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ON GENERALIZATIONS OF INJECTIVITY

LE DUC THOANG and LE VAN THUYET

Abstract. A ring R is called right GP-injective if for every nonzero element a in R, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^nR into R can be extended to one of R into R. A ring R is called right FSG if every finitely generated cofaithful right R-module is a generator in Mod-R. In this paper, we give some characterizations of PF rings, QF rings via GP-injective rings, FSG rings.

1. Introduction

Throughout this paper, R is an associative ring with identity $1 \neq 0$ and all modules considered are unitary modules. We write M_R (resp. ${}_RM$) to denote that M is a right (resp. left) R-module. The category of right (resp. left) R-module is denoted by Mod-R (resp. R-Mod). Unless otherwise mentioned, by a module we will mean a right R-module.

We recall some concepts and notations will be used in this paper. Let M be an R-module, we denote the Jacobson radical of M (resp. injective envelope, singular submodule and socle) of M by $\mathrm{Rad}(M)$ (resp. $E(M), Z(M)$ and $\mathrm{Soc}(M)$). When $M = R_R$, we write $\mathrm{Rad}(R_R) = J$ $(= \mathrm{Rad}({}_RR))$. If A is a submodule of M (resp. proper submodule), we denote by $A \leq M$ (resp. $A < M$). Moreover, we write $A \leq^e M$ to denote that A is an essential submodule of M. The right and left annihilators of a subset X of a ring R are denoted by $r(X)$ and $l(X)$, respectively.

A module M is called *uniform* if $M \neq 0$ and every non-zero submodule of M is essential in M. M has *finite Goldie dimension* n (finite uniform dimension) if there is a direct sum of n uniform submodules of M which is essential in M, or equivalently, there is a monomorphism from a direct sum of n uniform submodules of M to M such that its image is essential in M. We write $\mathrm{udim}(M) = n$ and call $\mathrm{udim}(M)$ to be finite Goldie dimension of M.

A ring R is called *quasi-Frobenius* (briefly, QF ring) if it is left and right artinian and left and right self-injective; or equivalently, if R has the ACC on right or left annihilators and is right or left self-injective. A ring R is called *right pseudo-Frobenius* (briefly, right PF) ring if every faithful right R-module is a generator; or

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equivalently, R is a semiperfect, right self-injective ring with essential right socle. A ring R is called *right finitely pseudo-Frobenius* (briefly, right FPF) ring if every finitely generated faithful right R-module is a generator.

We will consider a generalization of the concept of injectivity. Let M be an R-module and I a right ideal of R. We take an R-homomorphism f of I to M. Consider the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & R \arrow[dl, "h"] \\ & M & \end{tikzcd}$$

If there exists $h \in \operatorname{Hom}_R(R, M)$ for every principal (minimal, resp.) right ideal I in R and any $f \in \operatorname{Hom}_R(I, M)$, then we say that M is *P-injective (mininjective, resp.)*; or equivalently, $f = m\cdot$ is left multiplication by some element m of M. If for every $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism of a^nR into M can be extended to one of R into M, then M is called right *GP-injective.* A ring R is called right mininjective (resp. P-injective, GP-injective) if R_R is mininjective (resp. P-injective, GP-injective). A ring R is called a right *minannihilator ring* if every minimal right ideal H of R is an annihilator, equivalently, if $rl(H) = H$ and called a left *minsymmetric ring* if Rk is simple, $k \in R$, implies that kR is simple. For example, any left mininjective ring is left minsymmetric.

For the concepts and results are not shown in this paper, we will refer to Anderson and Fuller [**1**], Dung, Huynh, Smith and Wisbauer [**3**], Faith [**4**] and Wisbauer [**19**].

