Acta Math. Univ. Comenianae
Vol. LXXV, 2(2006), pp. 253–266

ON HYPERGEOMETRIC-TYPE GENERATING RELATIONS ASSOCIATED WITH THE GENERALIZED ZETA FUNCTION

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ABSTRACT. In this paper, we aim at introducing and studying two hypergeometric-type generating functions associated with the generalized zeta function; our goal is to derive their basic properties including integral representations, sums, series representations and generating functions. A number of (known and new) results are shown to follow as special cases of our formulae.

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

The generalized zeta function Φ_μ^* is defined by [**5**, p. 100, (1.5)]:

$$\Phi_\mu^*(x,z,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{(a+n)^z}\,\frac{x^n}{n!}, \qquad |x|<1, \quad \mathrm{Re}(a)>0, \quad \mu \geq 1, \tag{1.1}$$

with the Pochhammer symbols $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$ for $n = 0, 1, \ldots$, where $\Gamma(\lambda)$ denotes the gamma function. Equivalently, it has the integral expression

$$\Phi_\mu^*(x,z,a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-at}(1-xe^{-t})^{-\mu}\,dt, \tag{1.2}$$

provided that $\mu \geq 1$, $\mathrm{Re}\, a > 0$, $|x| \leq 1$, and either $\mathrm{Re}\, z > 0$ or $\mathrm{Re}\, z > \mathrm{Re}\, \mu$ according to $x \neq 1$ or $x = 1$.

Obviously, when $\mu = 1$, $\Phi_\mu^*(x,z,a)$ reduces to the zeta function $\Phi(x,z,a)$of Erdélyi [**2**, p. 27, (1)], and in particular (1.1) and (1.2) become

$$\Phi_1^*(x,z,a) = \Phi(x,z,a) = \sum_{n=0}^{\infty} \frac{x^n}{(a+n)^z}, \qquad |x|<1, \quad a \neq 0, -1, -2, \ldots, \tag{1.3}$$

and

$$\Phi_1^*(x,z,a) = \Phi(x,z,a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-at}(1-xe^{-t})^{-1}\,dt, \tag{1.4}$$

Received December 29, 2005.
2000 *Mathematics Subject Classification.* Primary 11M06, 11M35; Secondary 33C20.
Key words and phrases. Riemann and Hurwitz zeta functions, Euler integral, generating functions, hypergeometric functions.

respectively.

Moreover, $\Phi_\mu^*(x,z,a)$ reduces further to the Hurwitz's zeta-function $\zeta(z,a)$, and so to the Riemann zeta-function $\zeta(z)=\zeta(z,1)$, which asserts that cf. [**2**, p. 32]:

$$\Phi_1^*(1,z,1)=\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}, \tag{1.5}$$

and

$$\Phi_1^*(1,z,a)=\zeta(z,a)=\sum_{n=0}^{\infty}\frac{1}{(a+n)^z}. \tag{1.6}$$

In [**6**], Katsurada introduced two hypergeometric – type generating functions of the Riemann zeta function in the forms:

$$e_z(x)=\sum_{m=0}^{\infty}\zeta(z+m)\,\frac{x^n}{m!}, \qquad |x|<+\infty, \tag{1.7}$$

$$f_z(\nu;x)=\sum_{m=0}^{\infty}(\nu)_m\,\zeta(z+m)\,\frac{x^n}{m!}, \qquad |x|<1, \tag{1.8}$$

where ν and x are arbitrary fixed complex parameters.

Motivated by the work of Katsurada [**6**], Bin-Saad [**1**] subsequently proposed a unification (and generalization) of the generating functions $e_z(x)$ and $f_z(\nu;x)$ in the forms:

$$\zeta(x,y;z,a)=\sum_{m=0}^{\infty}\Phi(y,z+m,a)\,\frac{x^m}{m!}, \qquad |y|<1, \quad |x|<\infty, \tag{1.9}$$

and

$$\zeta_\nu(x,y;z,a)=\sum_{m=0}^{\infty}(\nu)_m\,\Phi(y,z+m,a)\,\frac{x^m}{m!}, \qquad |y|<1, \quad |x|<|a|, \tag{1.10}$$

respectively.

