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A STUDY ON THE ONE PARAMETER LORENTZIAN SPHERICAL MOTIONS

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ABSTRACT. In this paper we have introduced 1-parameter Lorentzian spherical motion. In addition to that we have given the relations between the absolute, relative and sliding velocities of these motions. Furthermore, the relations between fixed and moving pole curves in the Lorentzian spherical motions have also been obtained.

1. INTRODUCTION

The determination of a point or a set of points such that its velocity norm vanishes or that is a minimum has always aroused interest among kinematicians. The explanation of this is two-fold: points whose velocity, or acceleration, vanishes are important for they allow one to write simplified equations for the velocity and acceleration of any other point of the rigid body; and a point or a set of points with a minimum velocity norm locates the connecting place of a kinematic pair, in general a helicoidal pair, that connects the rigid body to the reference body. This connection produces a motion with the same characteristics, at least up to the first derivative of the original motion of the rigid body.

Indeed, the search for points of a rigid body with a minimum velocity norm has led to the description of the velocity of a rigid body in terms of infinitesimal screws, or helicoidal fields, and therefore to the definition of the instantaneous screw axis.

Muller has introduced one and two parameters planar motions and obtained the relations between absolute, relative, sliding velocity and pole curves of these motions, [**7**]. Lorentzian metric in 3-dimensional Minkowski space R_1^3 is indefinite. In the theory of relativity, geometry of indefinite metric is very crucial.Thus, by taking Lorentzian plane L^2 instead of Euclidean plane E^2, Ergin [**5**] has introduced 1-parameter planar motion in Lorentzian plane. Furthermore he gave the relation between the velocities, accelerations and pole curves of these motions.

To investigate the geometry of the motion of a line or a point in the motion of space is important in the study of space kinematics or spatial mechanisms or in physics. The geometry of such a motion of a point or a line has a number of

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applications in geometric modelling and model-based manufacturing of the mechanical products or in the design of robotic motions. These are specifically used to generate geometric models of shell-type objects and thick surfaces, [**4**], [**6**], [**9**].

This paper is organised as follows. In this first part, basic concepts have been given in Minkowski space IR_1^n. In the second part, 1-parameter Lorentzian spherical motions are defined. In doing so, the orthonormal frames of $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\left\{O; \vec{e_1}', \vec{e_2}', \vec{e_3}'\right\}$ are taken representing moving Lorentzian sphere S_1^2 and fixed Lorentzian sphere $\bar{S}_1^2$, respectively. Without making any of these privileged we have taken another orthonormal frame $\{O; \vec{r}_1, \vec{r}_2, \vec{r}_3\}$, called relative orthonormal frame, and given the Lorentzian spherical motions with respect to this new (relative) orthonormal frame. Furthermore the relations between absolute, relative and sliding velocities of 1-parameter Lorentzian spherical motions have been obtained. In the third part, the relations between the pole curves rolling on each other with respect to a spherical relative system have also been given.

We hope that these results will contribute to the study of space kinematics and physics applications.

2. Preliminaries

We start with preliminaries on the geometry of 3-dimensional Minkowski space. Let IR_1^n be a 3-dimensional Minkowski space endowed with Lorentzian inner product $\langle \, , \, \rangle$ of signature $(+, -, +)$. A vector $\vec{X} = (x_1, x_2, x_3)$ of IR_1^n is said to be time-like if $\langle \vec{X}, \vec{X} \rangle < 0$, space-like if $\langle \vec{X}, \vec{X} \rangle > 0$ and light-like (or null) if $\langle \vec{X}, \vec{X} \rangle = 0$. The set of all vector $\vec{X}$ such that $\langle \vec{X}, \vec{X} \rangle = 0$ is called the light-like (or null) cone and is denoted by Γ. The norm of a vector $\vec{X}$ is defined to be $\|\vec{X}\| = \sqrt{|\langle \vec{X}, \vec{X} \rangle|}$. Time orientation is defined as follows: A time-like vector $\vec{X} = (x_1, x_2, x_3)$ is future pointing (respectively past pointing) if and only if $x_2 > 0$ (respectively $x_2 < 0$), [**2**]. Let $\vec{X}$ be a future pointing time-like unit vector and $\vec{Y}$ also be a future pointing time-like unit vector. If the angle between $\vec{X}$ and $\vec{Y}$ is θ then we may have, [**2**], [**3**]