2. GP-injective rings with essential socles

Proposition 2.1. *The following conditions are equivalent for a right R-module M.*

(i) *M is GP-injective.*

(ii) *For each element $0 \neq a \in R$, there exists $n \in \mathbb{N}^*$ with $a^n \neq 0$, $l_M(r_R(a^n)) = Ma^n$.*

Proof. By [**15**, Lemma 1.3]. $\square$

A ring R is called right *generalized pseudo-Frobenius ring* (briefly, GPF-ring) if R is semiperfect, right P-injective and $\operatorname{Soc}(R_R)$ is essential as a right ideal. For convenience, we call a ring R *SGPE-ring* if R is semiperfect, right GP-injective and $\operatorname{Soc}(R_R)$ is essential as a right ideal. The following properties of a SGPE ring can be extended from properties of a GPF ring in [**12**], [**13**]. Some following properties were obtained in [**2**].

Proposition 2.2. *Let R be a right SGPE ring. Then the following statements hold:*

(i) *R is right and left Kasch.*
(ii) $\mathrm{Soc}\,(R_R) = \mathrm{Soc}\,({}_RR) = S$ *is essential in both R_R and ${}_RR$.*
(iii) *R is left finitely cogenerated.*
(iv) $l(S) = J = r(S)$ *and* $l(J) = S = r(J)$.
(v) $J = Z(R_R) = Z({}_RR)$.
(vi) $\mathrm{Soc}\,(Re) = Se$ *is simple and essential in Re for every local idempotent $e \in R$.*
(vii) $\mathrm{Soc}\,(eR)$ *is homogeneous and essential in eR for every local idempotent $e \in R$.*
(viii) *The map $K \mapsto r(K)$ and $T \mapsto l(T)$ are mutually inverse lattice isomorphisms between the simple left ideals K and the maximal right ideals T.*
(ix) *If $\{e_1, \ldots, e_n\}$ is a basic set of local idempotents, there exists elements $k_1, \ldots, k_n$ in R and a permutation σ of $\{1, 2, \ldots, n\}$ such that the following hold for all $i = 1, 2, \ldots, n$:*
   (*a*) $k_iR \subseteq e_iR$ *and* $Rk_i \subseteq Re_{\sigma i}$.
   (*b*) $k_iR \cong e_{\sigma i}R/e_{\sigma i}J$ *and* $Rk_i \cong Re_i/Je_i$.
   (*c*) $\{k_1R, \ldots, k_nR\}$ *and* $\{Rk_1, \ldots, Rk_n\}$ *are complete sets of distinct representatives of the simple right and left R-modules, respectively.*
   (*d*) $\mathrm{Soc}\,(Re_{\sigma i}) = Rk_i = Se_{\sigma i} \cong Re_i/Je_i$ *is simple and essential in $Re_{\sigma i}$ for each i.*
   (*e*) $\mathrm{Soc}\,(e_iR) \neq 0$ *is homogeneous and essential in e_iR with each simple submodule isomorphic to $e_{\sigma i}R/e_{\sigma i}J$.*

The following lemma is useful to prove the main result of this section.

Lemma 2.3. [**16**, Theorem 8], *Let R be a right artinian ring. The following conditions are equivalent:*

(i) *R is a quasi-Frobenius ring.*
(ii) (*a*) *R is a QF-2 ring.*
   (*b*) $\mathrm{Soc}\,(R_R) \leq \mathrm{Soc}\,({}_RR)$.
(iii) (*a*) $\mathrm{Soc}\,(eR)$ *is a minimal right ideal and* $\mathrm{Soc}\,(Re)$ *is a minimal left ideal for every local idempotent $e \in R$.*
   (*b*) $\mathrm{Soc}\,(R_R) \leq \mathrm{Soc}\,({}_RR)$.

Now we give some characterizations of a QF-ring via GP-injective rings.