In fact, it is easily verified by comparing (1.7) and (1.8) with (1.9) and (1.10) respectively that

$$\zeta(x,1;z,1)=e_z(x) \qquad \text{and} \qquad \zeta_\nu(x,1;z,1)=f_z(\nu;x). \tag{1.11}$$

The main object of the present paper is to investigate the functions $\zeta(x,y;z,a)$ and $\zeta_\nu(x,y;z,a)$ above, and their further generalizations defined by

$$\zeta_\mu^*(x,y;z,a)=\sum_{m=0}^{\infty}\Phi_\mu^*(y,z+m,a)\,\frac{x^m}{m!}, \qquad |y|<1, \quad |x|<\infty, \tag{1.12}$$

and

$$\zeta_{\mu,\,\nu}^*(x,y;z,a)=\sum_{m=0}^{\infty}(\nu)_m\Phi_\mu^*(y,z+m,a)\,\frac{x^m}{m!}, \qquad |y|<1, \quad |x|<|a|, \tag{1.13}$$

respectively, where a and z are complex parameters with $a \notin \{0, -1, -2, \ldots\}$, $\mathrm{Re}(z) > 1$, $\mu \geq 1$ and Φ_μ^* is the generalized zeta function defined by (1.1).

Clearly, on putting $\mu = 1$ in (1.12) and (1.13), we get the above mentioned definitions (1.9) and (1.10), respectively. For x =0, (1.12) and (1.13) reduce to (1.1), whereas, with $x = 0$ and $\mu = 1$, (1.12) and (1.13) reduce to (1.3).

Further, on putting $y = \mu = 1$ in definitions (1.12) and (1.13), we get the relations:

$$\zeta_1^*(x, 1; z, a) = \sum_{m=0}^{\infty} \Phi(1, z+m, a)\, \frac{x^m}{m!} = \sum_{m=0}^{\infty} \zeta(z+m, a)\, \frac{x^m}{m!}, \tag{1.14}$$

and

$$\zeta_{1,\nu}^*(x, 1; z, a) = \sum_{m=0}^{\infty} (\nu)_m\, \Phi(1, z+m, a)\, \frac{x^m}{m!} = \sum_{m=0}^{\infty} (\nu)_m\, \zeta(z+m, a)\, \frac{x^m}{m!}, \tag{1.15}$$

respectively, where $\zeta(z, a)$ is Hurwitz zeta function defined by (1.6).

Next, on putting $a = 1$ in equations (1.14) and (1.15), we get the functions (1.7) and (1.8). In fact, if we let $\nu = z$ in (1.8), this formula reduces to a well-known result of Ramanujan [**8**]:

$$\zeta(z, 1-x) = \sum_{m=0}^{\infty} (z)_m\, \zeta(z+m)\, \frac{x^m}{m!}, \qquad |x| < 1. \tag{1.16}$$

As an immediate consequence of the definitions (1.12) and (1.13), the following propositions are proved by substituting (1.1) and by changing the order of summation.

Proposition 1. *For any complex z, ν, μ and a with $a \notin \{0, -1, -2, \ldots\}$ and $\mu \geq 1$ we have*

$$\begin{aligned} \zeta_\mu^*(x, y; z, a) &= \sum_{m,n=0}^{\infty} \frac{(\mu)_n x^m y^n}{m!\, n!\, (a+n)^{z+m}} \\ &= \sum_{n=0}^{\infty} e^{x/(a+n)} \frac{(\mu)_n y^n}{n!(a+n)^z}, \qquad |x| < \infty,\ |y| < 1, \end{aligned} \tag{1.17}$$

$$\begin{aligned} \zeta_{\mu,\nu}^*(x, y; z, a) &= \sum_{m,n=0}^{\infty} \frac{(\nu)_m\, (\mu)_n\, x^m y^n}{m!\, n!\, (a+n)^{z+m}} \\ &= \sum_{n=0}^{\infty} \left[1 - \frac{x}{(a+n)}\right]^{-\nu} \frac{(\mu)_n y^n}{n!(a+n)^z}, \qquad |x| < |a|,\ |y| < 1. \end{aligned} \tag{1.18}$$

Proposition 2. *Under the same assumptions as in Proposition 1 with $y = 0$, we have*

$$\zeta_\mu^*(x, 0; z, a) = \zeta(x, 0; z, a) = a^{-z}\, e^{x/a}, \qquad |x| < \infty, \tag{1.19}$$

$$\zeta_{\mu,\nu}^*(x, 0; z, a) = \zeta_\nu(x, 0; z, a) = a^{-z} \left(1 - \frac{x}{a}\right)^{-\nu}, \qquad |x| < |a|, \tag{1.20}$$

while if $x = 0$, we have

$$\zeta^*_\mu(0, y; z, a) = \zeta^*_{\mu,\nu}(0, y; z, a) = \Phi^*_\mu(y, z, a). \tag{1.21}$$

2. Integral representations

We recall that if $\beta_j \neq 0, -1, -2, \dots, (j = 1, \dots, q)$, then the generalized hypergeometric series ${}_pF_q$ is defined by (see [**9**]):

$$ {}_pF_q\left(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z\right) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m ... (\alpha_p)_m}{(\beta_1)_m ... (\beta_q)_m} \frac{z^m}{m!}. \tag{2.1}$$

Important special cases of the series (2.1) are the Kummerian hypergeometric series ${}_1F_1(\alpha; \beta; z)$ and ${}_0F_1(-; \beta; z)$.

