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ON GENERALIZED EXTENDING MODULES

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Abstract. H. Hanada, J. Kado, and K. Oshiro have introduced, in a diagram of modules and homomorphisms, the concept of generalized M-injective modules. S. Mohamed, and B. Mueller have given a different characterization, based on an exchange property, of the generalized M-injective modules. Here we introduce the concept of M-jective modules, which is a generalization of Mohamed and Mueller concept for the generalized M-injectivity. The concept of M-jective modules is used here to solve the problem of finding a necessary and sufficient condition for a direct sum of extending modules to be extending. In fact, we show that relative jectivity is necessary and sufficient for a direct sum of two extending modules to be extending. We also introduced the concept of generalized extending modules, and give some properties of such modules in analougy with the known properties for extending modules.

1. Introduction.

In [**2**], H. Hanada, J. Kado, and K. Oshiro have introduced the concept of *generalized M-injective* modules, which is a generalization to the concept of *M-injective* modules. It was given and described in a diagram of modules and homomorphisms in the following sense, N is a generalized M-injective module if for any submodule X of M and any homomorphism $\varphi : X \to N$,there exist decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$ together with homomorphisms $\varphi_1 : M_1 \to N_1$ and $\varphi_2 : N_2 \to M_2$, such that φ_2 is one-to-one, and for $x = m_1 + m_2$ and $\varphi(x) = n_1 + n_2$ one has $n_1 = \varphi_1(m_1)$ and $m_2 = \varphi_2(n_2)$. In the honor of Oshiro, S. Mohamed and B. Mueller in [**12**] have used the name "*M-ojective modules*" for the "*generalized M-injective modules*". They have given an equivalent characterization for *M-ojective modules*, which is analogous to the observation in [**3**, Proposition 1.13] by Burgess and Raphael. In fact Mohamed and Mueller proved that if a module $A = M \oplus N$, then N is M-ojective if and only if for any complement C of N in A, A decomposes as $A = C \oplus M_1 \oplus N_1$, with $M_1$and N_1 are submodules of M and N respectively [**12**, Theorem 7]. This equivalent characterization for *M-ojective modules* is visible, and is easly checked in applications. It requires that every complement of N in A is a summand and has a complementary summand consists of a part of M and a part of N.

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Here we introduce the concept of *M-jectivity*, which is a generalization of *M-ojectivity*. In this new concept, we require that every complement of N in $M \oplus N$ is a summand and need not have a specific complementary summand in $M \oplus N$. In fact, A module N is *M-jective* if every complement of N in $M \oplus N$ is a direct summand. If N is *M-jective* and M is *N-jective*, we say that N and M are relatively jective.

The problem of finding a satisfactory necessary and sufficient condition for a direct sum of extending modules to be extending is still open and is annoying problem, as it mentioned in [**12**]. It has been investigated in a number of papers. In [**1**], and in [**11**], independently, it was shown that relative injectivity is sufficient but not necessary (as $Cp \oplus Cp^2$ is extending [**6**, Corollary 23]. In [**4**], it was shown that a direct sum of extending modules M_1 and M_2 is extending if and only if every closed submodule with zero intersection with M_1 or with M_2 is a summand (Lemma 7.9). In [**2**], H. Hanada, J. Kado, and K. Oshiro investigated a finite direct sum of modules which is exchangable for closed submodules. They claimed that relative ojectivity is necessary and sufficient for such a direct sum to be extending. In [**12**], they have given a proof of such a claim for a direct sum of two modules. The general case remains open.

Here we show that relative jectivity is necessary and sufficient for a direct sum of two extending modules to be extending. We also introduce the concept of generalized extending modules, and give some properties of such modules which are analogous to the properties which are known for extending modules.

By a module M we mean a unitary right module over a (not necessary comutative) ring with unity. A submodule A of a module M is essential in M, or M is an essential extension of A, if $A \cap B \neq 0$ for each nonzero submodule B of M. A is closed in M if it has no proper essential extensions in M. If A and B are submodules of M respectively, then A is a complement of B in M if A is a maximal in M with the property that $A \cap B = 0$. It is clear that every complement in M is a closed submodule of M. We use the notions $A \leq^e M$, and $A \leq^{\oplus} M$ to indicate that A is an essential submodule of M and A is a direct summand of M. A module B is said to be A-injective if every homomorphism from a submodule of A into B can be extended to A. It was observed in [**3**] that B is A-injective if and only if $M = C \oplus B$ holds for every complement C of B in $M = A \oplus B$.

