Acta Math. Univ. Comenianae
Vol. LXXVI, 1(2007), pp. 63–67
Proceedings of Equadiff 11

ON BLOW-UP AT SPACE INFINITY FOR SEMILINEAR HEAT EQUATIONS

Y. GIGA AND N. UMEDA

We are interested in solutions of semilinear heat equations which blow up at space infinity.

In [**7**], we considered a nonnegative blowing up solution of

$$u_t = \Delta u + u^p, \qquad x \in \mathbb{R}^n, \qquad t > 0$$

with initial data u_0 satisfying

$$0 \le u_0(x) \le M, \qquad u_0 \not\equiv M \quad \text{and} \quad \lim_{|x|\to\infty} u_0(x) = M,$$

where $p > 1$ and $M > 0$ is a constant. We proved in [**7**] that the solution u blows up exactly at the blow-up time for the spatially constant solution with initial data M. We moreover proved that u blows up only at the space infinity. In this paper we would like to generalize this result in the following directions.

(i) (Initial data) We consider more general initial data u_0 which may not converge to M for all directions of x, for example $u_0 \to M$ as $|x| \to \infty$ only for x in some sector. It is convenient to introduce a notion of blow up direction at the space infinity. We are able to give necessary and sufficient condition so that a particular direction is a blow-up direction.

(ii) (Nonlinear term) We extend the class of nonlinearities. It includes e^u and $u^p + u^q$ for p, $q > 1$.

In [**8**] we consider solutions of the initial value problem for the equation

$$\left\{\begin{array}{ll} u_t = \Delta u + f(u), & x \in \mathbb{R}^n, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n. \end{array}\right. \tag{1}$$

The nonlinear term f is assumed to be nonnegative and locally Lipschitz in $\mathbb{R}$ with the property that

$$\liminf_{b\ge b_0,\, \delta\in(\delta_0,1)} \frac{\delta^p f(b)}{f(\delta b)} > 0 \quad \text{for} \quad b_0 > 0, \quad \delta_0 \in (0,1), \quad p > 1. \tag{2}$$

Received November 30, 2005.

2000 *Mathematics Subject Classification.* Primary 35K05, 35K15, 35K55.

Partly supported by the Grant-in-Aid for Scientific Research, No.14204011 and by COE "Mathematics of Nonlinear Structures via Singularities" (JSPS).

Partly supported by COE "Mathematics of Nonlinear Structures via Singularities" and "New Mathematical Development Center to Support Scientific Technology" (JSPS).

We take two constants M and N satisfying $M+N>0$ and

(3) $$f(M)>0.$$

The initial data u_0 is assumed to be a measureable function in $\mathbb{R}^n$ satisfying

(4) $$-N \le u_0 \le M \text{ a.e.} \qquad \text{and} \qquad u_0 \not\equiv M \text{ a.e.}$$

We are interested in initial data such that $u_0 \to M$ as $|x| \to \infty$ for x in some sector of $\mathbb{R}^n$. We assume that

(5) $$\operatorname{essinf}_{x\in\tilde{B}_m}(u_0(x)-M_m(x-x_m)) \ge 0 \qquad \text{for } m=1,2,\dots,$$

where

(6) $$\tilde{B}_m = B_{r_m}(x_m)$$

with a sequence $\{r_m\}$ and a sequence of vectors $\{x_m\}_{m=1}^{\infty}$ and a sequence of functions $\{M_m(x)\}$ satisfying

$$\lim_{m\to\infty} r_m = \infty, \qquad M_m(x) \le M_{m+1}(x) \qquad \text{for } m \ge 1$$

$$\lim_{m\to\infty} \inf_{s\in[1,r_m]} \frac{1}{|B_s|}\int_{B_s(0)} M_m(x)\,\mathrm{d}x = M.$$

Here $B_r(x)$ denotes the closed ball of radius r centered at x. (In fact, it follows from (4) that $|x_m| \to \infty$ as $m \to \infty$.)

Problem (1) has a unique bounded solution at least locally in time. However, the solution may blow up in finite time. For a given initial value u_0 and nonlinear term f let $T^* = T^*(u_0,f)$ be the maximal existence time of the solution. If $T^* = \infty$, the solution exists globally in time. If $T^* < \infty$, we say that the solution blows up in finite time. It is well known that

(7) $$\limsup_{t\to T^*} \|u(\cdot,t)\|_\infty = \infty,$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm in space variables.

In this paper, we are interested in the behavior of a blowing up solution near space infinity as well as the location of blow-up points defined below. A point $x_{BU} \in \mathbb{R}^n$ is called a *blow-up point* (with value $\pm\infty$) if there exists a sequence $\{(x_m,t_m)\}_{m=1}^{\infty}$ such that

$$t_m \uparrow T^*, \quad x_m \to x_{BU} \quad \text{and} \quad u(x_m,t_m) \to \pm\infty \quad \text{as} \quad m\to\infty.$$

If there exists a sequence $\{(x_m,t_m)\}_{m=1}^{\infty}$ such that

$$t_m \uparrow T^*, \quad |x_m| \to \infty \quad \text{and} \quad u(x_m,t_m) \to \pm\infty \quad \text{as} \quad m\to\infty,$$

then we we say that the solution blows up to $\pm\infty$ at space infinity.