By using Eulerian integral formula of the second kind (see e.g.[**2**]]):

$$a^{-z}\Gamma(z) = \int_0^\infty e^{-at} t^{z-1} dt, \qquad \operatorname{Re}(z) > 0, \quad \operatorname{Re}(a) > 0, \tag{2.2}$$

it is easy to derive the following integral representations.

Theorem 1. *Let* $\operatorname{Re} a > 0$, $\operatorname{Re} \mu \geq 0$, $|x| < 1$, $|y| \leq 1$, *and either for* $\operatorname{Re} z > 0$ *or* $\operatorname{Re} z > Re\ \mu$ *according to* $y \neq 1$ *or* $y = 1$, *then*

$$\zeta^*_\mu(x, y; z, a) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-at} t^{z-1} (1 - ye^{-t})^{-\mu}\ {}_0F_1(-; z; xt)\, dt, \tag{2.3}$$

and

$$\zeta^*_{\mu,\,\nu}(x, y; z, a) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-at} t^{z-1} (1 - ye^{-t})^{-\mu}\ {}_1F_1(\nu; z; xt)\, dt. \tag{2.4}$$

Proof. Denote, for convenience, the right-hand side of equation (2.3) by I. Then it is easily seen that

$$I = \sum_{m=0}^{\infty} \frac{x^m}{m!(z)_m} \frac{1}{\Gamma(z)} \int_0^\infty e^{-at} t^{z+m-1} (1 - ye^{-t})^{-\mu} dt.$$

Since each term in the sum above can be evaluated by (1.2), we obtain (2.3) in view of the definition (1.12). In the same manner, one can derive the formula (2.4).

Moreover, by using the contour integral formula [**2**, p. 14, (4)]

$$2i\ \sin(\pi\, z)\, \Gamma(z) = -\int_\infty^{(0+)} (-t)^{z-1} e^{-t} dt, \qquad |\arg(-t)| \leq \pi, \tag{2.5}$$

one can derive the following contour integral representations. □

Theorem 2. *Let* $\mathrm{Re}(a) > 0$, $\mathrm{Re}(\mu) > 0$ *and* $|\arg(-t)| \leq \pi$, *then*

$$\zeta^*_{\mu}(x, y; z, a) = \frac{-\Gamma(1-z)}{2\pi\, i} \int_{\infty}^{(0+)} (-t)^{z-1} e^{-at} (1 - ye^{-t})^{-\mu}\; {}_0F_1(-; z; xt)\, dt, \tag{2.6}$$

and

$$\zeta^*_{\mu,\,\nu}(x, y; z, a) = \frac{-\Gamma(1-z)}{2\pi\, i} \int_{\infty}^{(0+)} (-t)^{z-1} e^{-at} (1 - ye^{-t})^{-\mu}\; {}_1F_1(\nu; z; xt)\, dt. \tag{2.7}$$

Proof. We start from the right-hand side of formula (2.6) and use (2.1) and the binomial expansion to get

$$\begin{aligned} &\frac{-\Gamma(1-z)}{2\pi i} \int_{\infty}^{(0+)} (-t)^{z-1}\, e^{-at}\, (1 - ye^{-t})^{-\mu}\; {}_0F_1[-; z; xt]\; dt \\ &\qquad = \sum_{m,n=0}^{\infty} \frac{(-1)^m (\mu)_n\, x^m y^n}{m! n!\, (z)_m} \cdot \frac{-\Gamma(1-z)}{2\pi i} \int_{\infty}^{(0+)} (-t)^{z+m-1}\, e^{-(a+n)t}\; dt. \end{aligned}$$

The desired result now follows from the first equality in (1.17), upon evaluating each integral above by (2.5) with the reflection formula $\Gamma(1-x)\,\Gamma(x) = \pi/\sin(\pi x)$ for the gamma function. The proof of (2.7) runs parallel to that of (2.6). □

If $x = 0$, (2.3) and (2.4) would immediately reduce to (1.2). Whereas, with $x = 0$ and $\mu = 1$, (2.6) and (2.7) reduce to another known result [**2**, p. 28, (5)].

