}_1)) - \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}(\mathbf{x}_2))\| &= \|\mathbf{x}_1 - \mathbf{x}_2 + \tilde{\mathbf{u}}(\mathbf{x}_1) - \tilde{\mathbf{u}}(\mathbf{x}_2)\| \\ &\geq \|\mathbf{x}_1 - \mathbf{x}_2\| - \sup\|\nabla\tilde{\mathbf{u}}\| \cdot \|\mathbf{x}_1 - \mathbf{x}_2\| \\ &> (1 - C(T_f))\|\mathbf{x}_1 - \mathbf{x}_2\| \ (\text{for } \mathbf{x}_1 \neq \mathbf{x}_2), \end{aligned} \tag{36}$$

$C(T_f)$ being the constant in the orientation preserving theorem.

The solution $(\mathbf{v}(\tilde{\mathbf{u}}), p(\tilde{\mathbf{u}}))$ of (18) satisfies the same type of equations as those in Theorem 3.1, with $\hat{\mathbf{f}} := \widetilde{\mathbf{f}^J}$, $\mathcal{N} := \mathcal{N}(\tilde{\mathbf{u}})$, $\mathcal{M} := \mathcal{M}(\tilde{\mathbf{u}})$ (see the Appendix for the properties of $\mathbf{M}$ and $\mathbf{N}$ in (11); thus, since $\boldsymbol{\gamma}^f$ in (18) *is* regular, the hypotheses in Theorem 3.1 are satisfied for $\mathcal{N}(\tilde{\mathbf{u}})$ and $\mathcal{M}(\tilde{\mathbf{u}})$, too). Then by Theorem 3.1 it follows that for any $\tilde{\mathbf{u}} \in \mathcal{U}_\chi$, $(\mathbf{v}(\tilde{\mathbf{u}}), p(\tilde{\mathbf{u}})) \in W^{2,p}(T_f) \times W^{1,p}(T_f)$ and

$$\|\mathbf{v}(\tilde{\mathbf{u}})\|_{\mathbf{W}^{2,p}(T_f)} + \|p(\tilde{\mathbf{u}})\|_{W^{1,p}(T_f)} \leq C_1\|\widetilde{\mathbf{f}^J}(\tilde{\mathbf{u}})\|_{\mathbf{L}^p(T_f)},$$

thus also

$$\|\mathbf{v}(\tilde{\mathbf{u}})\|_{\mathbf{W}^{2,p}(T_f)} + \|p(\tilde{\mathbf{u}})\|_{W^{1,p}(T_f)} \leq C(C_1, \chi)\|\mathbf{f}^{\tilde{\phi}}\|_{\mathbf{L}^p(\mathbf{R}^3)}. \tag{37}$$

We have $\mathbf{G}(\tilde{\mathbf{u}}) = p(\tilde{\mathbf{u}})\mathcal{M}(\tilde{\mathbf{u}}) \cdot \mathbf{n} - \nu(\mathcal{N}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f\nabla)\mathbf{v}(\tilde{\mathbf{u}}) \cdot \mathbf{n} \in \mathbf{W}^{1-1/p,p}(\Gamma_{fs})$.

$\mathbf{G}$ in (31) satisfies:

$$\begin{aligned} \|\mathbf{G}\|_{\mathbf{W}^{1-1/p,p}(\Gamma_{fs})} &\leq C(\|\mathbf{v}(\tilde{\mathbf{u}})\|_{\mathbf{W}^{2,p}(T_f)} + \|p(\tilde{\mathbf{u}})\|_{W^{1,p}(T_f)}) \\ &\leq C(C_1, \chi)\|\mathbf{f}^{\tilde{\phi}}\|_{\mathbf{L}^p}). \end{aligned}$$