$$\left\langle \vec{X}, \vec{X} \right\rangle = -\cosh\theta.$$

The Lorentzian sphere and hyperbolic sphere of radius 1 in IR_1^n are given by

$$S_1^2 = \left\{ \vec{X} = (x_1, x_2, x_3) \in IR_1^3 \middle| \left\langle \vec{X}, \vec{X} \right\rangle = 1 \right\}$$

and

$$H_0^2 = \left\{ \vec{X} = (x_1, x_2, x_3) \in IR_1^3 \middle| \left\langle \vec{X}, \vec{X} \right\rangle = -1 \right\}$$

respectively, [**8**].

H_0^2 consists of two connected components. The components of H_0^2 through (0,1,0) and (0,-1,0) are called the future-pointing hyperbolic unit sphere and past-pointing hyperbolic unit sphere and are denoted by H_0^{+2} and H_0^{-2}, respectively.

As in the case of Euclidean 3-dimensional space, the Lorentzian cross product of $\vec{X}$ and $\vec{Y}$ is defined by

$$\vec{X} \wedge \vec{Y} = (y_3x_2 - y_2x_3, y_3x_1 - y_1x_3, y_2x_1 - y_1x_2)$$

where $\vec{X} = (x_1, x_2, x_3)$ and $\vec{Y} = (y_1, y_2, y_3)$ are the vectors of the space IR_1^3, [**1**].

The matrix

$$B(\theta) = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

is called the Lorentzian rotation matrix in IR_1^2, where $\theta \in IR$, [**3**]. This matrix is similar to the rotation matrix, which is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

in E^2.

Lemma 1. *Time-like vectors are transformed to time-like vectors and space-like vectors are transformed to space-like vectors by B. That is, B conserves the orientation,* [**2**].

3. Lorentzian Spherical Motions and Their Velocities

Let S_1^2 and $\bar{S}_1^2$ be O-centered moving and fixed Lorentzian spheres, and related to these spheres $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{O; \vec{e_1}', \vec{e_2}', \vec{e_3}'\}$ be orthonormal coordinate frames moving related to each other, having the same centre O, respectively. Let assume that $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ represents the moving Lorentzian sphere S_1^2, whereas $\{O; \vec{e_1}', \vec{e_2}', \vec{e_3}'\}$ represents the fixed one (where base vectors $\vec{e}_1, \vec{e}_3$; $\vec{e_1}', \vec{e_3}'$ are space-like and the vectors $\vec{e}_2, \vec{e_2}'$ are time-like). Therefore,

$$\langle \vec{e}_i, \vec{e}_j \rangle = \langle \vec{e_i}', \vec{e_j}' \rangle = \varepsilon_i \delta_{ij}, \qquad \varepsilon_i = \begin{cases} 1, \text{are } \vec{e}_i \text{ or } \vec{e_i}' \text{ space-like} \\ -1, \text{are } \vec{e}_i \text{ or } \vec{e_i}' \text{ time-like} \end{cases}, \ 1 \leq i, j \leq 3$$

Adopting that none of these systems are privileged, we take another relative orthonormal frame, $\{O; \tilde{\mathrm{r}}_1, \vec{r}_2, \vec{r}_3\}$, in consideration and express the movement with respect to this relative one (where base vectors $\vec{r}_1, \vec{r}_3$ are space-like and the vectors $\vec{r}_2$ is time-like). Therefore,

$$\langle \vec{r}_i, \vec{r}_j \rangle = \varepsilon_i \delta_{ij} \quad , \quad \varepsilon_i = \begin{cases} 1, \text{ is } \vec{r}_i \text{ space-like} \\ -1 \ , \text{ is } \vec{r}_i \text{ time-like} \end{cases} \quad , \qquad 1 \leq i, j \leq 3$$

