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ON A NONLINEAR INTEGRAL EQUATION WITHOUT COMPACTNESS

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Abstract. The purpose of this paper is to obtain an existence result for the integral equation

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s))\, ds, \quad t \in [a, b]$$

where $\varphi : [a, b] \times \mathbb{R} \to \mathbb{R}$ and $\psi : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions which satisfy some special growth conditions. The main idea is to transform the integral equation into a fixed point problem for a condensing map $T : C[a, b] \to C[a, b]$. The "a priori estimate method" (which is a consequence of the invariance under homotopy of the degree defined for α-condensing perturbations of the identity) is used in order to prove the existence of fixed points for T. Note that the assumptions on functions φ and ψ do not generally assure the compactness of operator T, therefore the Leray-Schauder degree cannot be used (see K. Deimling [**2**, Example 9.1, p. 69]).

1. Introduction

The topological methods proved to be a powerful tool in the study of various problems which appear in nonlinear analysis. Particularly, the a priori estimate method (or the method of a priori bounds) has been often used in order to prove the existence of solutions for some boundary value problems for nonlinear differential equations or nonlinear partial differential equations. For example, J. Mawhin uses this method together with the coincidence degree and shows that under appropriate assumptions, the boundary value problem

$$\begin{cases} -x''(t) = f(t, x(t), x'(t)), & t \in [0, \pi] \\ x(0) = x(\pi) = 0 \end{cases}$$

and the problem

$$\begin{cases} x'(t) = f(t, x(t)), & t \in [0, 1] \\ x(0) = x(1) \end{cases}$$

admit solutions (see J. Mawhin [**6**, Sections V.2 and VI.2]). This method is also used (but together with the Leray-Schauder degree) in G. Dinca, P. Jebelean [**3**]

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and G. Dinca, P. Jebelean, J. Mawhin [**4**] to prove the existence of solutions for the problem

$$\begin{cases} -\Delta_p u & = f(t,u) \quad \text{in } \Omega \\ u\mid_{\partial\Omega} & = 0. \end{cases}$$

In the present paper, the a priori estimate method is used together with the degree for condensing maps in order to prove the existence of solutions for the integral equation

$$u(t) = \varphi(t,u(t)) + \int_a^b \psi(t,s,u(s))\,ds, \quad t \in [a,b], \tag{1}$$

under appropriate assumptions on functions φ and ψ. The result presented herein is in relation with a result of F. Isaia [**5**]. The hypothesis imposed on functions φ and ψ are stronger (and considerably simpler), but the result is stronger as well, namely the solution u of equation (1) is in $C[a,b]$, while in F. Isaia [**5**], we obtained $u \in L^p(a,b)$.

2. The topological degree for condensing maps

For a minute description of the following notions we refer the reader to K. Deimling [**2**].

In the following, X will be a Banach space and $\mathcal{B} \subset \mathcal{P}(X)$ will be the family of all its bounded sets.

Definition 1. The function $\alpha : \mathcal{B} \to \mathbb{R}_+$ defined by

$$\alpha(B) = \inf\{d > 0 : B \text{ admits a finite cover by sets of diameter} \leq d\}, \quad B \in \mathcal{B},$$

is called the (Kuratowski-) measure of noncompactness.

In the whole paper, the letter α will only be used in this context. We state without proof some properties of this measure.

Proposition 1. *The following assertions hold:*

(a) $\alpha(B) = 0$ *iff* B *is relatively compact.*
(b) α *is a seminorm, i.e.*

$$\alpha(\lambda B) = |\lambda|\,\alpha(B) \quad \textit{and} \quad \alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2).$$

(c) $B_1 \subset B_2$ *implies* $\alpha(B_1) \leq \alpha(B_2)$*;*

$$\alpha(B_1 \cup B_2) = \max\{\alpha(B_1), \alpha(B_2)\}.$$

(d) $\alpha(convB) = \alpha(B)$.
(e) $\alpha(\overline{B}) = \alpha(B)$.