3. Integrals involving $\zeta^*_{\mu}(x, y; z, a)$ and $\zeta^*_{\mu,\,\nu}(x, y; z, a)$

In this section we evaluate definite integrals involving the functions $\zeta^*_{\mu}(x, y; z, a)$ and $\zeta^*_{\mu,\,\nu}(x, y; z, a)$ in terms of the other kinds of zeta and hypergeometric functions.

At first, we obtain the following

Theorem 3. *Let* $\mathrm{Re}\, c > \mathrm{Re}(b) > 0$, $\mu \geq 1$, *then*

$$\begin{aligned} &\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\, \zeta^*_{\mu}(xt, y; z, a)\, dt \\ &\qquad = \sum_{n=0}^{\infty} {}_1F_1\left[b;\, c;\, \frac{x}{a+n}\right] \frac{(\mu)_n\, y^n}{n!(a+n)^z}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}\, \zeta^*_{\mu,\,\nu}(xt, y; z, a)\, dt \\ &\qquad = \sum_{n=0}^{\infty} {}_2F_1\left[b,\, \nu;\, c;\, \frac{x}{a+n}\right] \frac{(\mu)_n\, y^n}{n!(a+n)^z}. \end{aligned} \tag{3.2}$$

Proof. By using (1.12), we have

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,\zeta_\mu^*(xt,y;z,a)\,dt$$

$$=\frac{\Gamma(c)}{\Gamma(c-b)\,\Gamma(b)}\sum_{m,n=0}^{\infty}\frac{(\mu)_n\,x^m y^n}{m!n!\,(a+n)^{z+m}}\int_0^1 t^{b+m-1}\,(1-t)^{c-b-1}dt.$$

Now, with the help of the result

$$\int_0^1 t^{x-1}\,(1-t)^{y-1}\,dt=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},\qquad \mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0,$$

and (2.1), we get the expression on the right-hand side of (3.1), which completes the proof of (3.1). The proof of (3.2) runs parallel to that of (3.1), and then we ship the details. □

Note that, with $y=a=\mu=1$, (3.1) and (3.2) reduce to the known result (see [**6**, p. 24, (5.5) and (5.6)]):

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,e_z(xt)\,dt,=G_z(b;c;x), \tag{3.3}$$

and

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,f_z(\nu,xt)\,dt=G_{z,\,\nu}(b,\nu;c;x), \tag{3.4}$$

respectively, where

$$G_z(b;c;x)=\sum_{m=0}^{\infty}\frac{(b)_m}{(c)_m}\,\zeta(z+m)\,\frac{x^m}{m!},\qquad |x|<1, \tag{3.5}$$

and

$$G_{z,\,\mu}(b,\mu;c;x)=\sum_{m=0}^{\infty}\frac{(b)_m\,(\mu)_m}{(c)_m}\,\zeta(z+m)\,\frac{x^m}{m!},\qquad |x|<+\infty. \tag{3.6}$$

Further, in view of Proposition 2, if we let $y=0$ in (3.1) and (3.2) and replace x by xa, we get other known results (see [**10**, p. 31, (11) and p. 37, (6)]).

Now, other integral formulae would occur if we use the integral relation (2.2), and this asserts

Theorem 4. *Let* $\mathrm{Re}(z) > 0$, $\mathrm{Re}(\mu) > 0$ *and* $\mathrm{Re}(\lambda) < 1$, *then*

(3.7)
$$\frac{1}{\Gamma(1-\lambda)} \int_0^\infty t^{-\lambda}\, e^{-at} \zeta^*_\mu\left(\frac{x}{t},\, ye^{-t};\, z, a\right) dt = \Phi^*_\mu(y, z-\lambda+1, a)\, {}_0F_1(-;\lambda;-x),$$

(3.8)
$$\frac{1}{\Gamma(1-\lambda)} \int_0^\infty t^{-\lambda}\, e^{-at} \zeta^*_{\mu,\,\nu}\left(\frac{x}{t}, ye^{-t};\, z, a\right) dt = \Phi^*_\mu(y, z-\lambda+1, a)\, {}_1F_1(\nu;\lambda;-x),$$

(3.9)
$$\frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1}\, e^{-t} \zeta^*_\mu(xt\,, y; z, a)\, dt = \zeta^*_{\mu,\,\nu}(x, y; z, a),$$