Since each of these orthonormal frames has the same orientation, one frame is obtained by using another when rotated about O-point. Let A be a unit Lorentzian orthogonal matrix of type 3×3. That is, $A^t = \varepsilon A^{-1} \varepsilon$, where ε is a sign matrix defined as follows

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we use the following abbreviations

$$E = \begin{bmatrix} \vec{e_1} \\ \vec{e_2} \\ \vec{e_3} \end{bmatrix}, \qquad R = \begin{bmatrix} \vec{r_1} \\ \vec{r_2} \\ \vec{r_3} \end{bmatrix}, \qquad E' = \begin{bmatrix} \vec{e_1}' \\ \vec{e_2}' \\ \vec{e_3}' \end{bmatrix}$$

we get

$$R = AE, \qquad R = A'E'. \tag{1}$$

Here, the elements of the matrix A are not only continuous but all differentiable as well as we would like. Hence, 1-parameter motion is determined by the matrix $A = A(t)$ and called as 1-parameter Lorentzian spherical motion D_1.

Now, let us calculate the differentials of vectors $\vec{r_j}$ with respect to S_1^2 and $\bar{S}_1^2$, respectively. If we consider equation(1), then differential of the relative orthonormal coordinate frame R with respect to S_1^2 and $\bar{S}_1^2$ are

$$dR = dAA^{-1}R, \qquad d'R = dA'\left(A'\right)^{-1}R. \tag{2}$$

By choosing $dA \cdot A^{-1} = \Omega$ and $dA' \cdot \left(A'\right)^{-1} = \Omega'$ equation (2) can be rewritten as follows

$$dR = \Omega \mathrm{R}, \qquad d'R = \Omega' \mathrm{R}. \tag{3}$$

We can easily see that both Ω and Ω' matrices are anti-symmetric in the sense of Lorentzian, i.e., $\Omega^t = -\varepsilon\Omega\varepsilon$ where Ω^t is the transpose matrix of Ω and ε is sign matrix. Let assume that $\omega_{ij}\,(1 \leq i,j \leq 3)$ are the elements of Ω matrix. Let's denote the permutations of the indices $i,j,k = 1,2,3; 2,3,1; 3,1,2$, by $\omega_{ij} = \omega_k$. Then we can easily get that

$$\Omega = \begin{bmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & \omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{4}$$

In the similar way, anti-symmetric matrix Ω' in the sense of Lorentzian is obtained to be

$$\Omega' = \begin{bmatrix} 0 & \omega_3' & \omega_2' \\ \omega_3' & 0 & \omega_1' \\ -\omega_2' & \omega_1' & 0 \end{bmatrix}. \tag{5}$$

Let $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a point in the relative frame and configure the following vector

$$\overrightarrow{OX} = \vec{X} = X^t\mathrm{R}. \tag{6}$$

If the point X is a point on the unit Lorentzian sphere, then we have

$$\|\vec{X}\|^2 = x_1^2 - x_2^2 + x_3^2 = 1.$$

Now, we compute the differentials of X with respect to Lorentzian spheres S_1^2 (moving) and $\bar{S}_1^2$ (fixed). First of all, we evaluate the differentiation of X with respect to moving Lorentzian sphere S_1^2. If we consider equation (6), we obtain

$$d\vec{X} = dX^t R + X^t dR.$$

Substituting equation (3) in the last equation we have

$$d\vec{X} = \left(dX^t + X^t\Omega\right) R. \tag{7}$$

Therefore, relative velocity of X (i.e., velocity of X with respect to Lorentzian sphere S_1^2) is $\vec{V}_r = \frac{d\vec{X}}{dt}$. If $\vec{V}_r = 0$, i.e., $d\vec{X} = 0$, then the point X is fixed in the moving Lorentzian sphere S_1^2. Thus, from equation (7), the condition that the point X is fixed in S_1^2 is given by the following equation

