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CONGRUENCE KERNELS OF ORTHOIMPLICATION ALGEBRAS

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Abstract. Abstracting from certain properties of the implication operation in Boolean algebras leads to so-called orthoimplication algebras. These are in a natural one-to-one correspondence with families of compatible orthomodular lattices. It is proved that congruence kernels of orthoimplication algebras are in a natural one-to-one correspondence with families of compatible p-filters on the corresponding orthomodular lattices. Finally, it is proved that the lattice of all congruence kernels of an orthoimplication algebra is relatively pseudocomplemented and a simple description of the relative pseudocomplement is given.

In the literature many attempts were made in order to investigate properties of the implication operation in generalizations of Boolean algebras. These attempts led to different types of so-called implication algebras (cf. e. g. [**2**], [**5**] and [**6**]). It is interesting to note that these types of implication algebras are in a natural one-to-one correspondence with join-semilattices with 1 the principal filters of which are certain generalizations of Boolean algebras. Hence the question arises if there is a natural one-to-one correspondence between congruence kernels of these implication algebras on the one side and certain families of congruence kernels of the corresponding generalizations of Boolean algebras on the other side. We solve this problem for so-called orthoimplication algebras introduced in [**2**]. Moreover, we prove that the lattice of congruence kernels of orthoimplication algebras is relatively pseudocomplemented and we derive a simple description of the relative pseudocomplement.

In [**1**] implication algebras were introduced as algebras reflecting properties of the implication operation in Boolean algebras:

Definition 1. (cf. [**2**]) An orthoimplication algebra is an algebra $(A, \cdot, 1)$ of type $(2, 0)$ satisfying

$$xx = 1, \qquad (xy)x = x,$$
$$(xy)y = (yx)x, \qquad x((yx)z) = xz.$$

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Remark 1. In every orthoimplication algebra it holds $1x = x$ and $x1 = 1$ since $1x = (xx)x = x$ and $x1 = (1x)1 = 1$.

First we want to prove some congruence properties of the variety of orthoimplication algebras. For any algebra $\mathcal{B}$ let Con$\mathcal{B}$ denote the set of all congruences on $\mathcal{B}$.

Definition 2. Let $\mathcal{A}$ be an algebra with 1. $\mathcal{A}$ is called *weakly regular* if $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$ and $[1]\Theta = [1]\Phi$ together imply $\Theta = \Phi$. $\mathcal{A}$ is called *permutable at* 1 if $[1](\Theta \circ \Phi) = [1](\Phi \circ \Theta)$ for all $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$. $\mathcal{A}$ is called 3-*permutable* if $\Theta \circ \Phi \circ \Theta = \Phi \circ \Theta \circ \Phi$ for all $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$.

Theorem 1. *The variety $\mathcal{V}$ of orthoimplication algebras is weakly regular, permutable at* 1 *and* 3*-permutable.*

Proof. According to [**4**, Theorem 6.4.3], $\mathcal{V}$ is weakly regular if and only if there exist a positive integer n and binary terms $t_1, \ldots, t_n$ in $\mathcal{V}$ such that $t_1(x, y) = \ldots = t_n(x, y) = 1$ is equivalent to $x = y$. Now put $n := 2$, $t_1(x, y) := xy$ and $t_2(x, y) := yx$. Then $t_1(x, x) = t_2(x, x) = 1$. Conversely, if $t_1(x, y) = t_2(x, y) = 1$ then $x = 1x = (yx)x = (xy)y = 1y = y$. Hence $\mathcal{V}$ is weakly regular. According to [**4**, Theorem 6.6.11], $\mathcal{V}$ is permutable at 1 if and only if there exists a binary term t with $t(x, x) = 1$ and $t(x, 1) = x$. Now put $t(x, y) := yx$. Then $t(x, x) = 1$ and $t(x, 1) = x$ and hence $\mathcal{V}$ is permutable at 1. Finally, according to [**4**, Theorem 3.1.18], $\mathcal{V}$ is 3-permutable if and only if there exist ternary terms t_1, t_2 satisfying $t_1(x, z, z) = x$, $t_1(x, x, z) = t_2(x, z, z)$ and $t_2(x, x, z) = z$. Now put $t_1(x, y, z) := (zy)x$ and $t_2(x, y, z) := (xy)z$. Then

$$\begin{aligned} t_1(x, z, z) &= (zz)x = 1x = x, \\ t_1(x, x, z) &= (zx)x = (xz)z = t_2(x, z, z) \text{ and} \\ t_2(x, x, z) &= (xx)z = 1z = z \end{aligned}$$

and hence $\mathcal{V}$ is 3-permutable. □

In [**2**] a natural one-to-one correspondence between orthoimplication algebras and certain families of compatible orthomodular lattices was established. In order to be able to define these structures we first need the definition of an orthomodular lattice. (For the theory of orthomodular lattices we refer the reader to the monographs [**8**], [**3**] and [**9**].)

