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OVERDETERMINED PROBLEMS AND THE p-LAPLACIAN

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Abstract. In this lecture I report on essentially two results for overdetermined boundary value problems and the p-Laplace operator. The first one is joint work with H. Shahgholian on Bernoulli type free boundary problems that model for instance galvanization processes. For this family of problems the limits $p \to \infty$ and $p \to 1$ lead to interesting analytical and surprising geometric questions. In particular for the case $p \to 1$ I add more recent results, that are not contained in [**12**]. The second one is joint work with F. Gazzola and I. Fragalá [**6**]. It provides an alternative and more geometric proof of Serrin's seminal symmetry result for positive solutions to overdetermined boundary value problems. As a byproduct I give an analytical proof for the geometric statement that a closed plane curve of curvature not exceeding K must enclose a disk of radius $1/K$.

1. Bernoulli problems

It is well-known that minimizing the functional

$$E_p(v) = \int\limits_{\mathbb{R}^n} \frac{1}{p}\left(\frac{|\nabla v|}{a}\right)^p + \frac{p-1}{p}\chi_{\{v>0\}}\,\mathrm{d}x$$

on the set $\{v \in W^{1,p}(\mathbb{R}^n);\ v \equiv 1 \text{ on } K\}$ leads to the following Euler-Lagrange equation with overdetermined Bernoulli-type boundary condition

$$\Delta_p u_p = \mathrm{div}(|\nabla u_p|^{p-2}\nabla u_p) = 0 \quad \text{in } \{u_p > 0\} \setminus K, \tag{1.1}$$

$$u_p = 1 \quad \text{on } \partial K, \tag{1.2}$$

$$u_p = 0 \text{ and } |\nabla u_p| = a \quad \text{on } \partial\{u_p > 0\}. \tag{1.3}$$

In the first part of my lecture I report on a study of u_p as $p \to \infty$ or $p \to 1$, that was done with H. Shahgholian in [**12**]. For convex K the level sets of u_p are convex and u_p is monotone increasing in p. In this case the limits were identified in [**13**] as $u_\infty = \{1 - a\,\mathrm{dist}(x, \partial K)\}^+$ and $u_1 = \chi_K(x)$. What happens for nonconvex K? In that case the family of functionals $E_p(u)$ is monotone increasing in p. Therefore it is only natural to study the limits of these functionals: Fortunately there is the theory of Γ convergence available, see [**4**]. To apply it, the different functionals E_p must be redefined on a common domain of definition. From a priori estimates

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on the support of u_p one knows that it fits into a sufficiently large ball B. This justifies the choice of X_q and Y in the following subsections.

1.1. The case $p \to \infty$

For some $q > n$ set $X_q = \{v \in W_0^{1,q}(B);\ v \equiv 1 \text{ on } K\}$ and define E_p as

$$E_p(u) = \begin{cases} \int_B \left\{ \frac{1}{p} \left(\frac{|\nabla u|}{a} \right)^p + \frac{p-1}{p}\, \chi_{\{u>0\}} \right\} \mathrm{d}x & \text{if } u \in W_0^{1,p}(B) \cap X_q, \\ +\infty & \text{if } u \in X_q \setminus W_0^{1,p}(B). \end{cases} \tag{1.4}$$

Then the proof of the first statement in the following is fairly straightforward.

Theorem 1.1. *As $p \to \infty$, the functionals E_p Γ-converge on X_q to*

$$E_\infty(u) := \int_B \left\{ I_{[0,a]}(|\nabla u(x)|) + \chi_{\{u>0\}}(x) \right\} \mathrm{d}x \tag{1.5}$$

and, after possibly passing to a subsequence, u_p converges uniformly to a minimizer U_∞ of the limit functional E_∞. Moreover, U_∞ satisfies the differential equation $\Delta_\infty u = \nabla u D^2 u \nabla u = 0$ in the viscosity sense in $\{U_\infty > 0\} \setminus K$.

Here $I_B(y)$ is the indicator function of B, which vanishes in B and equals $+\infty$ off B. Therefore minimizers of E_∞ try to minimize the volume of their support under the side constraint $|\nabla u| \leq a$. One possible minimizer is $u_\infty = \{1 - a\,\mathrm{dist}(x, \partial K)\}^+$, **but in general $u_\infty \neq U_\infty$.** Let us demonstrate this with an example where K is the union of two disjoint disks at distance $d \in (\frac{1}{a}, \frac{2}{a})$ apart from each other.

