Acta Math. Univ. Comenianae
Vol. LXXV, 2(2006), pp. 227–232

SOME APPLICATIONS OF PARABOLIC COMPARISON PRINCIPLES TO THE STUDY OF DECAY ESTIMATES

C.-P. DANET

Abstract. This paper is concerned with the asymptotic behavior of solutions of general nonlinear parabolic equations. We consider a boundary value problem which was treated by Reynolds in a classical paper (J. Diff. Equations **12** (1972), 256–261). Our goal is to prove by different means a version of the main result in the above mentioned paper. We also point out that it remains valid under some weaker hypotheses if the working domain is cylindrical.

1. Introduction

We consider the problem:

$$\text{(1)}\qquad \begin{aligned} &\mathrm{Q}u = -D_t u + a^{ij}(x,t,u,Du)D_{ij}u + b(x,t,u,Du) = 0 \qquad &&\text{in}\,\Omega\times\mathbb{R}_+\\ &u = h &&\text{on}\,S,\end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and S are the "side walls" $\partial\Omega\times[0,\infty)$. Here $\mathbb{R}_+=\{t\in\mathbb{R}|t>0\}$, and $b(x,t,z,p)$ is differentiable with respect to the z and p variables in $\Omega\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^n$. The summation convention is followed throughout.

We make the following assumptions:
The operator Q is strictly parabolic in the sense that there exists a constant $\lambda>0$ such that,

$$\text{(2)}\qquad \lambda|\xi^2|\le a^{ij}(x,t,z,p)\xi_i\xi_j,$$

for all $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n\setminus\{0\}$ and for all $(x,t,z,p)\in\Omega\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^n$.

$$\text{(3)}\qquad \left|\frac{\partial b}{\partial p_i}\right| = |D_{p_i}b|\le\beta$$

in $\Omega\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^n$, for $i=1,\dots,n$, where $\beta>0$ is a constant.

$$\text{(4)}\qquad \frac{\partial b}{\partial z} = D_z b\le C = \frac{\beta+1+\delta}{e^{(\beta+1+\delta)\mathrm{diam}\Omega}}$$

Received May 29, 2005.
2000 *Mathematics Subject Classification.* Primary 35B40.
Key words and phrases. Comparison principles, decay estimates.

in $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n$ where diamΩ is the diameter of Ω, and δ is a strictly positive constant

$$|b(x,t,0,0)| \leq K_1 e^{-\mu_1 t} \tag{5}$$

and

$$|h(x,t)| \leq K_2 e^{-\mu_2 t} \tag{6}$$

in $\partial\Omega \times \mathbb{R}_+$, where K_1, K_2, μ_1, μ_2 are strictly positive constants.
Reynolds [**5**] proved (alongside with other relations) decay for the classical solution u of problem (1) when

$$\begin{aligned} D_z b &\leq C^*(x,t) && \text{in } \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n, \\ \limsup_{t\to\infty} C^*(x,t) &\leq 0 && \text{in } \Omega \times \mathbb{R}_+. \end{aligned} \tag{7}$$

Our main purpose here is to relax the condition (7) allowing $\limsup_{t\to\infty} C^*(x,t) \geq \alpha > 0$, where α is a constant (see condition (4)) and to note that the full conditions
(1.5.a) (i.e. $b(x,t,0,0)$ is continuous in $\Omega \times \mathbb{R}_+$),
(1.5.b) (i.e. a^{ij} are continuous in $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n, i,j = 1,\dots,n$),
(1.5.c) (i.e. $D_{p_i} b$ is continuous in $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n, i = 1,\dots,n$) and
(1.5.d) (i.e. $D_z b$ is continuous in $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n, i = 1,\dots,n$)
in [**5**] are not needed if the working domain is supposed cylindrical. Moreover our decay remains valid for strong solutions $u \in C^0(\overline{\Omega} \times \mathbb{R}_+) \cap W^{2,1}_{n+1,loc}(\Omega \times \mathbb{R}_+)$. $W^{2,1}_{n+1}(D), D \in \mathbb{R}^{n+1}$ is defined to be the completion of $C^\infty(\overline{D})$ under the norm

$$||u||_{W^{2,1}_{n+1}(D)} = ||D_t u||_{L^{n+1}(D)} + \sum ||D_{ij} u||_{L^{n+1}(D)} + \sum ||D_i u||_{L^{n+1}(D)} + ||u||_{C^0(\overline{D})}.$$