$$dX^t = -X^t\Omega. \tag{8}$$

Similarly, from equation (3), the differential of X with respect to fixed Lorentzian sphere $\bar{S}_1^2$ is

$$d'\vec{X} = \left(dX^t + X^t\Omega'\right) R. \tag{9}$$

So, absolute velocity vector (the velocity of the point X with respect to fixed Lorentzian sphere $\bar{S}_1^2$) is $\vec{V}_a = \frac{d'\vec{X}}{dt}$. If $\vec{V}_a = 0$, i.e., $d'\vec{X} = 0$, the point X is fixed in the fixed Lorentzian sphere $\bar{S}_1^2$.

Hence, the condition that the point X is fixed in $\bar{S}_1^2$ is given by

$$dX^t = -X^t\Omega'. \tag{10}$$

If the point X is fixed in moving Lorentzian sphere S_1^2 then the velocity of X with respect to $\bar{S}_1^2$ is called sliding velocity of X and denoted by $\vec{V}_f$. If equation (8) is substituted in (9) we get

$$\vec{V}_f = X^t\Psi R \tag{11}$$

where $\Psi = \Omega' - \Omega$.

If the Pfaffian vector $\vec{\Psi}$ is taken to be

$$\vec{\Psi} = \Psi_1\vec{r}_1 - \Psi_2\vec{r}_2 - \Psi_3\vec{r}_3 \quad , \quad \Psi_i = \omega_i' - \omega_i \quad , \quad 1 \le i \le 3 \tag{12}$$

then we get

$$\vec{V}_f = \vec{\Psi} \wedge \vec{X} \tag{13}$$

Taking equation (7) and equation (9) into account we can easily get

$$\vec{V}_f = d'\vec{X} - d\vec{X}.$$

From the last equation we may write

$$\vec{V}_a = \vec{V}_r + \vec{V}_f.$$

Therefore we give the following theorem.

Theorem 2. *In a 1-parameter Lorentzian spherical motion, absolute velocity vector of a point X is the sum of relative velocity vector and sliding velocity vector of it.*

Now, to understand the meaning of the pfaffian vector $\vec{\Psi}$ and equation (14) we emphasize the importance of Darboux rotation vector.

Let us consider a rotational motion about an axis. Assume that this axis passes through the origin and its direction be $\vec{d}$. We also assume that the angular velocity of this rotational motion is $\omega = \mp\|\vec{d}\|$.

Let apply this rotation motion to the point X with the position vector of $\overrightarrow{OX} = \vec{X}$ and let us define velocity vector $\vec{v}$ of this point X as follows

$$\vec{v} = \vec{d} \wedge \vec{X}.$$

The last equation implies that the vector $\vec{v}$ is orthogonal to both $\vec{X}$ and $\vec{d}$. If the angle between $\vec{d}$ and $\vec{X}$ is denoted by α and the distance of $\vec{X}$ from the rotation axis by r, then we can write, [**2**].

$$\|\vec{v}\| = \|\vec{d}\|\|\vec{X}\| \sinh\alpha = \mp\omega r.$$

It is very clear from this equation that $\vec{v}$ is the velocity vector of the point X on the rotation about the axis $\vec{d}$ with the angular velocity of $\mp\|\vec{d}\|$. Therefore, we call $\vec{\Psi}$ pfaffian vector as rotation vector of 1-parameter Lorentzian spherical motion D_1 at the time t. Thus we give the following theorem.