Definition 2. Consider $\Omega \subset X$ and $F : \Omega \to X$ a continuous bounded map. We say that F is α-Lipschitz if there exists $k \geq 0$ such that

$$\alpha(F(B)) \leq k\alpha(B) \quad (\forall)\, B \subset \Omega \text{ bounded}.$$

If, in addition, $k < 1$, then we say that F is a strict α-contraction. We say that F is α-condensing if

$$\alpha\left(F\left(B\right)\right) < \alpha\left(B\right) \quad \left(\forall\right) B \subset \Omega \text{ bounded with } \alpha\left(B\right) > 0.$$

In other words, $\alpha\left(F\left(B\right)\right) \geq \alpha\left(B\right)$ implies $\alpha\left(B\right) = 0$. The class of all strict α-contractions $F : \Omega \to X$ is denoted by $SC_\alpha\left(\Omega\right)$ and the class of all α-condensing maps $F : \Omega \to X$ is denoted by $C_\alpha\left(\Omega\right)$.

We remark that $SC_\alpha\left(\Omega\right) \subset C_\alpha\left(\Omega\right)$ and every $F \in C_\alpha\left(\Omega\right)$ is α-Lipschitz with constant $k = 1$. We also recall that $F : \Omega \to X$ is Lipschitz if there exists $k > 0$ such that

$$\|Fx - Fy\| \leq k\,\|x - y\| \quad \left(\forall\right) x, y \in \Omega$$

and that F is a strict contraction if $k < 1$.

Next, we state without proof some properties of the applications defined above.

Proposition 2. *If $F, G : \Omega \to X$ are α-Lipschitz maps with constants k, respectively k', then $F + G : \Omega \to X$ is α-Lipschitz with constant $k + k'$.*

Proposition 3. *If $F : \Omega \to X$ is compact, then F is α-Lipschitz with constant $k = 0$.*

Proposition 4. *If $F : \Omega \to X$ is Lipschitz with constant k, then F is α-Lipschitz with the same constant k.*

The theorem below asserts the existence and the basic properties of the topological degree for α-condensing perturbations of the identity.

Let

$$\mathcal{T} = \left\{ \begin{array}{ll} (I - F, \Omega, y) : & \Omega \subset X \text{ open and bounded,} \\ & F \in C_\alpha\left(\overline{\Omega}\right),\ y \in X \setminus \left(I - F\right)\left(\partial\Omega\right) \end{array} \right\}$$

be the family of the admissible triplets. There exists one degree function $D : \mathcal{T} \to \mathbb{Z}$ which satisfies the properties:

Theorem 1. (D1) *$D\left(I, \Omega, y\right) = 1$ for every $y \in \Omega$ (Normalization).*

(D2) *For every disjoint, open sets $\Omega_1, \Omega_2 \subset \Omega$ and every $y \notin (I - F)\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$ we have*

$$D\left(I - F, \Omega, y\right) = D\left(I - F, \Omega_1, y\right) + D\left(I - F, \Omega_2, y\right)$$

(Additivity on domain).

(D3) *$D(I - H(t, \cdot)\,, \Omega, y\left(t\right))$ is independent of $t \in [0, 1]$ for every continuous, bounded map $H : [0, 1] \times \overline{\Omega} \to X$ which satisfies*

$$\alpha\left(H\left(\left[0, 1\right] \times B\right)\right) < \alpha\left(B\right) \quad \left(\forall\right) B \subset \overline{\Omega} \text{ with } \alpha\left(B\right) > 0$$

and every continuous function $y : [0, 1] \to X$ which satisfies

$$y\left(t\right) \neq x - H\left(t, x\right) \quad \left(\forall\right) t \in \left[0, 1\right], \left(\forall\right) x \in \partial\Omega$$

(Invariance under homotopy).

(D4) *$D\left(I - F, \Omega, y\right) \neq 0$ implies $y \in \left(I - F\right)\left(\Omega\right)$ (Existence).*

(D5) *$D(I-F,\Omega,y) = D(I-F,\Omega_1,y)$ for every open set $\Omega_1 \subset \Omega$ and every $y \notin (I-F)\left(\overline{\Omega}\setminus\Omega_1\right)$ (Excision).*

Having in hand a degree function defined on $\mathcal{T}$, we study the usability of the "a priori estimate method" by means of this degree.