(3.10)
$$\frac{1}{\Gamma(1-\nu)} \int_0^\infty t^{-\nu}\, e^{-t} \zeta^*_{\mu,\,\nu}\left(-\frac{x}{t}\,, y; z, a\right)\, dt = \zeta^*_\mu(x, y; z, a).$$

Proof. Denote, for convenience, the left-hand side of equality (3.7) by I. Then in view of (1.12), it is easily seen that:

$$I = \sum_{m,n=0}^\infty \frac{(\mu)_n\, x^m y^n}{n!m!(a+n)^{z+m}} \frac{1}{\Gamma(1-\lambda)} \int_0^\infty t^{-\lambda-m}\, e^{-(a+n)t}\, dt\,.$$

Upon using the integral formula (2.2) and the definition (1.1), we are finally led to right-hand side of formula (3.7). It is equally straightforward in the same manner to derive the formulae (3.8), (3.9) and (3.10). □

Now, with $x = 0$ and $y = 1$, equation (3.7) and (3.8) reduce to the interesting result

$$\frac{1}{\Gamma(1-\lambda)} \int_0^\infty t^{-\lambda}\, e^{-at}\, \Phi^*_\mu(e^{-t}, z, a) dt = \Phi^*_\mu(1, z-\lambda+1, a). \tag{3.11}$$

For $\mu = 1$, (3.11) reduces to the elegant result

$$\frac{1}{\Gamma(1-\lambda)} \int_0^\infty t^{-\lambda}\, e^{-at}\, \Phi(e^{-t}, z, a) dt = \zeta(z-\lambda+1, a). \tag{3.12}$$

4. Sums

First, we derive the following basic sums of series.

Theorem 5. *Let* $z \neq 1, 2, 3, \ldots,$ *then*

$$\sum_{k=0}^{\infty} \zeta_{\mu}^{*}(x, y; z-k, a)\, \frac{w^k}{k!} = e^{aw} \zeta_{\mu}^{*}(x, ye^{w}; z, a), \tag{4.1}$$
$$|x| < +\infty,\ |y| < 1,\ |w| < +\infty,$$

$$\sum_{k=0}^{\infty} \zeta_{\mu,\,\nu}^{*}(x, y; z-k, a)\, \frac{w^k}{k!} = e^{aw} \zeta_{\mu,\,\nu}^{*}(x, ye^{w}; z, a), \tag{4.2}$$
$$|x| < |a|,\ |y| < 1,\ |w| < +\infty.$$

Proof. In formula (1.12), we replace z by $z-k$ with $k \in Z^{+} \cup \{0\}$, multiply both sides by $w^k/k!$ and then sum up with $k \in Z^{+} \cup \{0\}$ to get (4.1). The proof of (4.2) is similar to that of (4.1). □

Again, starting from (1.12) and (1.13), and changing the order of summation, we get

Theorem 6. *Let* $\mathrm{Re}(z) > 0,\ a \neq 0, -1, -2, \ldots,$ *then*

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \zeta_{\mu}^{*}(xw, y; z+k, a)\, \frac{w^k}{k!} \\ &= \sum_{k=0}^{\infty} \{ {}_2F_1[-k, 1-c-k; 1-b-k; -x] \cdot \Phi_{\mu}^{*}(y, z+k; a) \}\, \frac{(b)_k}{(c)_k}\, \frac{w^k}{k!}, \end{aligned} \tag{4.3}$$
$$|w| < 1,\ |x| < 1,$$

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(b)_k (d)_k}{(c)_k} \zeta_{\mu,\,\nu}^{*}(xw, y; z+k, a)\, \frac{w^k}{k!} \\ &= \sum_{k=0}^{\infty} \{ {}_3F_2[-k, \nu, 1-c-k; 1-b-k, 1-d-k; -x] \cdot \Phi_{\mu}^{*}(y, z+k; a) \} \\ &\quad \cdot \frac{(b)_k (d)_k}{(c)_k}\, \frac{w^k}{k!}, \qquad |w| < 1,\ |x| < |a|. \end{aligned} \tag{4.4}$$

Proof. By starting from the left-hand side of (4.3) and using (1.17), we get

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \zeta_{\mu}^{*}(xw, y; z+k, a)\, \frac{w^k}{k!} \\ &\qquad = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \left\{ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\mu)_n\, x^m y^n}{m!n!\,(a+n)^{z+k+m}} \right\}\, \frac{w^{k+m}}{k!}. \end{aligned}$$