A direction $\psi \in S^{n-1}$ is called a *blow-up direction for the value* $\pm\infty$ if there exists a sequence $\{(x_m,t_m)\}_{m=1}^{\infty}$ with $x_m \in \mathbb{R}^n$ and $t_m \in (0,T^*)$ such that $u(x_m,t_m) \to \pm\infty$ (as $m\to\infty$) and

(8) $$\frac{x_m}{|x_m|} \to \psi \qquad \text{as} \qquad m\to\infty.$$

We consider the solution $v(t)$ of an ordinary differential equation

$$\begin{cases} v_t = f(v), & t > 0, \\ v(0) = M. \end{cases} \tag{9}$$

Let $T_v = T^*(M, f)$ be the maximal existence time of the solution of (9), i. e.,

$$T_v = \int_M^\infty \frac{\mathrm{d}s}{f(s)}.$$

We are now in position to state our main results.

Theorem 1. *Assume that f is locally Lipschitz in $\mathbb{R}$ and satisfies (2) and (3). Let u_0 be a continuous function satisfying (4) and (5), and $T_v \leq T^*(-N, f)$. Then there exists a subsequence of $\{x_m\}_{m=1}^\infty$ (still denoted by $\{x_m\}$, independent of t) such that*

$$\lim_{m\to\infty} u(x_m, t) = v(t).$$

The convergence is uniform in every compact subset of $\{t : 0 \leq t < T_v\}$. Moreover, the solution blows up at T_v.

Remark. Our assumption $T_v \leq T^*(-N, f)$ says that the solution does not blow up to minus infinity before it blows up to plus infinity. From the condition (4), it follows that $\lim_{m\to\infty} |x_m| = \infty$.

This result in particular implies that

$$\sup_{0<t<T^*} v^{-1}(t)\|u(\cdot, t)\|_\infty < \infty. \tag{10}$$

When we set $f(u) = |u|^{p-1}u$, such a blow-up rate estimate is known for subcritical p; see e.g. [**3**], [**5**], [**6**] for general bounded initial data without assuming (4) and (5). However, for supercritical p such a blow-up rate estimate (10) may not hold in general; see e.g. [**1**], [**9**]. If one considers only radial solutions of (1) for supercritical p less than $1 + 4/(n - 4 - 2(n - 1)^{1/2})$ or $n \leq 10$, then the estimate (10) holds [**11**]. We would like to emphasize that Theorem 1 does not require any restriction on p.

Our second main result is on the location of blow-up points.

Theorem 2. *Assume the same hypotheses as in Theorem 1. Then the solution of (1) has no blow-up points with $+\infty$ in $\mathbb{R}^n$. (It blows up only at space infinity.)*

There is a huge literature on location of blow-up points since the work of Weissler [**13**] and Friedman-McLeod [**2**]. (We do not intend to list references exhaustively in this paper.) However, most results consider either bounded domains or solutions decaying at space infinity; such a solution does not blow up at space infinity [**4**].

As far as the authors know, before the result of [**7**] the only paper discussing blow-up at space infinity is the work of Lacey [**10**]. He considered the Dirichlet problem in a half line. He studied various nonlinear terms and proved that a solution blows up only at space infinity.

In particular, his result implies that the solution of

$$\begin{cases} u_t = u_{xx} + f(u), & x > 0, \quad t > 0, \\ u(0,t) = 1, & t > 0, \\ u(x,0) = u_0(x) \geq 1, & x > 0 \end{cases}$$

blows up only at space infinity, where u_0 satisfies $0 \leq u_0 \leq M$ with $M > 1$, and $f(s) = s^p$ and e^s.

His method is based on construction of suitable subsolutions and supersolutions. However, the construction heavily depends on the Dirichlet condition at $x = 0$ and does not apply to the Cauchy problem even for the case $n = 1$.

As previously described, the authors [**7**] proved the statement of Theorems 1 and 2 assuming that $u_0(x) \leq M$ for sufficiently large M for positive solutions of $u_t = \Delta u + u^p$. Later, Shimojyo [**12**] had the same results as in [**7**] by relaxing the assumptions of initial data $u_0 \geq 0$ which is similar to that in the present paper. His approach is a construction of a suitable supersolution which implies that $a \in \mathbb{R}^n$ is not a blow-up point. Although he restricted himself to $f(s) = s^p$, his idea works for our f under slightly stronger assumption on u_0. Here we give a different approach.

From Shimojyo's results [**12**], there arises a problem of "blow-up direction" defined in (8). We next study this "blow-up direction" for the value $+\infty$. Our third result is on this blow-up direction. It is convenient to introduce the function A_m defined by

$$A_m(s) = \frac{1}{|B_s(y_m)|} \int_{B_s(y_m)} u_0(z)\,\mathrm{d}z \tag{11}$$

for a given sequence $\{y_m\}_{m=1}^{\infty}$. This $A_m(s)$ represents the mean value of u_0 over the ball $B_s(y_m)$.