Theorem 2.4. *The following conditions are equivalent for a ring R:*

(i) *R is a quasi-Frobenius ring.*
(*ii*) *R is a right minannihilator, right GP-injective ring and R has ACC on right annihilators.*
(iii) *R is a left mininjective, right GP-injective ring and R has ACC on right annihilators.*
(iv) *R is a left minsymmetric, right GP-injective ring and R has ACC on right annihilators.*

(v) *R is a right GP-injective ring,* $\mathrm{Soc}\,(eR)$ *is simple for every local* $e \in R$ *and R has ACC on right annihilators.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii). We note that, if R is a right GP-injective ring satisfying ACC on right annihilators then R is left artinian by [**2**, Theorem 3.7]. Then R is a right SGPE ring. It follows from Propostion 2.2 that $\mathrm{Soc}\,(R_R) = \mathrm{Soc}\,({}_RR) = S$ is essential in both R_R and ${}_RR$. By [**14**, Corollary 2.5], R is a left mininjective ring.

(iii) $\Rightarrow$ (iv). Since R is left mininjective, R is left minsymmetric by [**14**, Theorem 1.14].

(iv) $\Rightarrow$ (v). Same argument of (ii) $\Rightarrow$ (iii), the ring R is left artinian, right and left Kasch and $\mathrm{Soc}\,(Re)$ is simple for every local idempotent $e \in R$. Since R is minsymmetric, $\mathrm{Soc}\,(eR)$ is also simple for every local idempotent $e \in R$.

(v) $\Rightarrow$ (i). Same argument of (ii) $\Rightarrow$ (iii), the ring R is left artinian. So R is a right SGPE ring and then by Proposition 2.2, $\mathrm{Soc}\,(R_R) = \mathrm{Soc}\,({}_RR) = S$, $\mathrm{Soc}\,(Re)$ is simple for every local idempotent $e \in R$. By assumption, $\mathrm{Soc}\,(eR)$ is simple for every local idempotent $e \in R$. Applying Lemma 2.3, R is QF. □

3. FSG, GP-injective rings and the Kasch condition

A ring R is called *right finitely subgenerator generator* (briefly, right FSG) if every finitely generated cofaithfull right R-module is a generator. FSG rings was introduced and investigated in [**18**]. It is well known that a ring R is right self-injective if and only if every cofaithful right R-module is a gennerator and a cofaithful module is faithful. Thus, right FSG ring is a generalization of both right FPF ring and right self-injective ring. For example, the ring of intergers $\mathbb{Z}$ is FSG and is not self-injective. Let D be a division ring (e.g. $D = \mathbb{R}$) and $S = \mathrm{End}_D(V)$, where V is an infinite dimensional vector space over D (e.g. $V = \mathbb{R}^{(N)}$). Then S is right FSG because of self-injectivity of S. Now, let $R = \mathbb{Z} \oplus S$. Then R is a right FSG ring which is neither self-injective nor FPF.

Lemma 3.1. [**18**, Corollary 5.10] *For a local ring R, the following conditions are equivalent:*

(i) *R is right FSG ring such that its Jacobson radical consists of zero divisors.*
(ii) *R is a right self-injective ring.*

Lemma 3.2. [**18**, Theorem 5.8] *Any semiperfect right FSG ring with nil Jacobson radical is right self-injective.*

Note 3.3. Let R be a semiperfect ring, and let $\{e_1, \dots, e_n\}$ be a set of orthogonal primitive idempotents of R. Then $R_R = e_1R \oplus \cdots \oplus e_nR$. Renumber idempotents if necessary so that $e_1R/e_1J, \dots, e_tR/e_tJ$ ($t \leqslant n$) constitute the isomorphism classes of simple right R-module. Thus, every simple right R-module is isomorphic to some e_iR/e_iJ with $i \leqslant t$. The right ideal $B = e_1R \oplus \cdots \oplus e_tR$ is called the *basic module* of R, $e_0 = e_1 + \cdots + e_t$ is then called the *basic idempotent*. We will keep the above notations up to the end of this paper.

