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ORIENTABLE SELF-EMBEDDINGS OF STEINER TRIPLE SYSTEMS OF ORDER 15

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ABSTRACT. It is shown that for 78 of the 80 isomorphism classes of Steiner triple systems of order 15 it is possible to find a face 2-colourable triangulation of the complete graph K_{15} in an orientable surface in which the colour classes both form representatives of the specified isomorphism class. For one of the two remaining isomorphism classes it is proved that this is not possible. We also discuss the remaining open case.

1. Introduction

It was shown in [**4**] that for every pair (including identical pairs) of the 80 isomorphism classes of Steiner triple systems of order 15, it is possible to find a face 2-colourable triangulation of the complete graph K_{15} in a non-orientable surface in which the two colour classes form representatives of the specified pair of isomorphism classes. The corresponding problem for orientable surfaces appears to be much harder, although it is known that at least one pair of isomorphism classes cannot be produced in such a manner from an orientable triangulation [**3**]. In the current paper, which may be seen as a companion to [**4**], we address the somewhat simpler question obtained by restricting analysis to the 80 identical pairs of isomorphism classes. We show that for 78 of the 80 isomorphism classes it is possible to find a face 2-colourable triangulation of K_{15} in an orientable surface in which the two colour classes both form representatives of the specified isomorphism class. We will call such a triangulation a *self-embedding* of the (isomorphism class of the) Steiner triple system involved. We also prove that for one of the two remaining isomorphism classes, there is no such self-embedding, and we then discuss the remaining open case. We assume that the reader is familiar with the description of topological embeddings by means of rotation schemes and the use of Ringel's rule R^* to determine whether or not the embedding is orientable. Details are given in [**10**]. We emphasize that we are concerned with embeddings in surfaces rather than in pseudosurfaces so that the rotation about each point comprises a single cycle.

It is well known that the complete graph K_n has a triangulation in an orientable surface if and only if $n \equiv 0, 3, 4$ or $7 \pmod{12}$ [**10**]. In any triangulation of K_n,

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the number of faces around each vertex is $n-1$. Hence if $n-1$ is even, i.e. if $n \equiv 3$ or 7 (mod 12), it may be possible to colour each face using one of two colours, say black or white, so that no two faces of the same colour are adjacent. We say that the triangulation is *(properly) face 2-colourable*. Such triangulations in an orientable surface are known to exist for all $n \equiv 3$ or 7 (mod 12), [**10**], [**11**]. Given a face 2-colourable triangulation of K_n, the set of faces of each colour class forms a *Steiner triple system of order* n, STS(n) for short, i.e. a collection of triples which have the property that every pair is contained in precisely one triple. We say that each STS(n) is *embedded*, and that the pair of STS(n)s is *biembedded* in the surface. It has been known for over 150 years, [**8**], that an STS(n) exists if and only if $n \equiv 1$ or 3 (mod 6), see also [**6**]. We are led to the following questions.

1. Which STS(n)s can be so embedded in an orientable surface?
2. Which pairs of STS(n)s can be embedded in an orientable surface?

The latter question needs clarification. Clearly an arbitrary pair of labelled STS(n)s will not, in general, be biembeddable; they may for example have a common triple. But this is not the spirit of the question. The triples of one of the Steiner triple systems can be thought of as being fixed and forming the black triangles of a possible biembedding. The question is then whether there exists a permutation of the points of the other STS(n) so that the resulting triples form the white triangles.

The numbers of non-isomorphic STS(n)s for $n = 3,\ 7$ and 15 are well-known; there are $1, 1$ and 80 respectively, [**9**]. More recently it has been shown that there are 11 084 874 829 non-isomorphic STS(19)s [**7**]. The case $n = 3$ is trivial; there is a unique biembedding of the system with itself (i.e. a self-embedding) in the sphere, with automorphism group S_3 of order 6. The case $n = 7$ is less trivial, but well-known; there is a unique self-embedding in the torus, with automorphism group AGL$(1, 7)$ of order 42. We include as automorphisms all mappings that either exchange the colour classes or reverse the orientation. In this paper we study the case $n = 15$. The case $n = 19$ is well beyond any exhaustive computational approach with current facilities.