Now apply Theorem 4.1 to get the existence of a unique solution $\mathbf{U}(\tilde{\mathbf{u}}) \in \mathbf{W}^{2,p}(T_s)$ of (19)–(21)) with

$$\|\mathbf{U}(\tilde{\mathbf{u}})\|_{\mathbf{W}^{2,p}(T_s)} \leq \mathrm{const}(C(C_1, \chi)\|\mathbf{f}^{\tilde{\phi}}\|_{\mathbf{L}^p} + \|\mathbf{g}\|_{\mathbf{L}^p(T_s)}). \tag{38}$$

We have thus constructed the mapping $\mathcal{U}_\chi \ni \tilde{\mathbf{u}} \overset{\mathcal{A}}{\mapsto} \mathbf{U}(\tilde{\mathbf{u}}) \in \mathcal{U}_\chi \subset \mathbf{W}^{2,p}(T_s)$. This mapping has a fixed point, by the theorem of Schauder:

- $\mathcal{A}$ is weakly sequentially continuous on $\mathbf{W}^{2,p}(T_s)$.

Indeed, let $\tilde{\mathbf{u}}_n \in \mathcal{U}_\chi$ with $\tilde{\mathbf{u}}_n \overset{n\to\infty}{\rightharpoonup} \tilde{\mathbf{u}}$ in $\mathbf{W}^{2,p}(T_s)$; by (37) and (38) it follows that $(\mathbf{v}(\tilde{\mathbf{u}}_n), p(\tilde{\mathbf{u}}_n), \mathbf{U}(\tilde{\mathbf{u}}_n))$ is (independently on n) bounded in $\mathbf{W}^{2,p}(T_f) \times W^{1,p}(T_f) \times \mathbf{W}^{2,p}(T_s)$, thus $(\exists)(\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{U}}) \in \mathbf{W}^{2,p}(T_f) \times W^{1,p}(T_f) \times \mathbf{W}^{2,p}(T_s)$ and there exists a subsequence $(\tilde{\mathbf{u}}_{nk})_k \subset (\tilde{\mathbf{u}}_n)_n$ with

$$\begin{aligned} \mathbf{v}(\tilde{\mathbf{u}}_{nk}) &\overset{k\to\infty}{\rightharpoonup} \hat{\mathbf{v}} && \text{in } \mathbf{W}^{2,p}(T_f) \\ p(\tilde{\mathbf{u}}_{nk}) &\overset{k\to\infty}{\rightharpoonup} \hat{p} && \text{in } W^{1,p}(T_f) \\ \mathbf{U}(\tilde{\mathbf{u}}_{nk}) &\overset{k\to\infty}{\rightharpoonup} \hat{\mathbf{U}} && \text{in } \mathbf{W}^{2,p}(T_s). \end{aligned}$$

We have to show that $\mathbf{U}(\tilde{\mathbf{u}}) = \hat{\mathbf{U}}$ and this will prove the weak continuity of $\mathcal{A}$, for then the sequence $\mathbf{U}(\tilde{\mathbf{u}}_n)$ will converge to $\mathbf{U}(\tilde{\mathbf{u}})$ in the weak topology of $\mathbf{W}^{2,p}(T_s)$, $\mathbf{U}(\tilde{\mathbf{u}})$ being the unique solution of (19)–(21) for $\tilde{\mathbf{u}}$ given.

We intend to pass to the limit in the equations satisfied by $\mathbf{v}(\tilde{\mathbf{u}}_{nk})$, $p(\tilde{\mathbf{u}}_{nk})$, $\mathbf{U}(\tilde{\mathbf{u}}_{nk})$.