A module M is extending (or a CS-module, or a module with (C_1)) if every submodule is essential in a direct summand (or equivalently, if $A \leq M$, then there is a decomposition $M = M_1 \oplus M_2$ such that $A \leq M_1$ and $A \oplus M_2 \leq^e M$). Extending modules generalize quasi-continuous modules, which, in turn, generalize quasi-injective modules. Many authors have studied them extensively.

The between brackets equivalent defining condition for extending modules can be generalized to the following condition:

$$(C_1^*) \qquad \text{If } A \leq M, \text{ then there is a decomposition } M = M_1 \oplus M_2 \text{ such that } A \cap M_2 = 0, \text{ and } A \oplus M_2 \leq^e M.$$

It is clear that every extending module must satisfies condition (C_1^*).

2. M-jective Modules

Lemma 2.1. [**12**, Theorem 7] *Let $M = A \oplus B$. Then B is A-ojective if and only if for any complement C of B, M decomposes as $M = C \oplus A_1 \oplus B_1$, with $A_1 \leq A$ and $B_1 \leq B$.*

As a generalization of Lemma 2.1, we introduce the following definition:

Definition 2.1. Let $M = A \oplus B$. Then B is called *A-jective* if every complement C of B in M is a direct summand.

Lemma 2.2. [**12**, Lemma 1] *Let A and B be submodules of a module M with $A \cap B = 0$. Then A is a complement of B in M if and only if A is a closed submodule of M and $A \oplus B$ is essential in M.*

Lemma 2.3. *Let $M = N \oplus K$. Let C be a complement in N of a submodule A of N. Then:*

(1) *$C \oplus K$ is a complement of A in M.*
(2) *C is a complement for $A \oplus K$ in M.*

Proof. (1): Let $C \oplus K \leq L \leq M$ such that $L \cap A = 0$. Since $(L \cap N) \cap A = 0$, and C is a complement of A in N, it is follows that $L \cap N = C$; and hence $L = K \oplus (L \cap N) = K \oplus C$.

(2): The fact the C is a complement of A in N implies that $C \oplus A \oplus K \leq^e M$. It is clear that if C is closed in N and $N \leq^{\oplus} M$ (N is closed in M), then C is closed in M. Then, by Lemma 2.2, and since C is closed in N (hence in M), C is a complement of $A \oplus K$ in M. □

Proposition 2.4. *Let $M = A \oplus B$, where B is A-jective. Let $A = A_1 \oplus A_2$, and $B = B_1 \oplus B_2$. Then (for i, $j = 1, 2$):*

(1) *B_i is A-jective;*
(2) *B is A_j-jective;*
(3) *B_i is A_j-jective.*

Proof. For (1), write $M = A \oplus B_1 \oplus B_2$. Let C be a complement of B_1 in $A \oplus B_1$. Then by (2) of Lemma 2.3, C is a complement of B in M. Since B is A-jective, then C is a summand.

For (2), write $M = A_1 \oplus A_2 \oplus B$. Let C be a complement of B in $A_1 \oplus B$. Then by (1) of Lemma 2.3, $C \oplus A_2$ is a complement of B in M. Since B is A-jective, $C \oplus A_2$ is a summand; and hence C is a summand of $A_1 \oplus B$.

(3): Follows from (1), and (2). □

Lemma 2.5. *Let $M = A \oplus B$, where B is A-jective. If A is extending, then every closed submodule C of M, with $C \cap B = 0$, is a summand of M.*

Proof. Since A is an extending module, we have $(C \oplus B) \cap A \leq^e A_1 \leq^{\oplus} A$, and hence $((C \oplus B) \cap A) \oplus B \leq^e A_1 \oplus B$. Since $C \oplus B = ((C \oplus B) \cap A) \oplus B$, we have $C \oplus B \leq^e A_1 \oplus B$. By Lemma 2.2, C is a complement of B in $A_1 \oplus B$. Proposition 2.4 tells us that B is A_1-jective. Therefore $C \leq^{\oplus} A_1 \oplus B \leq^{\oplus} M$. □

Lemma 2.6. [**4**, Lemma 7.9] *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are both extending modules. Then M is extending if and only if every closed submodule C of M such that $C \cap M_1 = 0$, or $C \cap M_2 = 0$, is a summand of M.*

The following is a necessary and sufficient condition of a direct sum of two extending modules to be extending.