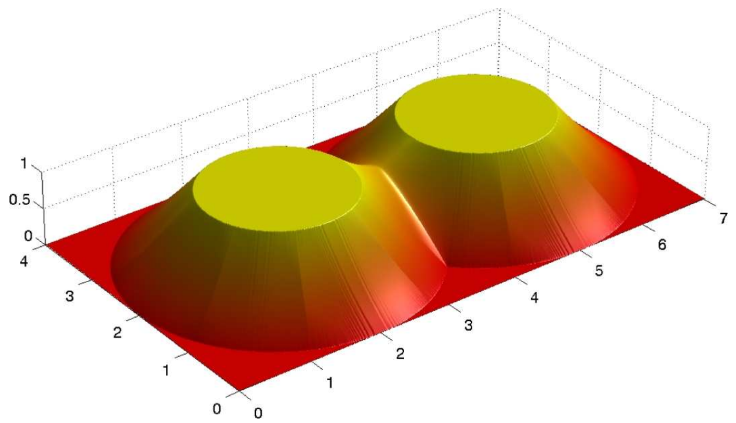

Figure 1.1. Graph of the function $U_\infty(x)$.

In this case u_∞ is the minimum of two cones, but U_∞ is infinity-harmonic and (due to a recent result of Savin [**15**]) must be of class C^1 in $\{U_\infty > 0\} \setminus K$. One can think of the shape that is depicted in Figure 1.1 as the lower part of two merging sandpiles. In general, dry sandpiles have constant slope, but when they merge, they behave differently.

1.2. The case $p \to 1$

This time we set $Y = \{v \in L^1(B);\ v \equiv 1 \text{ on } K\}$ and define E_p as

$$(1.6)\quad \mathrm{E}_p(u) = \begin{cases} \int_B \left\{ \frac{1}{p} \left(\frac{|\nabla u|}{a} \right)^p + \frac{p-1}{p}\ \chi_{\{u>0\}} \right\} \mathrm{d}x & \text{if } u \in W_0^{1,p}(B) \cap Y, \\ +\infty & \text{if } u \in Y \setminus W_0^{1,p}(B). \end{cases}$$

Theorem 1.2. *As $p \to 1$, the functionals E_p Γ-converge on Y to*

$$(1.7)\qquad E_1(u) := \begin{cases} \frac{1}{a} \int_B\ |Du|\, \mathrm{d}x & \text{if } u \in BV(B) \cap Y, \\ +\infty & \text{if } u \in Y \setminus BV(B). \end{cases}$$

and, after possibly passing to a subsequence, u_p converges in L^1 to a minimizer U_1 of the limit functional E_1.

How can one identify minimizers u_1 of E_1? Because of the coarea formula

$$(1.8)\qquad E_1(u) = \frac{1}{a} \int_{t=0}^{1} \mathrm{Per}\{u > t;\ \mathbb{R}^n\}\ dt,$$

so that minimizers of E_1 try to minimize the perimeter of their support and of almost all of its level sets under the side constraint that these level sets contain K. For convex sets K this is achieved by $u_1(x) = \chi_K(x)$. But in general the perimeter minimizers are not unique. As an example for nonuniqueness consider the case that K is the union of two disks of radius one, with their centers exactly d apart. Then the circumference of the two disks is 4π, while the circumference of their convex hull C, a stadium shaped domain, is $2\pi + 2d$. (In three dimensions, when K consists of two balls, one can construct a similar example using catenoids instead of line segments as minimial surfaces that wrap around K.) For $d < \pi$ the stadium has less perimeter than K, so that $U_1(x) = \chi_C(x)$ is the only minimizer of E_1. For $d > \pi$ the two disks that make up K have smaller perimeter than their convex hull, so that $U_1(x) = \chi_K(x)$. But for $d = \pi$ both the boundaries of K and C have equal length, and so both $u_1(x) = \chi_K(x)$ and $v_1 = \chi_C(x)$ (and even convex combinations thereof) are minimizers of E_1, but at present we are unable to identify the L^1-limit U_1 in this special case of Theorem 1.2. Since u_p is p-harmonic, one might hope that U_1 is 1-harmonic and satisfies

$$(1.9)\qquad \mathrm{div} \left(\frac{Du}{|Du|} \right) = 0 = \kappa(\{u = t\}) \quad \text{in } \Omega \setminus K.$$

At present, a proof of this does not seem to be easy. Notice that now u_p converges only in L^1 to U_1, whereas in Section 1.1 u_p converges in X_q and thus uniformly (to U_∞). And stability theorems for viscosity solutions are usually of the type: If u_p solves $F_p(Du, D^2u) = 0$ and $F_p(q, X)$ converges to $F_1(q, X)$ and u_p converges uniformly to u_1, then u_1 solves $F_1(Du, D^2u) = 0$. Here the apparent lack of uniform convergence poses an obstacle to the proof. In addition, the precise meaning of (1.9) is not clear in points where $Du = 0$. The interested reader might consult [**11**] for suitable interpretations of (1.9).