Now, $\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}_{nk}) \overset{k\to\infty}{\rightharpoonup} \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})$ in $\mathbf{W}^{2,p}(T_f)$. Since $p > 3$, $\mathbf{W}^{2,p}(T)$ is compactly imbedded in $\mathbf{C}^1(\bar{T})$ and therefore there exists a subsequence of $(\tilde{\mathbf{u}}_{nk})$, still denoted by $(\tilde{\mathbf{u}}_{nk})$ such that

$$\tilde{\mathbf{u}}_{nk} \overset{k\to\infty}{\to} \tilde{\mathbf{u}} \text{ in } \mathbf{C}^1(\bar{T}_s)$$

and

$$\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}_{nk}) \overset{k\to\infty}{\to} \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}) \text{ in } \mathbf{C}^1(\bar{T}_f).$$

It follows (see the definitions of $\mathcal{M}$ and $\mathcal{N}$ after (18)) that $\mathcal{M}(\tilde{\mathbf{u}}_{nk}) \overset{k\to\infty}{\to} \mathcal{M}(\tilde{\mathbf{u}})$ in $\mathbf{C}^0(\bar{T}_f)$, since $\mathbf{W}^{1,p}(T_f) \hookrightarrow \mathbf{C}^0(\bar{T}_f)$ $(p > 3)$.

Moreover, since $\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}_{nk})$ is invertible in $\mathbf{W}^{1,p}(T_f)$, thus also in $\mathbf{C}^0(\bar{T}_f)$ and since $C^0(\bar{T}_f) \ni \text{mapping} \mapsto \text{mapping}^{-1} \in C^0(\bar{T}_f)$, it also follows that $\mathcal{N}(\tilde{\mathbf{u}}_{nk}) \overset{k\to\infty}{\to} \mathcal{N}(\tilde{\mathbf{u}})$ in $\mathbf{C}^0(\bar{T}_f)$.

We also have that $\mathbf{f}^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}_{nk})) \overset{k\to\infty}{\to} \mathbf{f}^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}))$ in $\mathbf{L}^p(T_f)$. This is ensured by the following lemma (for a justification see, for instance, [**10**]):

Lemma 5.1. *Let $\boldsymbol{\psi} \in \mathbf{L}^p(\mathbf{R}^3)$. The mapping*

$$\mathbf{C}^1(\bar{T}_f) \ni \boldsymbol{\theta} \mapsto \boldsymbol{\psi} \circ (\mathrm{Id} + \boldsymbol{\theta}) \in \mathbf{L}^p(T_f)$$

is continuous at each point of the open ball $\{\boldsymbol{\theta} \in \mathbf{C}^1(\bar{T}_f),\ \|\nabla\boldsymbol{\theta}\|_{\mathbf{C}^0(\bar{T}_f)} < C(T_f)\}$, where $C(T_f)$ is the constant in the orientation preserving theorem for the mapping $\mathrm{Id} + \boldsymbol{\theta}$ (see (36) above).

We are able now to pass to the limit in the equations (18), (19)–(21) and due to the uniqueness of the solution to these equations we get $\hat{\mathbf{v}} = \mathbf{v}(\tilde{\mathbf{u}})$, $\hat{p} = p(\tilde{\mathbf{u}})$, $\hat{\mathbf{U}} = \mathbf{U}(\tilde{\mathbf{u}})$.

- $\mathcal{A}(\mathcal{U}_\chi) \subset \mathcal{U}_\chi$: by (35) and (38), with an adequate choice of C_{coupl}.
- $\mathcal{U}_\chi$ is convex and weakly compact in $\mathbf{W}^{2,p}(T_s)$ (this is straightforward).

Consequently, the hypotheses of Schauder's fixed-point theorem are satisfied and the conclusion follows. ☐

Now using the above results and transforming back to the original domain (see [**11**]), we obtain the following theorem for the fluid-structure interaction problem in the cylinder of length $L = \frac{2\pi}{a}$:

Theorem 5.2. *Let* $\mathbf{f}^{\tilde{\phi}} \in \mathbf{L}^p(\mathbf{R}^3)$ *and* $\mathbf{g} \in \mathbf{L}^p_{per}(C_s)$. *Assume there exists a constant* $\chi_1 > 0$ *with:*

$$K(\|\mathbf{f}^{\tilde{\phi}}\|_{\mathbf{L}^p(\mathbf{R}^3)} + \|\mathbf{g}\|_{\mathbf{L}^p(C_s)}) \leq \chi_1, \tag{39}$$

where K *is a constant depending on* a.