Theorem 3. *In 1-parameter Lorentzian spherical motion D_1 at the time t, for every point X there exists an infinitesimal rotational motion. In this rotational motion, pfaffian vector plays the role of Darboux rotation vector.*

Now we add an unit vector of $\vec{p}$ which is in the direction of the rotation vector $\vec{\Psi}$. Since we have

$$\|\vec{p}\| = 1$$

then we write

$$\vec{\Psi} = \vec{p}\sqrt{\Psi_1^2 - \Psi_2^2 + \Psi_3^2}$$

where $\Psi = \mp\|\vec{\Psi}\| = \sqrt{\Psi_1^2 - \Psi_2^2 + \Psi_3^2}$ demonstrates the infinitesimal rotational angle which produces the rotation in the time interval dt (the sign of Ψ depend on the direction of $\vec{p}$). The point P shown on the Lorentzian sphere $\left(\overrightarrow{OP} = \vec{p}\right)$ is an instantaneous rotation pole. As the point P is characterised by that the sliding velocity is equal to zero, according to the equation (13) if

$$\vec{\Psi} \wedge \vec{X} = 0, \qquad \|\vec{X}\|^2 = 1$$

then

$$\vec{X} = \mp\vec{p}.$$

Theorem 4. *In a 1-parameter Lorentzian spherical motion for any time t there exists a couple of points P, P' for each of which the sliding velocities are zero, where P is the rotational pole S_1^2 and P' is the rotational pole $\bar{S}_1^2$. Those points remain stable on both Lorentzian spheres at any time.*

Theorem 5. *Every point of moving Lorentzian sphere S_1^2 make a rotational motion (an instantaneous rotational motion) with angular velocity $\Psi : dt$ about the pole P (and its P' point) at every time t. Therefore, 1-parameter Lorentzian spherical motion is such a rotational motion of Lorentzian sphere S_1^2 with respect to fixed Lorentzian sphere $\bar{S}_1^2$ at a time t.*

4. Canonical Relative Frames and Rolling of the Pole Curves on Each Others

Now, let us choose a special relative frame that satisfies the following equation

$$\vec{p} = \vec{r}_3. \tag{14}$$

If we take $\vec{p} = \vec{r}_3$ then the vector $\vec{p}$ becomes orthogonal to $\vec{r}_1$ and $\vec{r}_2$. Therefore, since $\vec{\Psi} = \vec{p}\sqrt{\Psi_1^2 - \Psi_2^2 + \Psi_3^2}$, from the equation(12) we see that $\Psi_1 = 0$, $\Psi_2 = 0$. Since we have $\Psi = \Omega' - \Omega$, if we consider equations(4) and (5) we reach $\omega_1' = \omega_1$, $\omega_2' = \omega_2$. Thus, infinitesimal rotation angle of instantaneous rotation appears to be

$$\Psi = \Psi_3.$$

In this case, instantaneous rotation axis is expressed as follows

$$\vec{\Psi} = -\vec{r}_3\Psi_3 = -\vec{r}_3\left(\omega_3' - \omega_3\right).$$

From this point on, we assume that $\Psi_3 \neq 0$. We have not given the single meaning of relative frame by using equation(14), because the frame obtained from the condition of $\vec{p} = \vec{r}_3$ can be rotated arbitrarily about the $\vec{r}_3$-axis. Therefore, rotating the frames about $\vec{p} = \vec{r}_3$-axis by an angle of θ gives us (see Figure 1).

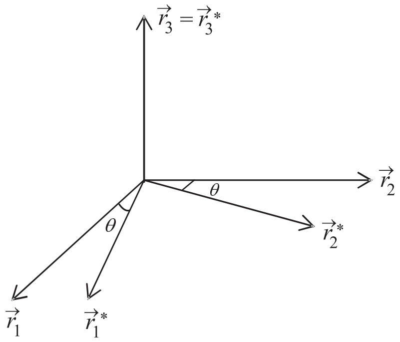

Figure 1

$$R^* = A(\theta)\,R \tag{15}$$

where $R^* = \begin{bmatrix} \vec{r}_1^{\,*} \\ \vec{r}_2^{\,*} \\ \vec{r}_3^{\,*} \end{bmatrix}$, $R = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix}$ and $A(\theta) = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

This new orthonormal frame $\{O; \vec{r}_1^*, \vec{r}_2^*, \vec{r}_3^*\}$ has the following differential equations, corresponding equation (3)

$$dR^* = \Omega^* R^* \quad , \quad d'R^* = \Omega'^* R^*. \tag{16}$$

Now we see the how we can obtain ω^*'s from ω's, i.e. we discuss the relationship between ω's and ω^*'s when the frame rotates by the angle of θ.