Using the standard numbering of the STS(15)s as in [**9**], the only known orientable biembeddings are those of the systems #1, #76 and #80, each with itself, i.e. self-embeddings, [**2**]. These are the only three of the 80 STS(15)s to have an automorphism of order 5 and the biembeddings can be obtained from index 3 current graphs. System #1 is the point-line design of the projective geometry PG$(3, 2)$ and it was shown in [**1**] that, up to isomorphism, there is precisely one orientable face 2-colourable triangular embedding of K_{15} in which both the black and the white systems are isomorphic to system #1. The only other result that appears to be known is that there is no orientable biembedding of system #1 with system #2 (the STS(15) obtained from system #1 by a *Pasch switch* i.e. replacing any Pasch configuration: $\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}$ with its "opposite": $\{x, y, z\}, \{x, b, c\}, \{a, y, c\}, \{a, b, z\}$), [**3**]. Hence, in answer to question 2, not every pair of STS(n)s, $n \equiv 3$ or 7 (mod 12), can be biembedded in an orientable surface although much further investigation is needed before any reasonable conjecture can be framed. We turn our attention now to a restricted version of question 2 by asking which of the 80 isomorphism classes of STS(15)s admit a self-embedding in an orientable surface.

2. Methodology

An algorithm for determining whether a pair of STS(n)s, say A and B, can be biembedded in an orientable surface is essentially straightforward. First, representations of systems A and B on the same base set V are chosen. System A is now held fixed with its triples forming the black triangles of a possible biembedding. Permutations of the base set V are then considered in turn and applied to system B. If π is any such permutation, one may then test whether the triples of system $\pi(B)$ can form the white triangles. This is easily done by forming the potential rotation scheme while checking that the rotation about each point is indeed a complete $n-1$ cycle and then checking that the entire scheme satisfies Ringel's rule R* for orientability. The difficulty in applying this algorithm for $n = 15$, even in the case of self-embeddings ($B = A$), lies in the number (15!) of permutations.

It was observed that each of the three self-embeddings of systems #1,#76 and #80 mentioned above has an automorphism of order 2, in each case an involution with one fixed point, which exchanges the colour classes. Motivated by this, a search for self-embeddings was started, restricting the permutations π to involutions with a single fixed point. The number of these is $\dfrac{15!}{(2^7 7!)} = 2\,027\,025$,. It was possible to examine all such permutations for every one of the 80 isomorphism classes.

In 78 of the 80 cases at least one self-embedding was thereby obtained. The exceptions are isomorphism classes #2 and #79. Appendix A, which originally appeared in the first author's doctoral thesis [**5**], gives a specimen self-embedding for each of the 78 successful cases. Table 1 gives the number of isomorphism classes of self-embeddings obtained in this way for each of the 80 isomorphism classes of STS(15)s. Across the columns we give the isomorphism class number of the STS(15), the order of its automorphism group, the number of isomorphism classes of orientable self-embeddings obtained by our method, and lastly the number of involutions having a single fixed point that give rise to orientable self-embeddings.

3. Self-embeddings of system #2

In [**1**] an adaptation of the algorithm described in the previous section is used to determine, up to isomorphism, all self-embeddings (both orientable and non-orientable) of STS(15) #1. The adaptation relies on the large automorphism group of this system. Briefly, without loss of generality, the rotation at 0 is taken to be

$$0 : 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14$$

with the black system A having triples $\{0, 2i-1, 2i\}$ and the white system B having triples $\{0, 2i, 2i+1\}$ for $i = 1, 2, \ldots, 7$ (with the point "15" replaced by "1"). For any realization of an STS(15) there are precisely $15.2^7.7!$ ways of mapping the triples through a single point onto the seven black triples described above. However, since STS(15) #1 has an automorphism group of order 20160, the number of