Most decay results (see [**2**], [**5**], [**7**]) are stated under the restriction "there exists (at least) an i such that a^{ii} is bounded below". We next show, using a method due to Hu and Yin ([**4**]), that a decay holds without this restriction. The proofs are based on the well known Nagumo-Westphal Lemma ([**6**, p. 187]) as well as on the following comparison principle:

Theorem 1. *Let $u, v \in C^0(\overline{\Omega_T}) \cap W^{2,1}_{n+1,loc}(\Omega_T)$ satisfy $\mathrm{Q}u \geq \mathrm{Q}v$ in $\Omega_T, u \leq v$ on S_T. Assume that*

i) *Q is uniformly parabolic in Ω_T,*
ii) *the coefficients a^{ij} are independent of z,*
iii) *the coefficient b is non-increasing in z for each $(x,t,p) \in \Omega_T \times \mathbb{R}^n$,*
iv) *the coefficients a^{ij}, b are continuously differentiable with respect to the p variables in $\Omega_T \times \mathbb{R} \times \mathbb{R}^n$.*

Then $u \leq v$ in $\overline{\Omega_T}$.
Here $\Omega_T = \Omega \times (0,T]$, $S_T = \Omega \times \{0\} \cup \partial\Omega \times [0,T]$.

Proof. We will imitate the proof of [**3**, Theorem 10.1, p. 263]. The details are left to the reader.
Step 1. Write $\mathrm{Q}u - \mathrm{Q}v = \mathrm{L}w = -D_t w + a^{ij}(x,t) D_{ij} w + b^i(x,t) D_i w \geq 0$ in $\Omega_T^+ = \{(x,t) \in \Omega_T |\, w(x,t) > 0\}$, where $w = u - v$.

Step 2. Prove a similar result to [**3**, Theorem 9.6, p. 235], i. e. if $u \in W^{2,1}_{n+1,loc}(\Omega_T)$ satisfies $\mathrm{L}u \geq 0$ in Ω_T, then u cannot achieve a maximum in Ω_T, unless it is a constant. Here L is uniformly parabolic in Ω_T and b^i are bounded in Ω_T To prove this result use an Alexandrov, Bakelman, Pucci, Krylov and Tso maximum principle (for example [**1**, Corollary 1.16, p. 548]), an auxiliary function $v(x,t) = e^{-\alpha[r^2+(t-t_0)^2]} - e^{-\alpha(R^2+T^2)}$, α large and imitate the proof of Theorem 9.6.

Step 3. Use *Step 1* and *Step 2* to conclude that

$$\max_{\overline{\Omega_T}^+} w = \max_{\partial\Omega_T^+} w.$$

Step 4. Use *Step 3*, the continuity of w and the boundary conditions to obtain

$$w \leq 0 \quad \text{in}\, \Omega_T.$$

□

2. Main results

We are now in position to prove our main results.

Theorem 2. *Let* (2)–(6) *hold. If u is a classical solution of* (1) *(i.e. $u \in C^0(\overline{\Omega} \times \mathbb{R}_+) \cap C^{2,1}(\Omega \times \mathbb{R}_+)$ then $\lim_{t\to\infty} |u(x,t)| = 0$ uniformly in $\Omega \times \mathbb{R}_+$*

Proof. We restrict ourselves to the case $a^{ij} = \delta^{ij}$. We assume initially that u solves $\mathrm{Q}u \geq 0$ in $\Omega \times \mathbb{R}_+$. We also assume that Ω lies in the strip $0 < x_1 < \mathrm{diam}\Omega$. We choose as comparison function, the strictly positive function

$$w(x,t) = e^{-rt}[\gamma - e^{\eta x_1}],$$

where the strictly positive constants r, η and γ are to be chosen below.
Hence

$$\mathrm{Q}w = e^{-rt}e^{\eta x_1}\left[r\left(\frac{\gamma}{e^{\eta x_1}} - 1\right) - \eta^2\right] + b(x,t,w,D_1w,0,\ldots,0,0).$$