Theorem 2. *Let $F : X \to X$ be α-condensing and*

$$S = \{x \in X : (\exists)\,\lambda \in [0,1] \text{ such that } x = \lambda F x\}.$$

If S is a bounded set in X, so there exists $r > 0$ such that $S \subset B_r(0)$, then

$$D(I-\lambda F, B_r(0), 0) = 1 \quad (\forall)\,\lambda \in [0,1].$$

Consequently, F has at least one fixed point and the set of the fixed points of F lies in $B_r(0)$.

Proof. First, we remark that every affine homotopy of α-condensing maps is an admissible homotopy. To see this, let us consider a bounded open set $\Omega \subset X$, the maps $F_1, F_2 \in C_\alpha\left(\overline{\Omega}\right)$ and let $H : [0,1]\times\overline{\Omega} \to X$ be defined by

$$H(t,x) = (1-t)F_1x + tF_2x.$$

For every $B \subset \overline{\Omega}$ with $\alpha(B) > 0$ we have

$$H([0,1]\times B) \subset \operatorname{conv}(F_1(B) \cup F_2(B))$$

and, using Proposition 1,

$$\begin{aligned} \alpha(H([0,1]\times B)) &\le \alpha(\operatorname{conv}(F_1(B)\cup F_2(B))) \\ &= \alpha(F_1(B)\cup F_2(B)) \\ &= \max\{\alpha(F_1(B)),\alpha(F_2(B))\} < \alpha(B). \end{aligned}$$

Next, we fix $\lambda \in [0,1]$ and we consider the affine homotopy between the α-condensing maps $\lambda F, 0 \in C_\alpha(X)$

$$H : [0,1]\times X \to X, \quad H(t,x) = (1-t)\,0x + t\lambda F x = t\lambda F x.$$

By the previous argument,

$$\alpha(H([0,1]\times B)) < \alpha(B) \quad (\forall)\,B \subset X \text{ bounded with } \alpha(B) > 0.$$

If $x \in X$ and $t \in [0,1]$ verify $x - H(t,x) = 0$, then $x \in S \subset B_r(0)$. Thus, we can use the properties (D3), (D1) of the degree and we obtain

$$\begin{aligned} D(I-\lambda F, B_r(0), 0) &= D(I - H(1,\cdot), B_r(0), 0) \\ &= D(I - H(0,\cdot), B_r(0), 0) \\ &= D(I, B_r(0), 0) = 1. \end{aligned}$$

Finally, the property (D4) of the degree is used. $\square$

3. The existence result

Consider equation (1)

$$u(t)=\varphi(t,u(t))+\int_a^b \psi(t,s,u(s))\,ds,\quad t\in[a,b],$$

where $\varphi:[a,b]\times\mathbb{R}\to\mathbb{R}$ and $\psi:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ are continuous functions which satisfy the following conditions:

(a) There exist $C_1, M_1\geq 0,\ q_1\in[0,1)$ such that

$$|\varphi(t,x)|\leq C_1|x|^{q_1}+M_1$$

for every $(t,x)\in[a,b]\times\mathbb{R}$.

(b) There exists $K_1\in[0,1)$ such that

$$|\varphi(t,x)-\varphi(t,y)|\leq K_1|x-y|$$

for every $(t,x),(t,y)\in[a,b]\times\mathbb{R}$.

(c) There exist $C_2, M_2\geq 0,\ q_2\in[0,1)$ such that

$$|\psi(t,s,x)|\leq C_2|x|^{q_2}+M_2$$

for every $(t,s,x)\in[a,b]\times[a,b]\times\mathbb{R}$.

Under these assumptions, we will show that equation (1) has at least one solution $u\in C[a,b]$.