System A	$\lvert \mathrm{Aut}(A) \rvert$	No. of Iso. classes	No. of involutions	System A	$\lvert \mathrm{Aut}(A) \rvert$	No. of Iso. classes	No. of involutions
1	20160	1	20160	41	1	29	29
2	192	0	0	42	2	14	28
3	96	3	288	43	6	12	72
4	8	7	56	44	2	19	38
5	32	1	32	45	1	32	32
6	24	9	216	46	1	29	29
7	288	2	576	47	1	23	23
8	4	9	36	48	1	33	33
9	2	16	32	49	1	23	23
10	2	19	38	50	1	25	25
11	2	12	24	51	1	29	29
12	3	13	39	52	1	34	34
13	8	12	96	53	1	22	22
14	12	4	48	54	1	22	22
15	4	9	36	55	1	28	28
16	168	1	168	56	1	22	22
17	24	3	72	57	1	21	21
18	4	10	40	58	1	27	27
19	12	5	60	59	3	8	24
20	3	11	33	60	1	27	27
21	3	11	33	61	21	1	21
22	3	12	36	62	3	5	15
23	1	29	29	63	3	8	24
24	1	53	53	64	3	12	36
25	1	25	25	65	1	41	41
26	1	31	31	66	1	35	35
27	1	44	44	67	1	19	19
28	1	27	27	68	1	22	22
29	3	16	48	69	1	25	25
30	2	15	30	70	1	25	25
31	4	12	48	71	1	28	28
32	1	28	28	72	1	26	26
33	1	27	27	73	4	6	24
34	1	29	29	74	4	6	24
35	3	12	36	75	3	27	81
36	4	4	16	76	5	9	45
37	12	1	12	77	3	13	39
38	1	27	27	78	4	8	32
39	1	38	38	79	36	0	0
40	1	25	25	80	60	1	60

Table 1. Analysis of our self-embeddings of the STS(15)s.

differently labelled copies of STS(15) #1 containing these seven specified triples is $\frac{15.2^7.7!}{20160} = 480$. From each of these 480 black systems we can derive exactly one white system containing the triples specified above by applying the permutation (0)(1 2 3 4 5 6 7 8 9 10 11 12 13 14) to the black system. Thus the maximum number of possible biembeddings of STS(15) #1 with itself is precisely 480×480, thereby significantly reducing the size of the search. The results of [**1**] are that #1 has, up to isomorphism, a unique orientable self-embedding and three further non-orientable self-embeddings.

In [**3**] this approach was modified to obtain all the biembeddings of STS(15) #1 with STS(15) #2. The modification relied on the Pasch switch described in the introduction. Since system #1 has 105 Pasch configurations within it, there are at most 105×480 differently labelled copies of system #2 with the blocks specified for B above. Thus the number of possible biembeddings of system #1 with system #2 is $480 \times (105 \times 480)$. The results of [**3**] are that the biembeddings of these two systems fall into five isomorphism classes, none of which is orientable.

In our work for the current paper the approach was modified further, and in the obvious way, to obtain all the self-embeddings of STS(15) #2. There are 105×480 realizations of system #2 containing the black triples $\{0, 2i-1, 2i\}$, $i = 1, \ldots, 7$. Applying the permutation (0)(1 2 3 4 5 6 7 8 9 10 11 12 13 14) to each of these yields 105×480 realizations of system #2 containing the white triples $\{0, 2i, 2i+1\}$, $i = 1, \ldots, 7$. We then attempt to form an embedding of every black system with every white system. By this method we have established that there are no orientable self-embeddings of STS(15) #2.

4. Self-embeddings of system #79

The search for orientable self-embeddings of STS(15) #79 using involutions with a single fixed point as described above was fruitless. No orientable self-embedding of STS(15) #79 having such an automorphism exists. The search was then extended to include involutions with three and with five fixed points. Again, no orientable self-embeddings were found. It is easily shown that any set of seven or more points of the base set of the STS(15) #79 contains all three points of at least one of the blocks of this system. Therefore applying any involution with seven or more fixed points would result in a system B which shares at least one block with system A, and a self-embedding (A with B) could not result.