Proposition 3.4. *Let R be a local ring. Then the following conditions are equivalent:*

(i) *R is right self-injective.*
(ii) *R is right P-injective, right FSG.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i). Let R be a right P-injective, right FSG ring. We will prove that for every x of R, $r(x) = 0$ if and only if there exists y of R such that $xy = 1$ (or $yx = 1$ because a local ring is directly finite). Let x be an element of R such that $r(x) = 0$, then $r(Rx) = 0$. It follows that $lr(Rx) = R$. However R is a right P-injective ring, $lr(Rx) = Rx$, hence $Rx = R$. Thus there exists y of R such that $yx = 1$.

Conversely, let $x \in R$ such that there exists y of R satisfying $xy = 1$ and hence $yx = 1$. If $z \in r(x)$, then $xz = 0$ and $yxz = 0$ hence $z = 0$. Thus $r(x) = 0$.

This establishes the previous claim.

Now, since R is a local ring, the Jacobson radical J of R consists of x such that x is not invertible. Thus J consists of zero divisors.

By Lemma 3.1, R is a right self-injective ring. $\square$

Proposition 3.5. *The following conditions are equivalent for a ring R:*

(i) *R is a QF ring.*
(ii) *R is a right GP-injective, right FSG ring such that R has ACC on right annihilators.*
(iii) *R is a semiperfect right GP-injective, right FSG ring such that $R/\operatorname{Soc}(R_R)$ is right Goldie.*
(iv) *R is a semiperfect right GP-injective, right FSG ring such that $R/\operatorname{Soc}(R_R)$ is left Goldie.*

Proof. (i) $\Rightarrow$ (ii), (iii) and (iv) are easy.

(ii) $\Rightarrow$ (i). Assume (ii). Then R is left artinian by [**2**, Theorem 3.7]. Then $J(R)$ is nilpotent. By Lemma 3.2, R is right self-injective.

(iii) $\Rightarrow$ (i). By [**15**, Corollary 2.11], $J(R)$ is nilpotent. By Lemma 3.2, R is right self-injective. Hence R is QF by [**5**, Theorem 4.1].

(iv) $\Rightarrow$ (i). Same argument of (iii) $\Rightarrow$ (i). $\square$

Motivated by [**21**, Theorem 1], we obtain the following result.

Theorem 3.6. *Let R be a semiperfect, right FSG ring. Then R is right self-injective if and only if $J(R) = Z(R_R)$.*

Proof. Suppose $J(R) = Z(R_R)$ and let $\{e_1, \ldots, e_n\}$ be a set of orthogonal primitive idempotents of R and the basic idempotent $e_0 = e_1 + \cdots + e_t$. To prove R is right self-injective, it is suffice to show that e_iR is injective for every $i = 1, \ldots, t$.

Let $E_1 = E(e_1R)$ be an injective hull of e_1R and y be any element of E_1, we prove that $y \in e_1R$ and e_1R is then injective. Proofs of injectivity of e_jR $(j = 2, \dots, t)$ are similar.

By [**18**, Theorem 5.4], e_1R is uniform. Hence $(yR + e_1R)$ is uniform. Let

$$M = (yR + e_1R) \oplus e_2R \oplus \cdots \oplus e_tR$$

is a finitely generated right R-module. Since R_R is always embedded in M^l $(l = n-t+1)$, hence M is a finitely generated cofaithfull right R-module. Since R is right FSG, hence M is a generator. Thus $M \cong e_1R \oplus \cdots \oplus e_nR \oplus X_R$ for some module X_R. By Krull-Schmidt Theorem, since $\mathrm{End}_R(e_1R)$ is local and $e_jR \ncong e_1R$ $(j = 2, \dots, t)$, it follows that $(yR + e_1R) \cong e_1R \oplus T_R$ for some module T_R. Since $yR + e_1R$ is uniform, $yR + e_1R \cong e_1R$ and hence $yR + e_1R$ is a local module. Let σ be an R-isomorphism between $yR + e_1R$ and e_1R. If $e_1R \neq yR + e_1R$, then

$$e_1R \leq J(yR{+}e_1R) \qquad \text{and} \qquad \sigma(e_1R) \leq J(e_1R) = e_1J(R) = e_1Z(R_R) \leq Z(R_R).$$