In the above relation, if we replace k by $k-m$, use the formulae

$$(\lambda)_{n-k} = \frac{(-1)^k\,(\lambda)_n}{(1-\lambda-n)_k} \qquad \text{and} \qquad (-n)_k = \frac{(-1)^n\,n!}{(n-k)!},$$

and note the definitions (1.1) and (2.1), we are led to the right-hand side of relation (4.3). This completes the proof of (4.3). The proof of (4.4) is similar to that of (4.3). □

If in (4.3) and (4.4), we let $x=0$, $y=a=\mu=1$ and use (1.5), then we obtain two Dirichlet series expressions due to Katsurada [**6**, p. 24, (5.3) and (5.4)]):

$$G_z(b;c;w) = \sum_{n=1}^{\infty} {}_1F_1\left(b;\ c;\ \frac{w}{n}\right)\ n^{-z}, \tag{4.5}$$

and

$$G_{z,\,d}(b,d;c;w) = \sum_{n=1}^{\infty} {}_2F_1\left(b,d;\ c;\ \frac{w}{n}\right)\ n^{-z}. \tag{4.6}$$

Next, if in (4.3) and (4.4), we let $b=c$, we obtain the following results

$$\sum_{k=0}^{\infty} \zeta^*_\mu(xw,y;z+k,a)\,\frac{w^k}{k!} = \sum_{k=0}^{\infty} (1+x)^k\,\Phi^*_\mu(y,z+k,a)\,\frac{w^k}{k!}, \tag{4.7}$$

and

$$\begin{aligned}&\sum_{k=0}^{\infty}(d)_k\ \zeta_{\mu,\,\nu}{}^*(xw,y;z+k,a)\,\frac{w^k}{k!}\\ &\quad= \sum_{k=0}^{\infty} {}_2F_1(-k,\nu;1-d-k;-x)\,\Phi^*_\mu(y,z+k,a)\,\frac{w^k}{k!},\end{aligned} \tag{4.8}$$

respectively.

Moreover, if in (4.4), we set $d=c$, $b=z$ and let $x=0$. Upon noting that

$$\left[1-\frac{w}{a+n}\right]^{-z} = (a+n)^z(a+n-w)^{-z},$$

the assertion (4.4) reduces to

$$\sum_{k=0}^{\infty} (z)_k\ \Phi^*_\mu(y,z+k,a)\frac{w^k}{k!} = \Phi^*_\mu(y,z,a-w), \tag{4.9}$$

which with $\mu=y=1$ reduces to a known result [**7**, p. 396, (6)]

$$\sum_{k=0}^{\infty} (z)_k\ \zeta(z+k,a)\frac{w^k}{k!} = \zeta(z,a-w). \tag{4.10}$$

Note that, with $a=1$ (4.10) reduces to (1.16).

Further, from the definitions (1.12) and (1.13), we easily have the following interesting series relation.

Theorem 7. *Let $|x| < 1$, $|y| < 1$, $|w| < |a|$, $|t| < |a|$, then for any complex number z and b*

$$\zeta^*_\mu(x, y; z, a - w) = \sum_{k=0}^{\infty} \zeta^*_{\mu,\, z+k}(w, y; z + k, a)\frac{x^k}{k!}, \tag{4.11}$$

$$\zeta^*_{\mu,\, b}(x, y; z, a - w) = \sum_{k=0}^{\infty} (b)_k\, \zeta^*_{\mu,\, z+k}(w, y; z + k, a)\frac{x^k}{k!}. \tag{4.12}$$

Proof. By starting from (1.17) and according to the result

$$(a + n - w)^{-\lambda} = \sum_{m=0}^{\infty} \frac{(\lambda)_m w^m}{m!(a + n)^{\lambda+m}},$$

it is easily seen that

$$\zeta^*_\mu(x, y; z, a - w) = \sum_{k=0}^{\infty} \sum_{m,n=0}^{\infty} \frac{(z + k)_m(\mu)_n\, w^m y^n}{m!n!(a + n)^{z+m+k}}\; \frac{x^k}{k!},$$

which, in view of (1.13), yields the right-hand side of (4.11). In the same manner one can prove the relation (4.12). □

Note that, for $w \to 0$, (see equation (1.21)), the formulae (4.11) and (4.12) reduce immediately to the results (1.12) and (1.13) respectively. Moreover, for $x \to 0$, equations (4.11) and (4.12) yield the following interesting identity

$$\Phi^*_\mu(y, z, a - w) = \zeta^*_{\mu,\, z}(w, y; z, a). \tag{4.13}$$

For $\mu = y = 1$, (4.13) reduces to a known result mentioned in (4.10).