By the mean value theorem we get

$$b(x,t,w,D_1w,0,\ldots,0,0) = b(x,t,0,\ldots,0,0) + wD_z b(\xi) + D_1wD_{p_1}b(\xi).$$

By (3), (4) and (5)

$$b(x,t,w,D_1w,0,\ldots,0,0) \leq K_1e^{-\mu_1 t} + Cw + \beta|D_1w|$$

in $\Omega \times \mathbb{R}_+$. We now have

$$\mathrm{Q}w \leq e^{-rt}e^{\eta x_1}\left[r\left(\frac{\gamma}{e^{\eta x_1}} - 1\right) - \eta^2 + C\left(\frac{\gamma}{e^{\eta x_1}} - 1\right) + \beta\eta + K_1e^{(r-\mu_1)t}\right].$$

We select r small such that

$$r\left(\frac{\gamma}{e^{\eta x_1}} - 1\right) << 1 \quad \text{in}\, \Omega$$

and

$$0 < r < \min\{1, \mu_1, \mu_2\}$$

to obtain

$$\mathrm{Q}w \leq e^{-rt}e^{\eta x_1}\left[\delta - \eta^2 + C(\gamma - 1) + \beta\eta\right]$$

in $\Omega \times [\sigma, \infty)$, where $\delta > 0$ is any positive constant and σ is a sufficiently large constant.

Choose $\eta = \beta + 1 + \delta$ and $\gamma = e^{\eta \operatorname{diam}\Omega} + 1$.
It follows that

$$Qw < 0 \le Qu$$

in $\Omega \times [\sigma, \infty)$. The Nagumo-Westphal Lemma tells us that $u < w$ in $\Omega \times [\sigma, \infty)$.
Since $-u$ solves a similar equation we obtain $|u| < w$ in $\Omega \times [\sigma, \infty)$,
and the result follows. □

In Theorem 1, the condition "there exist an i such that $a^{ii} > \lambda$ in $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n$" cannot be relaxed to allow $a^{ii} > 0$, $i = 1, 2, \dots n$. This is possible in

Theorem 3. *Suppose that the matrix $[a^{ij}]$ is semipositive definite and that relation (3) holds. If in addition the following assumptions are satisfied*

(8) $\quad a^{ij}$ *are bounded in* $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n$ *for* $i \ne j$, $i, j = 1, \dots, n$.

(9) $\quad a^{ii}$ *are bounded above in* $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n$ *for* $i, j = 1, \dots, n$.

$$D_z b \le \frac{K_1}{t^{2+\delta}} \text{ in } \quad \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n. \tag{10}$$

$$b(x, t, 0, 0) \le \frac{K_2}{t^{2+\delta}} \text{ in } \quad \Omega \times \mathbb{R}_+, \tag{11}$$

where K_1, K_2 and δ are strictly positive constants,
then the classical solution of problem (1) *satisfies* $\lim_{t\to\infty} |u(x,t)| = 0$ *uniformly in* $\Omega \times \mathbb{R}_+$.

Proof. For the sake of simplicity we take $a^{ij} = \delta^{ij}$. Let us assume initially that Ω is of class C^2.

We define the distance function $d(x) = \operatorname{dist}(x, \partial\Omega)$. For $\mu > 0$ small (μ need to be less than $\frac{1}{K}$ where K is an upper bound for the normal curvatures of Ω) we set $\Omega_\mu = \{x \in \Omega | d(x) < \mu\}$. [**3**, Lemma 14.16, p. 335] tells us that the function d is smooth, namely $d \in C^2(\overline{\Omega_\mu})$.

In a principal coordinate system (see [**3**, p. 354]) we have for small enough μ

$$\Delta d^2 + 2\beta d \sum |D_i d| + 2 = 2(1 + d\Delta d) + 2\beta d + 2 \le 6 \qquad \text{in } \Omega_\mu.$$

We extend the function d to a strictly positive function in Ω, belonging to $C^2(\Omega)$, which we still denote by d, such that

$$\Delta d^2 + 2\beta d \sum |D_i d| + 2 \le \frac{C}{2} \qquad \text{in } \Omega,$$

for some $C > 0$.