Define operators

$$\begin{aligned} F &: C[a,b]\to C[a,b], && (Fu)(t)=\varphi(t,u(t)),\ t\in[a,b],\\ G &: C[a,b]\to C[a,b], && (Gu)(t)=\int_a^b \psi(t,s,u(s))\,ds,\ t\in[a,b],\\ T &: C[a,b]\to C[a,b], && Tu=Fu+Gu. \end{aligned}$$

Then, equation (1) can be written as

$$u=Tu. \tag{2}$$

Thus, the existence of a solution for equation (1) is equivalent to the existence of a fixed point for operator T.

Proposition 5. *The operator $F:C[a,b]\to C[a,b]$ is Lipschitz with constant K_1. Consequently F is α-Lipschitz with the same constant K_1.*

Proof. From (b), we have

$$\begin{aligned} \|Fu-Fv\|_{C[a,b]} &= \sup_{t\in[a,b]}|(Fu)(t)-(Fv)(t)|\\ &= \sup_{t\in[a,b]}|\varphi(t,u(t))-\varphi(t,v(t))|\\ &\leq K_1\sup_{t\in[a,b]}|u(t)-v(t)|=K_1\|u-v\|_{C[a,b]}, \end{aligned}$$

for every $u,v\in C[a,b]$. By Proposition 4, F is α-Lipschitz with constant K_1.

Moreover, F satisfies the following growth condition:

$$\|Fu\|_{C[a,b]} \leq C_1 \|u\|_{C[a,b]}^{q_1} + M_1, \tag{3}$$

for every $u \in C[a,b]$. Relation (3) is a simple consequence of condition (a). $\square$

Proposition 6. *The operator $G : C[a,b] \to C[a,b]$ is compact. Consequently G is α-Lipschitz with zero constant.*

Proof. First, we prove the continuity of G. Let $(u_n) \subset C[a,b]$, $u \in C[a,b]$ be such that $\|u_n - u\|_{C[a,b]} \to 0$. We have to show that $\|Gu_n - Gu\|_{C[a,b]} \to 0$. Fix $\varepsilon > 0$. There exists a constant $K \geq 0$ such that

$$\begin{aligned} \|u_n\|_{C[a,b]} &\leq K \quad (\forall)\, n \in \mathbb{N}^*, \\ \|u\|_{C[a,b]} &\leq K. \end{aligned}$$

Using the uniform continuity of ψ on $[a,b] \times [a,b] \times [-K,K]$, we derive that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|\psi(t_1, s_1, x_1) - \psi(t_2, s_2, x_2)| \leq \frac{\varepsilon}{b-a},$$

for every $(t_1, s_1, x_1), (t_2, s_2, x_2) \in [a,b] \times [a,b] \times [-K,K]$ such that $|t_1 - t_2| + |s_1 - s_2| + |x_1 - x_2| < \delta$. From $\|u_n - u\|_{C[a,b]} \to 0$, it follows that there exists $N = N(\varepsilon) \in \mathbb{N}^*$ such that

$$\sup_{t \in [a,b]} |u_n(t) - u(t)| < \delta,$$

for every $n \geq N$. Consequently,

$$\begin{aligned} \|Gu_n - Gu\|_{C[a,b]} &= \sup_{t \in [a,b]} \left| \int_a^b \psi(t, s, u_n(s))\, ds - \int_a^b \psi(t, s, u(s))\, ds \right| \\ &\leq \sup_{t \in [a,b]} \int_a^b |\psi(t, s, u_n(s)) - \psi(t, s, u(s))|\, ds < \varepsilon, \end{aligned}$$

for every $n \geq N$. The continuity of G is proved.

Moreover, G satisfies the following growth condition:

$$\|Gu\|_{C[a,b]} \leq C_2 (b-a) \|u\|_{C[a,b]}^{q_2} + (b-a) M_2, \tag{4}$$

for every $u \in C[a,b]$. Relation (4) is a simple consequence of condition (c).