Let me stress the point that finding the support of U_1 or the minimal surface that wraps around K is an interesting geometric variational problem in itself. Once the support of U_1 is known, however, it can be shown as in [**10**] that U_1 is locally a function of least gradient, in the sense that for all $C \subset\subset (\Omega \setminus K)$ and all $v \in BV(C)$ with $v = U_1$ on ∂C

$$\|U_1\|_{BV(C)} \leq \|v\|_{BV(C)}.$$

Therefore, for nonconvex K the question of uniqueness and identifiability of $\lim_{p\to 1} u_p(x)$ remains in general an open problem.

2. Symmetry result

In the second half of my lecture I present a new and more geometric proof of the following result:

Theorem 2.1. *If the overdetermined elliptic boundary value problem*

$$-\operatorname{div}(A(|\nabla u|)\nabla u) = 1 \qquad \text{in } \Omega \tag{2.1}$$

$$u = 0, \quad \text{and} \quad |\nabla u| = a \qquad \text{on } \partial\Omega \tag{2.2}$$

has a solution in a simply connected bounded domain of class $C^{2,\alpha}$, then Ω is a ball.

For classical solutions of strongly elliptic equations on sufficiently smooth domains this is a celebrated result of Serrin [**17**]. To prove it, Serrin introduced the PDE community to Alexandrov's moving plane method, and the proof applied to even more general equations with classical solutions. For $A(|\nabla u|) \equiv 1$ Weinberger provided a much simpler proof, and there have been several attempts ([**8**], [**5**] and [**2**]) to extend Weinberger's approach or Serrin's result to more general equations. In [**6**] F. Gazzola, I. Fragalá and I were able to provide a fairly simple and geometric proof that applies to degenerate equations such as

$$-\Delta_p u = 1 \qquad \text{in } \Omega, \tag{2.3}$$

for which $A(|\nabla u|) = |\nabla u|^{p-2}$. However, we will have to pay a price in form of an additional starshapedness assumption on Ω if $n \geq 3$. In what follows, I will outline the proof only for this equation in the range $p \in (1, \infty)$, because then the individual steps will not be obscured by technicalities. For the benefit of the reader, however, I should at least list the general assumptions on A that were made in [**6**]:

$$A \in C^1(0, +\infty), \quad \lim_{t\to 0^+} tA(t) = 0 \qquad \text{and} \qquad \big(tA(t)\big)' > 0 \text{ for } t > 0.$$

As explained in [**6**], these assumptions are less stringent than the ones in any of [**8**], [**5**] and [**2**]. Here is the sketch of proof for the special case $A(t) = t^{p-2}$ with $p \in (1, \infty)$.

Step 1: An integration of the differential equation gives a relation between perimeter and volume of Ω:

$$a^{p-1}|\partial\Omega| = |\Omega|. \tag{2.4}$$

Step 2: The function $P(x) := \frac{2(p-1)}{p}|\nabla u(x)|^p + \frac{2}{n}u(x)$ attains its maximum over $\overline{\Omega}$ on $\partial\Omega$. This is shown by a Bernstein type argument.

Step 3: The fact that $P_\nu(x) \geq 0$ on $\partial\Omega$ follows from Step 2 and translates into a mean curvature bound on $\partial\Omega$:

$$H(x) \leq \frac{1}{n}a^{1-p}\ . \tag{2.5}$$

Step 4: Minkowski's identity, estimate (2.5) and integration by parts gives (for Ω starshaped with respect to x_0)

$$|\partial\Omega| = \int_{\partial\Omega} H(x)(x-x_0,\nu)\,\mathrm{d}s \leq \frac{1}{n}a^{1-p}\int_{\Omega} \operatorname{div}(x-x_0)\ \mathrm{d}x \leq a^{1-p}|\Omega|. \tag{2.6}$$

Step 5: (2.6) and (2.4) imply equality in (2.5) everywhere on $\partial\Omega$. So $H = $ const. and $\partial\Omega$ has constant mean curvature. But then by [**1**] Ω is necessarily a ball.[1]

Now I want to explain a few steps in more detail and point out how to get rid of the starshapedness assumption for plane domains, i.e. if $n = 2$. (This was the physically relevant case for which the symmetry of Ω was first conjectured.)