If we take into account equation (15) we can write

$$dR^* = dA(\theta) R + A(\theta) dR.$$

Substituting equation (3) in the last equation we obtain

$$dR^* = (dA(\theta) + A(\theta)\Omega) R \tag{17}$$

and using (15) and (16) equations we have the following

$$\Omega^* A(\theta) = dA(\theta) + A(\theta)\Omega. \tag{18}$$

If we write this last equation in matrix form we can easily see that

$$\begin{aligned} \omega_1^* &= \omega_1 \cosh\theta + \omega_2 \sinh\theta \\ \omega_2^* &= \omega_1 \sinh\theta + \omega_2 \cosh\theta \\ \omega_3^* &= \omega_3 + d\theta. \end{aligned}$$

So, in this type of rotation of the frame, pfaffian forms transform as unit vectors $\vec{r}_1$ and $\vec{r}_2$.

Now, to normalise the relative system we choose the rotation angle θ in such that

$$\omega_1^* = \omega_1 \cosh\theta + \omega_2 \sinh\theta = 0. \tag{19}$$

The equation (19) is a conditional equation for the rotation angle θ. At this point we suppose that the relative frame is rotated about $\vec{r}_3$ by the angle θ which satisfies equation (19) and omit the asterixes. Thus, we can rewrite the equation (16) and (19) for the canonical relative frame as follows. Differentiation with respect to S_1^2 is

$$\begin{bmatrix} d\vec{r}_1 \\ d\vec{r}_2 \\ d\vec{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & 0 \\ -\omega_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix} \tag{20}$$

and the differentiation with respect to $\bar{S}_1^2$ is

$$\begin{bmatrix} d'\vec{r}_1 \\ d'\vec{r}_2 \\ d'\vec{r}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega_3' & \omega_2' \\ \omega_3' & 0 & 0 \\ -\omega_2' & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{bmatrix} \tag{21}$$

$\vec{p} = \vec{r}_3$ vector draws a curve (P) on the moving sphere S_1^2, we call this curve as moving pole curve centrode of 1-parameter Lorentzian movement D_1. From the equation(20) we have the following equation

$$\frac{d\vec{r}_3}{\omega_2} = \frac{d\vec{r}_3}{ds} = -\vec{r}_1.$$

This last equation tells us that the unit tangential vector of moving pole curve (P) is $(-\vec{r}_1)$ and $\omega_2 = ds$ is the arc element of (P).

In the same manner, end point of the vector $\vec{p} = \vec{r}_3$ draws a constant pole curve (P') on the sphere $\bar{S}_1^2$. On this curve unit tangential vector at the point P is $(-\vec{r}_1)$ and arc element is $\omega_2 = ds'$ (here we took equation(21) into account). So, we can give the following theorems.

Theorem 6. *Velocity vectors of the rotating pole (P) are the same at any time when the pole on the moving and constant sphere draw pole curves (P) and (P'), respectively.*

Theorem 7. *In a 1-parameter spherical Lorentzian movement D_1, spherical moving pole curve (P) of S_1^2 rolls on constant pole curve (P') of $\bar{S}_1^2$ with no slide.*

Theorem 8. *In the reverse movement of 1-parameter spherical rotation motion the spherical surfaces of S_1^2, $\bar{S}_1^2$ and spherical pole curve (P) and (P') changes their roles*

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