In order to prove the compactness of G, we consider a bounded set $M \subset C[a,b]$ and we will show that $G(M)$ is relatively compact in $C[a,b]$ with the help of Arzela-Ascoli theorem. Let $\overline{K} \geq 0$ be such that

$$\|u\|_{C[a,b]} \leq \overline{K},$$

for every $u \in M$. By (4), we have

$$\|Gu\|_{C[a,b]} \leq (b-a) \left[C_2 \overline{K}^{q_2} + M_2 \right],$$

for every $u \in M$, so $G(M)$ is bounded in $C[a,b]$. Fix $\varepsilon > 0$. Using the uniform continuity of ψ on $[a,b] \times [a,b] \times \left[-\overline{K}, \overline{K}\right]$, we derive that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|\psi(t_1, s_1, x_1) - \psi(t_2, s_2, x_2)| \leq \frac{\varepsilon}{b-a},$$

for every $(t_1, s_1, x_1), (t_2, s_2, x_2) \in [a,b] \times [a,b] \times \left[-\overline{K}, \overline{K}\right]$ such that $|t_1 - t_2| + |s_1 - s_2| + |x_1 - x_2| < \delta$. If $t_1, t_2 \in [a,b]$ satisfy $|t_1 - t_2| < \delta$, then

$$|(Gu)(t_1) - (Gu)(t_2)| \leq \int_a^b |\psi(t_1, s, u(s)) - \psi(t_2, s, u(s))| \, ds < \varepsilon,$$

for every $u \in M$. The set $G(M) \subset C[a,b]$ satisfies the hypothesis of Arzela-Ascoli theorem, so $G(M)$ is relatively compact in $C[a,b]$.

By Proposition 3, G is α-Lipschitz with zero constant. □

Now, we have the possibility to prove the main result of this paper.

Theorem 3. *If the functions $\varphi : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $\psi : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions (a), (b),(c), then the integral equation*

$$u(t) = \varphi(t, u(t)) + \int_a^b \psi(t, s, u(s)) \, ds, \quad t \in [a,b],$$

has at least one solution $u \in C[a,b]$ and the set of the solutions of equation (1) *is bounded in $C[a,b]$.*

Proof. Let $F, G, T : C[a,b] \to C[a,b]$ be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, F is α-Lipschitz with constant $K_1 \in [0,1)$ and G is α-Lipschitz with zero constant (see Propositions 5 and 6). Proposition 2 shows us that T is a strict α-contraction with constant K_1.

Set

$$S = \{u \in C[a,b] : (\exists)\, \lambda \in [0,1] \text{ such that } u = \lambda T u\}.$$

Next, we prove that S is bounded in $C[a,b]$. Consider $u \in S$ and $\lambda \in [0,1]$ such that $u = \lambda T u$. It follows from (3) and (4) that

$$\begin{aligned} \|u\|_{C[a,b]} &= \lambda \|Tu\|_{C[a,b]} \leq \lambda \left(\|Fu\|_{C[a,b]} + \|Gu\|_{C[a,b]} \right) \\ &\leq \lambda \left[C_1 \|u\|_{C[a,b]}^{q_1} + C_2 (b-a) \|u\|_{C[a,b]}^{q_2} + M_1 + (b-a) M_2 \right]. \end{aligned}$$

This inequality, together with $q_1 < 1$, $q_2 < 1$, shows us that S is bounded in $C[a,b]$.

Consequently, by Theorem 2 we deduce that T has at least one fixed point and the set of the fixed points of T is bounded in $C[a,b]$. □

Remark 1.

(i) if the growth condition (a) is formulated for $q_1 = 1$, then the conclusions of Theorem 3 remain valid provided that $C_1 < 1$;

(ii) if the growth condition (c) is formulated for $q_2 = 1$, then the conclusions of Theorem 3 remain valid provided that $(b-a) C_2 < 1$;

(iii) if the growth conditions (a) and (c) are formulated for $q_1 = 1$ and $q_2 = 1$, then the conclusions of Theorem 3 remain valid provided that

$$C_1 + (b - a)\, C_2 < 1.$$

Remark 2. The conclusions of Theorem 3 remain valid provided that equation (1) is replaced by

$$u(t) = \varphi(t, u(t)) + \int_a^t \psi(t, s, u(s))\, ds, \quad t \in [a, b].$$

Only slight modifications in the proof of Proposition 6 are needed.

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