5. Series Representations and Generating functions

By means of the integral representations (2.3) and (2.4) and Euler integral formula (2.2), we now proceed to establish a new representations of the functions ζ^*_μ and $\zeta^*_{\mu\,,\,\nu}$ in terms of Humbert's series Ψ_1 and Appell's series F_2 (see e.g. [**9**]).

For the purpose of the present study, we recall a known result of Exton [**3**, p. 147, (3)]:

$$\exp\left[s + u - \frac{wu}{s}\right] = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{s^m}{\Gamma(m + 1)}\, \frac{u^n}{\Gamma(n + 1)}\; {}_1F_1[-n; m + 1; w].$$

On putting $a = 1 - s - u + wu/s$ in (2.3) and making use of the above formula, we find that

$$\zeta^*_\mu\left(\, x, y;\, z, 1 - s - u + \frac{wu}{s}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{s^m}{\Gamma(m + 1)}\frac{u^n}{\Gamma(n + 1)}$$

$$\cdot \frac{1}{\Gamma(z)} \int_0^\infty e^{-t} t^{z+m+n-1}(1 - ye^{-t})^{-\mu}\, {}_0F_1(-; z; xt)\, {}_1F_1(-n; m + 1; wt)\, dt. \tag{5.1}$$

Making use of the series representation (2.1) and expanding the function $(1-ye^{-t})^{-\mu}$, we can integrate the resulting series term-by-term by means of the result (2.2). We thus find that

$$\zeta^*_\mu\left(x, y;\ z,\ 1-s-u+\frac{wu}{s}\right)$$
$$(5.2)\qquad =\sum_{m=-\infty}^{\infty}\sum_{n,p=0}^{\infty}\frac{(z)_{m+n}(\mu)_p}{\Gamma(m+1)\,\Gamma(n+1)\,\Gamma(p+1)\ (1+p)^z}\left(\frac{s}{1+p}\right)^m\left(\frac{u}{1+p}\right)^n y^p$$
$$\cdot\,\Psi_1\left[z+m+n, -n; m+1, z;\ \frac{w}{1+p}, \frac{x}{1+p}\right],$$

where Ψ_1 is Humbert's confluent function of two variables (see [**2**, p. 225, 5.7.1(23)]).

Similarly, for the function $\zeta^*_{\mu,\,\nu}$, we can show that

$$\zeta^*_{\mu,\,\nu}\left(x, y;\ z,\ 1-s-u+\frac{wu}{s}\right)$$
$$(5.3)\qquad =\sum_{m=-\infty}^{\infty}\sum_{n,p=0}^{\infty}\frac{(z)_{m+n}\ (\mu)_p}{\Gamma(m+1)\Gamma(n+1)\Gamma(p+1)\ (1+p)^z}\left(\frac{s}{1+p}\right)^m\left(\frac{u}{1+p}\right)^n y^p$$
$$\cdot\,F_2\left[z+m+n, -n, \nu; m+1, z;\ \frac{w}{1+p}, \frac{x}{1+p}\right],$$

where F_2 is Appell's function of two variables (see [**9**, p. 23, (3)]).

Some special cases of equations (5.2) and (5.3) are of interest. First, setting $w=0$, equations (5.2) and (5.3) would reduce to the following interesting representations

$$\zeta^*_\mu(x, y; z, 1-s-u)$$
$$(5.4)\qquad =\sum_{m=-\infty}^{\infty}\sum_{n,p=0}^{\infty}\frac{(z)_{m+n}\ (\mu)_p}{\Gamma(m+1)\,\Gamma(n+1)\,\Gamma(p+1)(1+p)^z}\left(\frac{s}{1+p}\right)^m\left(\frac{u}{1+p}\right)^n y^p$$
$$\cdot\ {}_1F_1\left[z+m+n; z;\ \frac{x}{1+p}\right],$$

and

$$\zeta^*_{\mu,\,\nu}(x, y; z, 1-s-u)$$
$$(5.5)\qquad =\sum_{m=-\infty}^{\infty}\sum_{n,p=0}^{\infty}\frac{(z)_{m+n}\ (\mu)_p}{\Gamma(m+1)\,\Gamma(n+1)\,\Gamma(p+1)\ (1+p)^z}\left(\frac{s}{1+p}\right)^m\left(\frac{u}{1+p}\right)^n y^p$$
$$\cdot\ {}_2F_1\left[z+m+n, \nu; z;\ \frac{x}{1+p}\right],$$

respectively.