Theorem 2.7. *Let $M = M_1 \oplus M_2$. Then M is extending if and only if the M_i is extending, and is M_j-jective, $i \neq j (= 1, 2)$.*

Proof. Follows from Lemma 2.5, and Lemma 2.6. □

Corollary 2.1. *A module M with the condition (C_1^*) is extending if and only if M has the property that A is B-jective for every decomposition of $M = A \oplus B$.*

Proof. By the condition (C_1^*), every closed submodule of M is a complement of a summand of M. Hence, by assumption, every closed submodule is a summand. Therefore M is extending. The converse is obvious. □

Remark. 1. If M is a module with the property that A is B-jective for every decomposition of $M = A \oplus B$; then M need not have the condition (C_1^*). In fact indecomposable modules need not satisfy the condition (C_1^*).
2. The fact that essential extensions have the same complements in any module M, allows us to replace submodules in the condition (C_1^*) by closed submodules.

3. Generalized Extending Modules

Definition 3.1. A module M is called a *generalized* extending module (for short a *GE*-module) if the following condition is satisfied: If $M = M_1 \oplus M_2$, and $A \leq M$, then there exist $C_i \leq^{\oplus} M_i$ $(i = 1, 2)$ such that $C_1 \oplus C_2$ is a complement of A in M.

Observe that in The condition (C_1^*), according to Lemma 2.2, the M_2 is a complement of A in M. Hence The condition (C_1^*) is equavalent to the following: every submodule has a complement in M which is a summand. Observe also that, from the definition of *GE*-modules, the equivanlent condition to (C_1^*) holds in every *GE*-module.

In the following, we are going to show that every extending module is a *GE*-module, and also give the relation between modules with (C_1^*) and *GE*-modules.

Lemma 3.1. *The following are equivalent for a module $M = A \oplus B$:*
(1) *A has (C_1^*);*
(2) *For every closed submodule C of M, with $C \cap B = 0$, there exists $A_1 \leq^{\oplus} A$ such that $A_1 \oplus B$ is a complement of C in M.*

Proof. (1) $\Rightarrow$ (2): Let C be a closed submodule of M, with $C \cap B = 0$. By the condition (C_1^*) for A, there exists $A_1 \leq^{\oplus} A$ such that A_1 is a complement of $(C \oplus B) \cap A$ in A. As $[(C \oplus B) \cap A] \oplus A_1 \leq^{e} A$, we have that

$[(C \oplus B) \cap A] \oplus A_1 \oplus B \leq^e M$. Since $C \oplus B = [(C \oplus B) \cap A] \oplus B$, it follows that $C \oplus B \oplus A_1 \leq^e M$. Thus, by Lemma 2.2, $A_1 \oplus B$ is a complement of C in M.

(2) $\Rightarrow$ (1): Let C be a closed submodule of A. Since a closed submodule in a summand of M is closed in M, it follows that C is closed in M. By (2), there exists $A_1 \leq^{\oplus} A$ such that $A_1 \oplus B$ is a complement of C in M. It follows that $C \oplus A_1 \oplus B \leq^e M = A \oplus B$, and hence $C \oplus A_1 \leq^e A$. By Lemma 2.2, A_1 is a complement of C in A, and therefore A has (C_1^*). □

It is known that direct sums of two extending modules need not be extending. In the following theorem we show that direct sums of two modules with (C_1^*) are modules with (C_1^*).