Then there exists a solution $(\mathbf{v}, p, \mathbf{U})$ *of the equations* (2)–(5)*),* (12)–(15) *with* $\mathbf{v} \in \mathbf{W}^{2,p}_{per}(C_f) \cap \mathbf{W}$, $p \in W^{1,p}_{per}(C_f)$ *and* $\mathbf{U}$ *sufficiently small in* $\mathbf{W}^{2,p}_{per}(C_s)$.

Remark. Remember the way $\mathbf{f}^J$ in (12) was defined. For the definitions of the involved spaces see the Appendix.

Appendix

Let $\tilde{C}$ be an infinite cylindrical pipe like in Section 3 and $\tilde{\Gamma}$ be its boundary. For $a \in \mathbf{R}_+ \setminus \{0\}$ and the finite cylinder C with boundary Γ like in Section 3, we define

$$\begin{aligned}
C^\infty_{0,per}(C) &:= \{f \in C^\infty_{per}(C) \ : \ \text{supp } (f) \cap (\bar{C} - \Gamma) \text{ is compact in } C_f\},\\
L^p_{per}(C) &:= \text{ the closure of } C^\infty_{per}(C) \text{ in } L^p(C)\\
W^{m,p}_{per}(C) &:= \text{ the closure of } C^\infty_{per}(C) \text{ in } W^{m,p}(C),\\
W^{m,p}_{0,per}(C) &:= \text{ the closure of } C^\infty_{0,per}(C) \text{ in } W^{m,p}(C),\\
\tilde{\mathbf{W}} &:= \{\mathbf{F} \in C^\infty_{0,per}(C) : \nabla \cdot \mathbf{F} = 0\},\\
\mathbf{W} &:= \text{ the closure of } \tilde{\mathbf{W}} \text{ in } \mathbf{W}^{1,p}_{0,per}(C).
\end{aligned}$$

Observe that

$$\mathbf{W} = \{\mathbf{v} \in \mathbf{W}^{1,p}_{0,per}(C) \ : \ \nabla \cdot \mathbf{v} = 0\}$$

and that, by the Poincaré inequality, the inner product in $\mathbf{W}^{1,p}_{0,per}(C)$ is equivalent to the inner product

$$((\mathbf{v}, \mathbf{w})) := \int_C \frac{\partial v_j}{\partial x_i} \frac{\partial w_j}{\partial x_i} d\mathbf{x}, \quad \mathbf{v}, \mathbf{w} \in \mathbf{W}^{1,p}_{0,per}(C).$$

***

The following lemma gives some properties of the mappings $\mathbf{N}$, $\mathbf{M}$ and J, which were defined in (11) and (10).

Lemma 5.2. *The mappings* $\mathbf{M}$, $J : W^{2,p}(T_s) \to W^{1,p}(T_f)$ *are of class* C^∞. $\mathbf{N} : \mathcal{U}_\chi \to W^{1,p}(T_f)$ *is also* $C^\infty(\mathcal{U}_\chi)$ *and it also satisfies an ellipticity condition:*

$$\exists \ \zeta > 0 \ \textit{such that} \ \mathbf{N}(\tilde{\mathbf{u}}) \geq \zeta \mathbf{I}, \ \forall \tilde{\mathbf{u}} \in \mathcal{U}_\chi, \ \forall \mathbf{x} \in \bar{T}_f.$$

Proof. The proof is a straightforward adaptation of the proof of Lemma 3 in [**10**]. It relies on the properties of $\tilde{\boldsymbol{\phi}}$, on the fact that the mapping $W^{1,p}(T_f) \ni M \to M^{-1} \in W^{1,p}(T_f)$ is C^∞ at any invertible matrix of $W^{1,p}(T_f)$ and on the compact embedding of $\mathbf{W}^{1,p}(T_f)$ in $\mathbf{C}^0(T_f)$. The fact that $p > 3$ is essential. $\square$

Remark 5.3. The fact that $\tilde{\mathbf{u}} \in \mathcal{U}_\chi$ and the above lemma ensure that the hypothesis (H_1) in Theorem 3.1 is satisfied. In order to be able to apply Theorem 3.1 in the proof of Theorem 5.1, hypothesis (H_3) must be satisfied, too.