Consider next the possibility of an orientable self-embedding of system #79 having <u>any</u> colour-exchanging automorphism. If A, B are representatives of this system which together form a biembedding, and if π is any colour-exchanging automorphism of the biembedding, then $\pi(A) = B$ and $\pi(B) = A$. Hence $\pi^2 \in \mathrm{Aut}(A)$, the automorphism group of A, and similarly $\pi^2 \in \mathrm{Aut}(B)$. The results of the previous paragraph enable us to conclude that $\pi^2 \neq e$ (the identity permutation), but there remains the possibility that π^2 is some other element of $\mathrm{Aut}(A)$. In [**9**], the cycle types of automorphisms of system #79 are specified; they are $1^{15}, 1^3 2^6, 1^3 3^4, 1^1 2^1 4^3$ and 3^5, where a^i denotes i cycles of length a. Setting aside the case 1^{15} which corresponds to the identity permutation, suppose firstly that π^2 has cycle type $1^3 2^6$. Then the three fixed points of π^2 must form a block of A and likewise of B. Thus A and B have a common block and so cannot form a biembedding. The same argument excludes the possibility that π^2 has cycle type $1^3 3^4$. The next case is excluded simply because there are no permutations π for which π^2 has the cycle type $1^1 2^1 4^3$. Finally, consider the case when π^2 has cycle type 3^5. Then π has one of the types $3^5, 3^3 6^1$ or $3^1 6^2$; in the first of these cases $\pi = \pi^{-2}$ and so $\pi \in \mathrm{Aut}(A)$ and $B = A$, which is impossible, while in the second and third cases π^3 is a colour-exchanging involution which has already been shown not to exist. We may therefore conclude that any orientable self-embedding of system #79 has no colour-exchanging automorphisms.

Suppose now that π is any colour-preserving automorphism of a biembedding in which one colour class represents system #79. Then π must stabilize both constituent systems A and B, that is to say $\pi \in \mathrm{Aut}(A) \cap \mathrm{Aut}(B)$. Automorphisms π having cycle types $1^3 2^6, 1^3 3^4$ and $1^1 2^1 4^3$ force a common block in A and B, a contradiction. Thus the only possible non-trivial colour-preserving automorphisms of a biembedding of system #79 with any STS(15) are those with cycle type 3^5. Taking system A to be the realization of STS(15) #79 given in [**9**], it is easy to show that $\mathrm{Aut}(A)$ contains only four permutations having cycle type 3^5, namely

$\pi_1 = (1\ 6\ 7)(3\ 9\ 11)(2\ 5\ 4)(8\ 12\ 13)(10\ 15\ 14)$ and

$\pi_2 = (1\ 6\ 7)(3\ 11\ 9)(2\ 12\ 14)(5\ 13\ 10)(4\ 8\ 15)$,

together with their squares π_1^2 and π_2^2. Then by computer search we find that for each of $i = 1, 2$ there are precisely 8 958 candidates for system B, that is Steiner triple systems of order 15 which have $\pi_i \in \mathrm{Aut}(B)$ and are block disjoint from A. Of these, in each case 726 produce biembeddings with system A, but only six of these are orientable. For each of the twelve orientable biembeddings, system

1	:	2	3	5	4	15	14	8	9	7	6	11	10	13	12
2	:	3	1	12	9	10	8	5	7	11	14	6	4	13	15
3	:	4	8	10	5	1	2	15	6	7	13	9	11	12	14
4	:	5	12	10	11	7	8	3	14	13	2	6	9	15	1
5	:	6	13	11	15	12	4	1	3	10	7	2	8	14	9
6	:	7	3	15	8	13	5	9	4	2	14	10	12	11	1
7	:	8	4	11	2	5	10	15	13	3	6	1	9	14	12
8	:	9	1	14	5	2	10	3	4	7	12	13	6	15	11
9	:	10	2	12	15	4	6	5	14	7	1	8	11	3	13
10	:	11	4	12	6	14	15	7	5	3	8	2	9	13	1
11	:	12	3	9	8	15	5	13	14	2	7	4	10	1	6
12	:	13	8	7	14	3	11	6	10	4	5	15	9	2	1
13	:	14	11	5	6	8	12	1	10	9	3	7	15	2	4
14	:	15	10	6	2	11	13	4	3	12	7	9	5	8	1
15	:	1	4	9	12	5	11	8	6	3	2	13	7	10	14

Table 2. Orientable biembedding of systems #79 and #77.

B is found to be a copy of STS(15) #77, and all twelve biembeddings lie in a single isomorphism class. Thus we are able to state that if any orientable self-embedding of system #79 exists, then it has only a trivial automorphism group. For completeness, a representative of the orientable biembedding of system #79 with #77 is specified in Table 2 by its rotation scheme.

5. Concluding remarks

We summarize the currently known results concerning biembeddings of the 80 isomorphism classes of STS(15)s. As stated in the introduction, every pair (including identical pairs) can be biembedded in a non-orientable surface, [**4**]. There are three non-isomorphic self-embeddings of system #1 in a non-orientable surface and five non-isomorphic biembeddings of system#1 with system #2 in a non-orientable surface.