Further, in view of the result (1.21), we find from (5.2) and (5.3) that

$$\Phi^*_\mu\left(y; z,\ 1-s-u+\frac{wu}{s}\right)$$

$$(5.6)\qquad =\sum_{m=-\infty}^{\infty}\sum_{n,p=0}^{\infty}\frac{(z)_{m+n}(\mu)_p}{\Gamma(m+1)\,\Gamma(n+1)\,\Gamma(p+1)(1+p)^z}\left(\frac{s}{1+p}\right)^m\left(\frac{u}{1+p}\right)^n y^p$$

$$\cdot\ {}_2F_1\left[z+m+n,-n;m+1;\ \frac{w}{1+p}\right].$$

More interestingly, for $y=\mu=1$ and $s=u=w/2$ (in conjunction with (1.5)) equation (5.6) yields the following elegant representation relation for the Riemann zeta function $\zeta(z)$

$$\zeta(z)=\sum_{m=-\infty}^{\infty}\sum_{n,p=0}^{\infty}\frac{(z)_{m+n}}{2^{m+n}\Gamma(m+1)\ \Gamma(n+1)\ \ (1+p)^z}\left(\frac{w}{1+p}\right)^{m+n}$$

$$(5.7)\qquad \cdot\ {}_2F_1\left[z+m+n,-n;m+1;\ \frac{w}{1+p}\right].$$

Still, other interesting special cases of the assertions (5.2) and (5.3) occur when we employ (1.19) and (1.20). We thus find that

$$\left(1-s\ -\ u+\frac{wu}{s}\right)^{-z}\exp\left(\frac{x}{1-s-u+wu/s}\right)$$

$$(5.8)\qquad =\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(z)_{m+n}\frac{s^m}{\Gamma(m+1)}\frac{u^n}{\Gamma(n+1)}\Psi_1\left[z+m+n,-n;m+1,z;w,x\right],$$

and

$$\left(1-s\ -\ u+\frac{wu}{s}\right)^{-z}\left(1-\frac{x}{1-s-u+wu/s}\right)^{-\nu}$$

$$(5.9)\qquad =\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(z)_{m+n}\frac{s^m}{\Gamma(m{+}1)}\frac{u^n}{\Gamma(n{+}1)}F_2\left[z+m+n,-n,\nu;m+1,z;w,x\right].$$

For $x=0$, equations (5.8) and (5.9) reduce to the known result of Exton [**4**, p. 174, (5.1)]:

$$\left(1-s\ -u+\ \frac{wu}{s}\right)^{-z}$$

$$(5.10)\qquad =\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}(z)_{m+n}\frac{s^m}{\Gamma(m{+}1)}\ \frac{u^n}{\Gamma(n{+}1)}{}_2F_1\ \left[z+m+n,-n;m+1;w\right],$$

which is exactly the same as the result of Yasmeen [**11**, Eq. (5.2.3)].

Finally, according to the formulae (1.17) and (1.18) and the identity

$$(\lambda+m)_n=\frac{(\lambda)_n(\lambda+n)_m}{(\lambda)_m},$$

we easily have the following generating relations of the functions ζ^*_μ and $\zeta^*_{\mu,\nu}$

$$\sum_{m_1,\dots,m_r=0}^{\infty} \zeta^*_{\mu+M}(x,y;z+M,a)\prod_{j=1}^{r}\left\{(\alpha_j)_{m_j}\frac{w_j^{m_j}}{m_j!}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(\mu)_n}{(a+n)^z}\,\mathrm{e}^{x/(a+n)}\,F_D^{(r)}\left[\mu+n,\alpha_1,\dots,\alpha_r;\mu;\frac{w_1}{a+n},\dots,\frac{w_r}{a+n}\right]\frac{y^n}{n!}, \tag{5.11}$$

and

$$\sum_{m_1,\dots,m_r=0}^{\infty} \zeta^*_{\mu+M,\nu+M}(x,y;z,a)\prod_{j=1}^{r}\left\{(\alpha_j)_{m_j}\frac{w_j^{m_j}}{m_j!}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(\mu)_n}{(a+n)^z}\left[1-\frac{x}{a+n}\right]^{-\nu} \tag{5.12}$$

$$\cdot\; F_D^{(r)}\left[\mu+n,\alpha_1,\dots,\alpha_r;\mu;\frac{w_1}{1-x/(a+n)},\dots,\frac{w_r}{1-x/(a+n)}\right]\frac{y^n}{n!},$$

with $M = m_1 + \ldots + m_r$, where $F_D^{(r)}$ is Lauricella's hypergeometric function in r-variables (see [**9**, p. 33, (1)]).

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