Now $r(e_1) = r(\sigma(e_1))$ which is right essential in R_R, a contradiction. Thus $y \in e_1R$. This complete the proof. □

Corollary 3.7. *Let R be a semiperfect ring. Then the following conditions are equivalent:*

(i) *R is QF.*
(ii) *R is a right FSG ring, $J(R) = Z(R_R)$ and R has ACC on right or left annihilators.*
(iii) *R is a right FSG, right P-ring and R has ACC on right or left annihilators.*
(iv) *R is a right FSG ring, $J(R) = Z(R_R)$ and R has DCC on essential right or left ideals.*
(v) *R is a right FSG, right P-ring and R has DCC on essential right or left ideals.*
(vi) *R is a right FSG ring, $J(R) = Z(R_R)$ and R has ACC on essential right or left ideals.*
(vii) *R is a right FSG, right P-ring and R has ACC on essential right or left ideals.*
(viii) *R is a right FSG ring, $J(R) = Z(R_R)$ and $R/\mathrm{Soc}(R_R)$ is right Goldie.*
(ix) *R is a right FSG, right P-ring and $R/\mathrm{Soc}(R_R)$ is right Goldie.*
(x) *R is a right FSG ring, $J(R) = Z(R_R)$ and $R/\mathrm{Soc}(R_R)$ is left Goldie.*
(xi) *R is a right FSG, right P-ring and $R/\mathrm{Soc}(R_R)$ is left Goldie.*

Proof. By Proposition 3.4, Theorem 3.6 and [**5**, Theorem 4.1]. □

The following result extends [**6**, Lemma 5.2]

Theorem 3.8. *The following conditions are equivalent for a ring R:*

(i) *R is right PF.*
(ii) *R is a semiperfect, right FPF ring with essential right socle.*
(iii) *R is a semiperfect, right FSG ring with essential right socle.*

Proof. (i) $\Rightarrow$ (iii) is clear and (ii) $\Leftrightarrow$ (i) is [**6**, Lemma 5.2].

(iii) $\Rightarrow$ (ii). Let $\{e_1, \ldots, e_n\}$ be a set of orthogonal primitive idempotents of R. Since R is semiperfect right FSG, by [**18**, Theorem 5.4], $R = \oplus_{i=1}^n e_i R$, each $e_i R$ is uniform. From this and the fact that R has essential right socle, it follows that $\mathrm{Soc}(R_R)$ is finitely generated. Now let M_R be any finitely generated faithful right R-module, by the Beachy's Theorem (see [**4**, Theorem 19.13A], M_R is cofaithful. So M_R is a generator, and R is then a right FPF ring. $\square$

Corollary 3.9. [**18**, Theorem 5.11] *For a left perfect ring R, the following conditions are equivalent:*

(i) *R is right PF.*
(ii) *R is right FPF.*
(iii) *R is right FSG.*

Proof. Given (iii). Let $\{e_1, \ldots, e_n\}$ be a set of orthogonal primitive idempotents of R, by [**18**, Theorem 5.4], $R = \oplus_{i=1}^n e_i R$, each $e_i R$ is uniform. By the Bass's Theorem (see [**4**, 18.27.3]), it implies that R has essential right socle, and (i) follows from Theorem 3.8. $\square$

The following result extends [**18**, Corollary 5.13].

Corollary 3.10. *A right PF ring R is left PF if and only if R is left FSG.*

Proof. Since R is right PF ring, it's right SGPE, and hence $\mathrm{Soc}({}_R R) \leq^e {}_R R$ by Proposition 2.2. Thus R is left PF by Theorem 3.8. $\square$

The following result extends [**5**, Corollary 2.3 and 2.7].

Corollary 3.11. *A left* (*or right*) *perfect, right and left FSG ring R is QF.*

Proof. Since R is left perfect, right FSG, it follows from Corollary 3.9 that R is right PF. In addition, since R is left FSG, R is PF by Corollary 3.10. Thus R is QF by [**5**, Theorem 2.3] $\square$

The following result extends [**10**, Proposition 14].