The main result of the current paper is that for 78 of the 80 isomorphism classes of STS(15)s, there exists a self-embedding in an orientable surface. In the case of system #1, this self-embedding is unique up to isomorphism, [**1**]. All of these 78 self-embeddings have a colour-reversing automorphism. The two remaining isomorphism classes are exceptions to this pattern.

There is no self-embedding of system #2 in an orientable surface. In fact, no biembedding in an orientable surface of system #2 with any system is known. We have shown, by similar methods to those used in the discussion regarding system #79 in the preceding section, that if a biembedding of system #2 in an orientable surface exists, then it must have a trivial automorphism group. Examination of this possibility is an important open problem since system #2 is the only remaining system for which an orientable biembedding is unknown.

Although we are currently unable to determine whether there is a self-embedding of system #79 in an orientable surface, we have shown that if one exists it has only a trivial automorphism group. We have also exhibited an orientable biembedding of system #79 with system #77. This is the first known biembedding of different isomorphism classes of STS(15)s in an orientable surface, although such biembeddings are well-known for Steiner triple systems of orders 19 and 31, [**5**]. Moreover, this is the only orientable biembedding having a non-trivial automorphism in which one of the STS(15)s is system #79. As with system #2, an examination of the case when the automorphism group is trivial would be of considerable interest.

The ultimate goal, possibly unattainable with present knowledge and computing power, is to determine for each pair of isomorphism classes of STS(15)s whether or not the classes possess a biembedding in an orientable surface. However, it appears that in the case when one of the STS(15)s has a "large" automorphism group, the exploitation of this fact together with careful analysis and programming could yield all the orientable biembeddings involving that system.

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Appendix A
Orientable self-embeddings of STS(15)s

The STS(15)s used in this appendix, and the numbering assigned to them, are those given in [**9**] with the exception that the element 15 is replaced with the element 0. The permutations are given in abbreviated 2-line format; for example row 1 represents the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 0 & 3 & 2 & 5 & 4 & 10 & 14 & 11 & 12 & 6 & 8 & 9 & 13 & 7 \end{pmatrix}$$

The given permutation π, when applied to the STS(15) A indicated, will produce a copy $\pi(A)$ of that system such that A and $\pi(A)$ can be orientably biembedded. Systems #2 and #79 are omitted.

System A	Permutation														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	0	3	2	5	4	10	14	11	12	6	8	9	13	7
3	1	0	4	12	2	14	13	7	11	10	9	8	3	6	5
4	1	0	8	12	13	9	14	10	2	5	7	11	3	4	6
5	4	2	1	10	0	7	6	5	9	8	3	13	14	11	12
6	1	0	4	3	2	8	13	10	5	11	7	9	14	6	12
7	1	0	4	7	2	9	11	3	12	5	10	6	8	14	13
8	1	0	13	7	12	5	8	3	6	11	14	9	4	2	10
9	1	0	8	14	6	7	4	5	2	9	12	13	10	11	3
10	1	0	3	2	8	10	9	12	4	6	5	11	7	14	13
11	1	0	13	6	5	4	3	14	9	8	10	12	11	2	7
12	1	0	3	2	13	5	11	10	14	12	7	6	9	4	8
13	1	0	3	2	13	9	6	11	12	5	14	7	8	4	10
14	1	0	9	11	4	10	7	6	12	2	5	3	8	14	13
15	1	0	12	8	5	4	7	6	3	9	13	14	2	10	11
16	1	0	3	2	13	6	5	14	10	11	8	9	12	4	7
17	1	0	7	3	8	14	9	2	4	6	12	13	10	11	5
18	1	0	3	2	9	7	14	5	12	4	11	10	8	13	6
19	1	0	6	3	11	14	2	12	13	10	9	4	7	8	5
20	1	0	5	8	14	2	12	9	3	7	13	11	6	10	4
21	1	0	14	12	11	13	6	9	10	7	8	4	3	5	2
22	1	0	4	12	2	7	14	5	8	10	9	13	3	11	6
23	1	0	14	12	11	7	6	5	13	10	9	4	3	8	2
24	1	0	6	12	11	9	2	14	10	5	8	4	3	13	7
25	2	7	0	10	12	13	14	1	8	11	3	9	4	5	6
26	1	0	3	2	11	5	7	6	14	10	9	4	13	12	8
27	1	0	12	14	10	5	7	6	11	13	4	8	2	9	3
28	1	0	13	14	5	4	7	6	9	8	11	10	12	2	3
29	1	0	4	5	2	3	12	14	11	10	9	8	6	13	7
30	1	0	3	2	11	5	14	10	9	8	7	4	13	12	6
31	1	0	5	14	12	2	6	13	10	11	8	9	4	7	3
32	1	0	3	2	6	7	4	5	10	11	8	9	14	13	12
33	1	0	6	12	13	8	2	9	5	7	14	11	3	4	10
34	1	0	8	3	10	14	11	13	2	12	4	6	9	7	5
35	1	0	4	12	2	5	11	8	7	10	9	6	3	14	13
36	2	3	0	1	13	6	5	14	9	8	11	10	12	4	7
37	1	0	3	2	4	7	10	5	9	8	6	14	13	12	11
38	1	0	6	11	14	5	2	10	12	13	7	3	8	9	4
39	2	3	0	1	11	5	10	8	7	13	6	4	14	9	12
40	1	0	6	11	12	9	2	8	7	5	14	3	4	13	10