For Step 2 one wants to use a differential inequality for P, but this would u require to be of class C^3, while in fact solutions of (2.3) are in general not more regular than $C^{1,\alpha}$. This technicality is overcome by a suitable regularization and approximation argument.

How does one get the curvature bound in Step 3? Observe that

$$P = \frac{2(p-1)}{p}|u_\nu|^p + \frac{2}{n}u(x)$$

so that a combination of

$$P_\nu = 2(p-1)|u_\nu|^{p-2}u_\nu u_{\nu\nu} + \frac{2}{n}u_\nu = \left[(p-1)|u_\nu|^{p-2}u_{\nu\nu} + \frac{1}{n}\right]2u_\nu \geq 0 \quad \text{on } \partial\Omega$$

and

$$-\Delta_p u = -(p-1)|u_\nu|^{p-2}u_{\nu\nu} - (n-1)H|u_\nu|^{p-2}u_\nu = 1 \quad \text{on } \partial\Omega,$$

lead to the bound (2.5) on H.

What if Ω is not starshaped? Then for $\mathbf{n = 2}$ the curvature bound implies that Ω contains a disc D of radius 2 a^{p-1}. This simple geometric fact does not seem to be so well-known, although it is recorded (without proof) for instance in [**3**] Section 30.4.1, and the proofs in [**16**] and [**9**] are not so easily available. Therefore I take the liberty of providing a more analytical proof here.

Lemma 2.2. *If $\Omega \subset \mathbb{R}^2$ is a plane domain with boundary of class C^2, and if the boundary has curvature $\kappa \leq K$, then Ω contains a disk of radius $1/K$.*

The proof is illustrated by Figure 2.1. Suppose the Lemma is false. Then there is a point $P \in \partial\Omega$ at which a circle with radius $1/K$ touches $\partial\Omega$ tangentially and another one, $Q \in \partial\Omega$, where it intersects $\partial\Omega$ transversally. I denote the circular arc (in clockwise direction) from P to Q by C, and the corresponding part of $\partial\Omega$

[1] Meanwhile (2006) the starshapedness assumption in Step 4 of the proof of Theorem 2.1 could be removed, see [**7**]

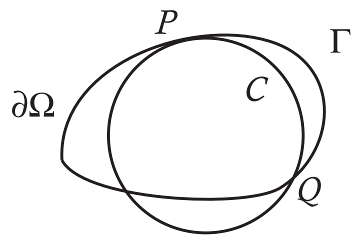

Figure 2.1. An impossible situation.

by Γ. Then $D_\Gamma := \int\limits_\Gamma \kappa \ \mathrm{d}\theta$ denotes the angular difference of tangents to $\partial\Omega$ in Q and P, while $D_C := \int\limits_C K \ \mathrm{d}\theta$ denotes the angular difference of tangents to C in Q and P. Clearly Γ has to bend more than C to reach Q, and therefore

$$D_C < D_\Gamma. \tag{2.7}$$

So the range of θ over which Γ can be parametrized is larger than for C, and the chain of inequalities $D_\Gamma = \int\limits_\Gamma \kappa \ \mathrm{d}\theta \leq \int\limits_\Gamma K \ \mathrm{d}\theta < \int_C K \ \mathrm{d}\theta = D_C$ contradicts (2.7). This proves the Lemma.

With this Lemma at hand, we can now follow a "method of moving disks" D of suitable radius $2a^{p-1}$. The radial solution v of

$$-\Delta_p v = 1 \qquad \text{in } D, \tag{2.8}$$

$$v = 0 \qquad \text{on } \partial D, \tag{2.9}$$

happens to satisfy $|\nabla v| = a$. By moving discs (rather than moving planes) and comparison arguments similar to the ones in [**17**] one can then show that $\Omega = D$.

What did Weinberger do (for general n)? For $p = 2$ and $n \geq 3$ he did not need a starshapedness assumption on Ω. From Step 2 he concluded that either P is constant, and then u is (fairly easily shown to be) radial, or

$$P(x) < \frac{2(p-1)}{p} a^p \qquad \text{in } \Omega. \tag{2.10}$$

In the second case, an integration of (2.10) over Ω gives a relation between $\int |\nabla u|^p$, $\int u$ and $|\Omega|$, while testing the PDE with u gives another relation of this nature, namely $\int |\nabla u|^p = \int u$. This and a relation between $|\Omega|$ and $\int u$ (that seems to work only for $p = 2$) led Weinberger to a contradiction.

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