Theorem 3.2. *If $M = M_1 \oplus M_2$, where M_1 and M_2 are both have the condition (C_1^*), then M has (C_1^*).*

Proof. Let C be a closed submodule of M, and let C_1 be a maximal essential extension of $C \cap M_1$ in C. It is clear that C_1 is closed in M with $C_1 \cap M_2 = 0$. Hence by Lemma 3.1, there exists a complement of C_1 in M of the form $N_1 \oplus M_2$ such that $N_1 \leq^{\oplus} M_1$. As $C_1 \oplus N_1 \oplus M_2 \leq^e M$, we have that $C_1 \oplus [C \cap (N_1 \oplus M_2)] \leq^e C$. Let C_2 be a maximal essential extension of $C \cap (N_1 \oplus M_2)$ in C. It is clear that C_2 is a closed submodule of M with $C_2 \cap M_1 = 0$ (due to $C \cap (N_1 \oplus M_2) \cap M_1 = C \cap N_1 \leq C_1$). Hence, again by Lemma 3.1, there exists a complement of C_2 in M of the form $M_1 \oplus N_2$ such that $N_2 \leq^{\oplus} M_2$. It is easy to see that the sum $C_1 + C_2 + N_1 + N_2$ is a direct sum. Since$(C \cap M_1) \oplus N_1 \oplus C_2 \oplus N_2 \leq^e M_1 \oplus C_2 \oplus N_2 \leq^e M$, it follows that $C \oplus N_1 \oplus N_2 \leq^e M$, and thus, by Lemma 2.2, $N_1 \oplus N_2$ is a complement of C in M. Therefore C has a complement in M which is a summand of M. □

Observe that in the proof of Theorem 3.2 we obtained a complement of the form $N_1 \oplus N_2$, where $N_i \leq^{\oplus} M_i (i = 1, 2)$, for an arbitrary closed submodule C of $M = M_1 \oplus M_2$. An immediate consequence of this observation is the following corollary.

Corollary 3.3. *The following are equivalent for a module M:*

(1) *M is a GE-module.*
(2) *Every direct summand of M has (C_1^*).*

Corollary 3.4. *Direct summands of a GE-module are GE-modules.*

Proof. Is an immediate consequence of Corollary 3.3. □

Corollary 3.5. *Every extending module is a GE-module.*

Proof. Since every direct summand of an extending module is extending, hence has (C_1^*). □

Corollary 3.6. *Every finite uniform dimensional module M with (C_1^*) is a GE-module.*

Proof. By induction on the uniform dimension of M. It is clear that every uniform module is a GE-module. Now let M be a module of uniform dimension n. Since every nonzero proper summand submodule of M has uniform dimension less than n, by induction it is a GE-module; and hence has (C_1^*). Therefore by Corollary 3.3 M is a GE-module. $\square$

The following implications are now clear for a module M:

$$\begin{aligned} M \text{ is an extending module} &\implies M \text{ is a } GE\text{-module} \\ &\implies M \text{ has the condition } (C_1^*). \end{aligned}$$

Corollary 3.7. *The following are equivalent for a module $M = \oplus_{i=1}^n M_i$:*

(1) *The M_i $(i = 1, 2, \ldots, n)$ has the condition (C_1^*);*
(2) *Each closed submodule of M has a complement in M of the form $\oplus_{i=1}^n N_i$, where $N_i \leq^{\oplus} M_i$ $(i = 1, 2, \ldots, n)$.*

Proof. (1) $\Rightarrow$ (2): By induction on the number n of the summands M_i, s of M, and by using the observation followed after Theorem 3.2.

(2) $\Rightarrow$ (1): follows from the fact that each closed submodule of M_i is closed in M, hence apply the modular law. $\square$

Definition 3.2. A module M is called an *absolute relative jective* module (for short ARJ-module) if M_i is M_j-jective $(i \neq j)$; whenever $M = M_1 \oplus M_2$.

Clearly every extending module is an ARJ-module (Theorem 2.7), and any indecomposable module is obviously an ARJ-module, which is not extending. The following proposition gives the relation between Extending modules and ARJ--modules.

Proposition 3.8. *The following are equivalent for a module M:*

(1) *M is an extending module;*
(2) *M is an ARJ-module and satisfies the condition (C_1^*).*

Proof. (1) $\Rightarrow$ (2): From Theorem 2.7, and since extending modules satisfy the condition (C_1^*).