By the previous lemma, the mappings $\mathbf{M}$ and $\mathbf{N}$ are of class C^∞ and clearly $\mathbf{M}(\mathbf{0}) = \mathbf{I},\ \mathbf{N}(\mathbf{0}) = \mathbf{I}$.

Thus we write the Taylor seria for $\mathbf{N}$ and $\mathbf{M}$ and get

$$\|\mathbf{N}(\tilde{\mathbf{u}}) - \mathbf{I}\|_{W^{1,p}(T_f)} \leq [[D\mathbf{N}]] \cdot \|\tilde{\mathbf{u}}\|_{W^{2,p}(T_s)},$$

respectively

$$\|\mathbf{M}(\tilde{\mathbf{u}}) - \mathbf{I}\|_{W^{1,p}(T_f)} \leq [[D\mathbf{M}]] \cdot \|\tilde{\mathbf{u}}\|_{W^{2,p}(T_s)},$$

where $[[D\mathbf{N}]] := \sup_{\mathbf{u}\in\mathcal{U}_\xi} \|D\mathbf{N}(\mathbf{u})\|_{\mathcal{L}(\mathbf{W}^{2,p}(T_f),\mathbf{W}^{1,p}(T_f))}$.
Choose χ such that $\chi \max\{[[D\mathbf{N}]], [[D\mathbf{M}]]\} \leq C$, where C is the constant in hypothesis (H_3) of Theorem 3.1.

References

1. Adams R., *Sobolev Spaces*, Academic Press, New York, 1975.
2. Agmon S., Douglis A. and Nirenberg L., *Estimates near the boundary of elliptic partial differential equations satisfying general boundary conditions II*, Comm. Pure Appl. Math. **17** (1964), 35–92.
3. Agmon S., *Lectures on Elliptic Boundary Value Problems*, Mathematical Studies, Van Nostrand 1965.
4. Bayada G., Chambat M., Cid B. and Vázquez C., *On the existence of solution for a non-homogeneous Stokes-rod coupled problem*, preprint 362 of the research team "Analyse Numérique Lyon-Saint Etienne" (UMR 5585), 2003.
5. Ciarlet P. G., *Mathematical Elasticity. Volume I: Three-Dimensional Elasticity*, North-Holland, 1988.
6. Constantin P. and Foias C., *Navier-Stokes Equations*. The University of Chicago Press, Chicago 1988.
7. Galdi G. P., *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol.I: Linearized Steady Problems*, Springer Tracts in Natural Philosophy **38**, Springer, 1994.
8. Ghidaglia J.-M., *Régularité des solutions des certaines problèmes aux limites liés aux equations d'Euler*, Comm. Part. Diff. Eq. **9** (1984), 1237–1264.
9. Grandmont C., *Éxistence et unicité des solutions d'un problème de couplage fluide-structure bidimensionnel stationnaire*, Note aux CRAS, Paris, **326**, Série I, 1998.
10. ______, *Existence for a Three-Dimensional Steady-State Fluid-Structure Interaction Problem*, Cahiers du CEREMADE, nr. 0011 (2000).
11. Mahalov A., Titi E. S. and Leibovich S., *Invariant Helical Subspaces for the Navier-Stokes Equations*, Arch. Rat. Mech. Anal., **112** 3 (1990), 193–222.
12. Temam R., *Navier-Stokes Equations*, North Holland, Amsterdam 1984.

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