We choose w as comparison function, where

$$w(x,t) = \varepsilon - \frac{1}{d^2 + Ct + 1}.$$

Here ε is any strictly positive constant. Of course $w(x,t) > 0$ in $\Omega \times [\sigma, \infty)$, for sufficiently large σ.
We get

$$\mathrm{Q}w \le \frac{-C}{(d^2+Ct+1)^2} + \frac{1}{(d^2+Ct+1)^2}\left[\Delta d^2 - \frac{8d^2|Dd|^2}{d^2+Ct+1}\right] + b(x,t,w,Dw)$$

in $\Omega \times [\sigma, \infty)$.

Using the mean value theorem, (10) and (11) we obtain

$$\begin{aligned} \mathrm{Q}w \quad \le \quad & \frac{-1}{(d^2+Ct+1)^2}\left[C - \left(\Delta d^2 - \frac{8d^2|Dd|^2}{d^2+Ct+1}\right)\right. \\ & \left. - \frac{\varepsilon K_1(d^2+Ct+1)^2}{t^{2+\delta}} - \frac{K_2(d^2+Ct+1)^2}{t^{2+\delta}} - 2\beta d \sum |D_i d|\right] \end{aligned}$$

in $\Omega \times [\sigma, \infty)$.
Hence $\mathrm{Q}u < 0 \le \mathrm{Q}w$ in $\Omega \times [\sigma, \infty)$ and the proof follows by the Nagumo-Westphal Lemma for smooth domains.

To remove the above restriction on Ω we approximate Ω by smooth domains. □

By virtue of Theorem 1 it is easy to check that the conclusion of Theorem 2 and Theorem 3 remain valid for solutions $u \in C^0(\overline{\Omega} \times (0,\infty)) \cap W^{2,1}_{n+1,loc}(\Omega \times (0,\infty))$. Similar decay estimates for fully nonlinear parabolic operators defined on non cylindrical domains can be inferred from the corresponding results for quasilinear equations. One can easily check that

$$-D_t u + F(x,t,u,Du,D^2u) = -D_t u + a^{ij}(x,t,u,Du)D_{ij}u + b(x,t,u,Du)$$

where,

$$\begin{aligned} a^{ij}(x,t,z,p) &= \int_0^1 F_{ij}(x,t,z,p,sD^2u)ds, \\ b(x,t,z,p) &= F(x,t,z,p,0). \end{aligned}$$

Here $F = F(x,t,z,p,r)$, $r = [r_{ij}]$ is a matrix and $F_{ij} = \frac{\partial F}{\partial r_{ij}}$.

References

1. Cabré X., *On the Alexandroff-Bakelman-Pucci estimate and the reversed Hölder inequality for solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math. **X**LVIII (1995), 539–570.
2. Ewer J. P. G., *On the asymptotic properties of a class of nonlinear parabolic equations*, Appl. Anal. **1**3 (1982), 249–260.
3. Gilbarg D. and Trudinger N. S., *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer Verlag, 2001.
4. Hu B. and Yin H. M., *Blow up of solution for the heat equation with a nonlinear boundary condition*, Comparison Methods and Stability Theory, Lecture Notes in Pure and Applied Mathematics **1**62 (1994), 189–198.
5. Reynolds A., *Asymptotic behavior of solutions of nonlinear parabolic equations*, J. Diff. Equations **1**2 (1972), 256–261.
6. Walter W., *Differential and Integral Inequalities*, Ergebnisse d. Mathematik u. Ihrer Grenzgebiete **V**ol. 55, Springer Verlag, 1970.

7. Wiegner M., *On the asymptotic behaviour of solutions of nonlinear parabolic equations*, Math. Z. **1**88 (1984), 3–22.

C.-P. Danet, Department of Applied Mathematics, University of Craiova, Al. I. Cuza St. 13, 200585 Craiova, Romania, *e-mail*: `cristiandanet@yahoo.com`