System A	Permutation														
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
41	1	0	8	9	7	14	12	4	2	3	11	10	6	13	5
42	1	0	11	9	14	7	6	5	13	3	12	2	10	8	4
43	1	0	11	3	9	6	5	13	10	4	8	2	14	7	12
44	1	0	4	14	2	8	11	12	5	9	13	6	7	10	3
45	1	0	3	2	9	12	6	11	10	4	8	7	5	14	13
46	1	0	3	2	9	7	12	5	8	4	13	14	6	10	11
47	3	1	4	0	2	11	9	12	10	6	8	5	7	14	13
48	1	0	4	6	2	12	3	11	13	14	10	7	5	8	9
49	1	0	9	14	10	13	11	7	12	2	4	6	8	5	3
50	1	0	7	12	4	14	8	2	6	13	11	10	3	9	5
51	1	0	8	10	6	14	4	13	2	11	3	9	12	7	5
52	1	0	4	9	2	14	12	13	8	3	11	10	6	7	5
53	1	0	6	12	7	11	2	4	8	14	13	5	3	10	9
54	2	3	0	1	10	13	11	9	14	7	4	6	12	5	8
55	1	0	3	2	6	12	4	9	10	7	8	11	5	14	13
56	1	0	8	6	9	5	3	10	2	4	7	14	13	12	11
57	1	0	6	9	14	10	2	12	8	3	5	13	7	11	4
58	1	0	3	2	11	14	7	6	13	9	12	4	10	8	5
59	1	0	10	14	9	13	12	7	11	4	2	8	6	5	3
60	1	0	7	10	6	12	4	2	14	13	3	11	5	9	8
61	1	0	4	12	2	10	14	11	13	9	5	7	3	8	6
62	1	0	5	3	8	2	14	9	4	7	11	10	13	12	6
63	1	0	10	9	5	4	13	7	12	3	2	14	8	6	11
64	1	0	3	2	8	10	6	14	4	13	5	12	11	9	7
65	1	0	6	8	12	5	2	14	3	13	11	10	4	9	7
66	1	0	3	2	4	14	10	8	7	11	6	9	13	12	5
67	1	0	13	9	7	12	14	4	10	3	8	11	5	2	6
68	1	0	11	7	13	14	9	3	8	6	12	2	10	4	5
69	1	0	9	7	5	4	14	3	11	2	10	8	13	12	6
70	1	0	5	14	12	2	6	9	13	7	11	10	4	8	3
71	1	0	7	14	11	13	9	2	8	6	12	4	10	5	3
72	1	0	6	10	9	5	2	12	14	4	3	13	7	11	8
73	1	0	3	2	8	13	6	12	4	14	11	10	7	5	9
74	2	6	0	14	10	7	1	5	12	13	4	11	8	9	3
75	1	0	14	10	13	11	7	6	9	8	3	5	12	4	2
76	1	0	13	6	11	7	3	5	9	8	10	4	14	2	12
77	1	0	6	5	12	3	2	7	14	10	9	13	4	11	8
78	1	0	3	2	11	7	13	5	8	14	12	4	10	6	9
80	1	0	8	7	5	4	12	3	2	10	9	11	6	14	13

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