Theorem 3.12. *The following conditions are equivalent for a ring R:*

(i) *R is right PF.*
(ii) *R is a right SGPE, right FSG ring.*
(iii) *R is a semiperfect, right FSG ring, and satisfies* $\mathrm{Soc}(R_R) \leq^e \mathrm{Soc}(R_R)$.
(iv) *R is a semiperfect, right FSG, left and right P-injective, left Kasch ring.*
(v) *R is a semiperfect, right FSG, left GP-injective, left Kasch ring.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$(iii), (iv) $\Rightarrow$ (v) are clear.

(iii) $\Rightarrow$ (i). By Theorem 3.8.

(i) $\Rightarrow$ (iv). Given (i). Then all conditions in (*iv*) are satisfied immediately exception for R being left P-injective, and it is satisfied by [**12**, Lemma 5.21].

(iv) $\Rightarrow$ (iii). Since R is left GP-injective, left Kasch ring, it follows that $\mathrm{Soc}(R_R) \leq^e R_R$ by [**2**, Theorem 2.3]. $\square$

4. Goldie dimension and some application to FSG rings

Lemma 4.1. *Let $N_R \leq M_R$ be R-modules. Then:*

(i) *If M has finite Goldie dimension, then N has finite Golodie dimension and* $\operatorname{udim}(N) \leqslant \operatorname{udim}(M)$.

(ii) *If $N \leq^e M$ then M has finite Goldie dimension if and only if N has finite Goldie dimension, and in this case* $\operatorname{udim}(M) = \operatorname{udim}(N)$.
Conversely, if M has finite Goldie dimension and $\operatorname{udim}(M) = \operatorname{udim}(N)$, *then*
$N \leq^e M$.

Proof. (i) is easy, (ii) is a part of [**3**, 5.8]. □

Lemma 4.2. *Let R be a semiperfect, right FSG ring with set of orthogonal primitive idempotents $\{e_1, \dots, e_n\}$, the basic idempotent $e_0 = e_1 + \cdots + e_t$. If R contain t non-isomorphic minimal right ideals, then* $\operatorname{udim}(\operatorname{Soc}(R_R)) = n$.

Proof. Note that, for every $i = 1, \dots, n$, $\operatorname{Soc}(e_iR)$ is either simple or zero by [**18**, Theorem 5.4].

Firstly, we prove that $\operatorname{Soc}(e_iR)$ is simple for every $1 \leqslant i \leqslant t$.

Assume on the contrary. Then there exists a positive integer i, $1 \leqslant i \leqslant t$, such that $\operatorname{Soc}(e_iR) = 0$. On the other hand, for every k, $t+1 \leqslant k \leqslant n$. Since $e_kR \cong e_jR$ for some $j \in \{1, \dots, t\}$, hence $\operatorname{Soc}(e_kR) \cong \operatorname{Soc}(e_jR)$. This contradicts to the fact that R contain t non-isomorphic minimal right ideals.

By the same argument, it implies that $\operatorname{Soc}(e_kR)$ is simple for every k, $t+1 \leqslant k \leqslant n$. Thus $\operatorname{udim}(\operatorname{Soc}(R_R)) = n$. □

Lemma 4.3. *Let R be a semiperfect, left mininjective ring. Then R is left Kasch if and only if $e\operatorname{Soc}({}_RR)$ is simple for every local idempotent e in R.*

Proof. It is straightforward from [**12**, Theorem 3.2]. □

Theorem 4.4. *The following conditions are equivalent for a ring R:*

(i) *R is right PF.*
(ii) *R is a semiperfect, right FSG ring and* $\operatorname{Soc}(R_R) \leq^e R_R$.
(iii) *R is a semiperfect, right FSG, right Kasch ring.*
(iv) *R is a semiperfect, right FSG ring and* $\operatorname{Soc}(R_R) \leq^e {}_RR$.
(v) *R is a semiperfect, right FSG, left Kasch, left mininjective ring.*

Proof. Let $\{e_1, \dots, e_n\}$ be a set of orthogonal primitive idempotents and $e_0 = e_1 + \cdots + e_t$ is the basic idempotent of R.