(2) $\Rightarrow$ (1): Let C be a closed submodule of M. By the condition (C_1^*), we have that C has a complement in M which is a summand; i.e. M has a decomposition $M = M_1 \oplus M_2$, where $M_2 \oplus C \leq^e M$. Since M is an ARJ-module, M_2 is M_1-jective. From Lemma 2.2 C is a complement of M_2 in M, and hence from the definition of relative jectivity, $C \leq^{\oplus} M$. Therefore M is extending. $\square$

Proposition 3.9. *Every indecomposable module M with the condition (C_1^*) is uniform.*

Proof. Let A be a nonzero submodule of M. By (C_1^*), there exists a decomposition of M as $M = M_1 \oplus M_2$ such that $A \oplus M_2 \leq^e M$. Since M is indecomposable, we have $M_2 = 0$; and hence $A \leq^e M$. $\square$

Proposition 3.10. *If M has (C_1^*), then it has a decomposition $M = M_1 \oplus M_2$, where* $\mathrm{Soc}(M) \leq^e M_1$.

Proof. By (C_1^*), there exists a submodule M_2 of M such that $M = M_1 \oplus M_2$, and $\mathrm{Soc}(M) \oplus M_2 \leq^e M$. It is clear that $\mathrm{Soc}(M_2) = 0$, and $\mathrm{Soc}(M) \leq^e M_1$. $\square$

Proposition 3.11. *Let M be an R-module. If M has (C_1^*), then the second singular submodule $Z_2(M)$ of M splits.*

Proof. By (C_1^*), there exists a complement K of $Z_2(M)$ in M which is a summand of M. Write $M = K \oplus L$. It is clear that K is nonsingular. Hence $Z_2(M) \leq L$. Since $Z_2(M) \oplus K \leq^e M$, then $Z_2(M) \leq^e L$. Since $Z_2(M)$ is closed in M, we have that $Z_2(M) = L \leq^{\oplus} M$. $\square$

In the following Proposition we show that arbitrary direct sums of uniform modules must have (C_1^*).

Proposition 3.12. *Direct sums of uniform modules have (C_1^*).*

Proof. Let $M = \bigoplus_{i \in I} U_i$, where the U_i are uniforms, and let A be a submodule of M. By Zorn's Lemma, there exists $J \subseteq I$ maximal with respect to $A \cap (\bigoplus_{i \in J} U_i) = 0$. Since $A \oplus (\bigoplus_{i \in J} U_i) \leq^e M$, it follows, by Lemma 2.2, $\bigoplus_{i \in J} U_i$ is a complement of A in M. $\square$

Remark. There are GE-modules which are not extending. In fact, Corollary 3.6 tells us that the $\mathbb{Z}$-module $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ is a GE-module, while M is not an extending module (see M. Kamal [**5**]).

Proposition 3.13. *If M is a GE-module with finite uniform dimension, then M is a direct sum of uniform submodules.*

Proof. Since M has a finite uniform dimension, then M is a direct sum of indecomposable submodules. By Corollary 3.4, the indecomposable summand of M are GE-modules, and hence, by Proposition 3.9, they are uniform modules. $\square$

Remark. Consider a direct sum of uniform submodules, which contains an indecomposble and not uniform summand submodule. This module has (C_1^*) (by Proposition 3.12), which is not a GE-module. This also shows that direct summands of modules with (C_1^*) need not have (C_1^*).

Lemma 3.14. [**12**, Lemma 2] *Let $A \leq B \leq M$. If C is a complement of A in M, then $C \cap B$ is a complement of A in B.*

Conside the following condition for a module M:

$$(*) \quad \begin{array}{c} \text{If } A \text{ and } B \text{ are summands of } M, \text{ with } A \cap B \text{ closed in } M, \\ \text{then } A \cap B \text{ is a summand of } M. \end{array}$$

Proposition 3.14. *If M has (C_1^*), and satisfies the condition $(*)$, then M is a GE-module.*

Proof. Let $B \leq^{\oplus} M$, and A be a closed submodule of B. It follows that A is closed in M. By (C_1^*) for M, there exists a complement K of A in M such that $K \leq^{\oplus} M$. By Lemma 3.14, we have $K \cap B$ is a complement of A in B; and hence a closed submodule of B. By the given condition $(*)$, and since $K \leq^{\oplus} M$, $B \leq^{\oplus} M$ with $K \cap B \leq_c M$; it follows that $K \cap B \leq \oplus M$. This shows that any summand B of M has (C_1^*). Therefore M is a GE-module. □

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