Theorem 3. *Assume the same hypotheses as in Theorem 1 and let* $\{s_m\}_{m=1}^{\infty}$ *be a sequence diverging to* ∞ *in* $\mathbb{R}$. *For a given direction* $\psi \in S^{n-1}$, *the following alternatives hold.*

(i) *If there exists a sequence* $\{y_m\}_{m=1}^{\infty}$ *satisfying* $\lim_{m\to\infty} y_m/|y_m| = \psi$ *it holds that*

$$\limsup_{m\to\infty} \inf_{s\in(1,s_m)} A_m(s) = M,$$

then ψ *is a blow-up direction.*

(ii) *If for any sequence* $\{y_m\}_{m=1}^{\infty}$ *satisfying* $\lim_{m\to\infty} y_m/|y_m| = \psi$ *there exists a constant* $c \in (1/(M+N), \infty)$ *such that*

$$\limsup_{m\to\infty} \inf_{s\in(1,c)} A_m(s) \leq M - \frac{1}{c},$$

then ψ *is not a blow-up direction.*

This characterizes blow up directions by profiles of initial data. This is a new result even if $f(u) = |u|^{p-1}u$ or $n = 1$.

Here are the main ideas of the proofs. To prove Theorem 1 we construct a suitable subsolution. To prove Theorem 2 we derive a non blow-up criterion. We do not appeal any energy arguments for rescaled function as is done in our previous paper [**7**]. Our argument consists of two parts. First we observe that

$$u(x,t) \leq \delta v(t)$$

near a point $a \in \mathbb{R}^n$ with some $\delta \in (0,1)$ when t is close to the blow-up time. By a bootstrap argument we derive that u is actually bounded near a when t is close to the blow-up time. To prove Theorem 3 we use a comparison argument as in Theorems 1 and 2 and a non blow-up criterion as in the proof of Theorem 2. Moreover, we give conditions on the direction $\psi \in S^{n-1}$ for being the blow-up direction or not cover all of S^{n-1} exclusively.

The detailed proofs will be discussed in paper [**8**].

References

1. Filippas S., Herrero M. A. and Velázquez J. J. L., *Fast blow-up mechanisms for sign-chainging solutions of a semilinear parabolic equation with critical nonlinearity,* Proc R. Soc. Lond. Ser. A **456** (2000), 2957–2982.

2. Friedman A. and McLeod B., *Blow-up of positive solutions of semilinear heat equations,* Indiana Univ. Math. J. **34(2)** (1985), 425–447.

3. Giga Y. and Kohn R. V., *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. **36** (1987), 1–40.

4. ______, *Nondegeneracy of blowup for semilinear heat equations,* Comm. Pure Appl. Math. **42(6)** (1989), 845–884.

5. Giga Y., Matsui S. and Sasayama S. *Blow up rate for semilinear heat equations with subcritical nonlinearity.* Indiana Univ. Math. J. **53(2)** (2004), 483–514.

6. ______, *On blow-up rate for sign-changing solutions in a convex domain,* Math. Meth. Appl. Sci. **27** (2004), 1771–1782.

7. Giga Y., Umeda N., *On Blow-up at Space Infinity for Semilinear Heat Equations*, J. Math. Anal. Appl., **316** (2006), 538–555.

8. ______, *Blow-up directions at space infinity for solutions of semilinear heat equations*, Bol. Soc. Paran. Mat., **23** (2005), 9–28.

9. Herrero M. A. and Velázquez J. J. L., *Explosion de solutions d' quations paraboliques semilin aires supercritiques,* C. R. Acad. Sci. Paris, S r. I **319** (1994), 141–145.

10. Lacey A. A., *The form of blow-up for nonlinear parabolic equations,* Proc. Roy. Soc. Edinburgh Sect. A **98(1–2)** (1984), 183–202.

11. Matano H. and Merle F., *On nonexistence of type II blowup for a supercritical nonlinear heat equation,* Comm. Pure Appl. Math. **57(11)** (2004), 1494–1541.

12. Shimojyo M., *On blow-up phenomenon at space infinity and its locality for semilinear heat equations,* (In Japanese), Master's Thesis, The University of Tokyo (2005).

13. Weissler F. B., *Single point blow-up for a semilinear initial value problem*, J. Differential Equation **55**, 1984 204–224.

Y. Giga, Graduate School of Mathematical Sciences, The University of Tokyo 3-8-1, Komaba, Meguro-ku, Tokyo 153-8914, Japan, *e-mail*: `labgiga@ms.u-tokyo.ac.jp`

N. Umeda, Graduate School of Mathematical Sciences, The University of Tokyo 3-8-1, Komaba, Meguro-ku, Tokyo 153-8914, Japan, *e-mail*: `umeda@ms.u-tokyo.ac.jp`