(i) ⇒ (iv), (v) by Theorem 3.12.

(v) ⇒ (ii). Since R is a semiperfect, right FSG ring, each e_iR is uniform, hence $\operatorname{udim}(R_R) = n \geqslant \operatorname{udim}(\operatorname{Soc}(R_R))$ by Lemma 4.1. To prove $\operatorname{Soc}(R_R) \leq^e R_R$, it's suffice to show that $\operatorname{udim}(\operatorname{Soc}(R_R)) = n$. Indeed, since R is left mininjective, $\operatorname{Soc}({}_RR) \leq \operatorname{Soc}(R_R)$ by [**12**, Theorem 2.21]. Since R is a semiperfect, left Kasch ring, $e_i\operatorname{Soc}({}_RR)$ is simple for every $i = 1, \dots, n$ by Lemma 4.3. It follows that

$e_i \operatorname{Soc}({}_RR) \neq e_j \operatorname{Soc}({}_RR)$, $(i \neq j)$ and hence $e_i \operatorname{Soc}({}_RR) \cap e_j \operatorname{Soc}({}_RR) = 0$, $(i \neq j)$. Then

$$\operatorname{udim}(\operatorname{Soc}({}_RR)) = \operatorname{udim}((\sum_{i=1}^{n} e_i) \operatorname{Soc}({}_RR)) = n \leqslant \operatorname{udim}(\operatorname{Soc}(R_R)).$$

Thus $\operatorname{udim}(\operatorname{Soc}(R_R)) = n$ as desired.

(ii) $\Rightarrow$ (i). By Theorem 3.8.

(iv) $\Rightarrow$ (iii) By [**12**, Lemma 1.48].

(iii) $\Rightarrow$ (ii). Since R is right Kasch, every simple right R-module isomorphic to a minimal right ideal of R.

Consider the following commutative diagram:

$$(*) \qquad \begin{tikzcd} e_iR/e_iJ \arrow[r, hook, "\iota_i"] \arrow[d, "j_i"'] & \bigoplus_{j=1}^{n}(e_jR/e_jJ) \arrow[dl, "\wr"] \\ R_R & \end{tikzcd}$$

in which j_i is an embedding morphism and ι_i is a canonical embedding morphism for every $i \in \{1, \dots, n\}$.

From the fact that $\bigoplus_{j=1}^{n}(e_jR/e_jJ)$ contain t non-isomorphic simple right R-module and the commutative diagram $(*)$, it follows that R contains t non-isomorphic minimal right ideals. Thus $\operatorname{udim}\operatorname{Soc}(R_R) = n$ by Lemma 4.2 and hence $\operatorname{Soc}(R_R) \leq^e R_R$ by Lemma 4.1. $\square$

Note. The conditions (ii), (iii) and (iv) of Theorem 4.4 are extensions of [**6**, Theorem 5.1]. Related to (v), we have a question: *Is a semiperfect right FSG, left Kasch ring necessarily right PF?*

Corollary 4.5. *The following conditions are equivalent for a ring R:*

(i) *R is PF.*

(ii) *R is a semiperfect, right and left FSG, right Kasch ring.*

(iii) *R is a semiperfect, right and left FSG, left Kasch ring.*

Proof. (ii), (iii) $\Rightarrow$ (i): By Theorem 4.4 and Corollary 3.10. $\square$

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Le Duc Thoang, Department of Mathematics, Hue University, Vietnam,
e-mail: `sciuni@dng.vnn.vn`

Le Van Thuyet, Department of Mathematics, Hue University, Vietnam,
e-mail: `sciuni@dng.vnn.vn`