Definition 3. An *orthomodular lattice* is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and

$$\begin{aligned} &x \vee x' = 1, && x \wedge x' = 0, \\ &(x \vee y)' = x' \wedge y', && (x \wedge y)' = x' \vee y', \\ &(x')' = x, && x \leq y \text{ implies } y = x \vee (y \wedge x'). \end{aligned}$$

The third and fourth condition are the well-known De Morgan laws and the last condition is the so-called orthomodular law.

Now we are able to define the order-theoretical counterpart of orthoimplication algebras introduced in [**2**]:

Definition 4. (cf. [**2**]) A *semi-orthomodular lattice* is a partial algebra $(A, \vee, ({}^x; x \in A), 1)$ such that $(A, \vee, 1)$ is a join-semilattice with 1, for each $x \in A$, ^{x} is a unary operation on $[x, 1]$ such that $([x, 1], \vee, \wedge, {}^x, x, 1)$ is an orthomodular lattice and the compatibility condition

(CC) $$z^y = z^x \vee y \text{ for all } x, y, z \in A \text{ with } x \leq y \leq z$$

is satisfied.

Remark 2. It should be remarked that for $a \in A$ the meet-operation $\wedge_a$ in the orthomodular lattice $([a, 1], \vee, \wedge_a, {}^a, a, 1)$ does not depend on a in the following sense:

$$\text{If } a, b, x, y \in A \text{ and } a, b \leq x, y \text{ then } x \wedge_a y = x \wedge_b y.$$

This can be seen as follows: First assume $a \leq b$. Then $x \wedge_b y$ is a lower bound of x and y in $([b, 1], \leq)$ and hence also in $([a, 1], \leq)$. If c is an arbitrary lower bound of x and y in $([a, 1], \leq)$ then x and y are upper bounds of b and c and hence $b \vee c \leq x$ and $b \vee c \leq y$. This shows that $b \vee c$ is a lower bound of x and y in $([b, 1], \leq)$ and hence $b \vee c \leq x \wedge_b y$ which implies $c \leq x \wedge_b y$. This proves $x \wedge_b y = x \wedge_a y$. If now $a, b, x, y \in A$ and $a, b \leq x, y$ then $x \wedge_a y = x \wedge_{a \vee b} y = x \wedge_b y$. Obviously, $x \wedge y$ exists in $(A, \leq)$ if and only if x and y have a common lower bound.

Remark 3. *According to the De Morgan laws $x \wedge y = (x^a \vee y^a)^a$ holds for all $a \in A$ and $x, y \in [a, 1]$.*

The natural one-to-one correspondence between orthoimplication algebras and semi-orthomodular lattices can be formulated as follows (for other types of implication algebras and their corresponding order-theoretical counterparts cf. e. g. [**1**] and [**5**]):

Theorem 2 (cf. [**2**]). *For every fixed set A the formulas*

$$x \vee y = (xy)y, \qquad x^y = xy,$$

respectively

$$xy = (x \vee y)^y$$

induce mutually inverse bijections between the set of all orthoimplication algebras over A and the set of all semi-orthomodular lattices over A.

In what follows, let $\mathcal{A} = (A, \cdot, 1)$ be an arbitrary, but fixed orthoimplication algebra and $\mathcal{S} = (A, \vee, ({}^x; x \in A), 1)$ its corresponding semi-orthomodular lattice.

For semi-orthomodular lattices we need a certain notion corresponding to the notion of a congruence.

Definition 5. A *compatible congruence family* on $\mathcal{S}$ is a family $(\Theta_x; x \in A)$ of congruences Θ_x on $([x, 1], \vee, \wedge, {}^x, x, 1)$ such that $\Theta_y = \Theta_x \cap [y, 1]^2$ for all $x, y \in A$

with $x \le y$. Let $\mathrm{CCF}(\mathcal{S})$ denote the set of all compatible congruence families on $\mathcal{S}$. On $\mathrm{CCF}(\mathcal{S})$ we define a binary relation $\le$ by

$$(\Theta_x; x \in A) \le (\Phi_x; x \in A) \text{ if } \Theta_x \subseteq \Phi_x \text{ for all } x \in A.$$

Remark 4. $(\mathrm{CCF}(\mathcal{S}), \le)$ is a complete lattice.

Now we can formulate the natural one-to-one correspondence between congruences on $\mathcal{A}$ and compatible congruence families on $\mathcal{S}$:

Theorem 3. *The formulas*

$$\Theta_x = \Theta \cap [x,1]^2$$

and

$$\Theta = \{(x,y) \in A^2 \mid (x, x \vee y) \in \Theta_x \textit{ and } (x \vee y, y) \in \Theta_y\}$$

induce mutually inverse isomorphisms between $(\mathrm{Con}\mathcal{A}, \subseteq)$ *and* $(\mathrm{CCF}(\mathcal{S}), \le)$.

Proof. Let $a,b,c,d \in A$. If $\Theta \in \mathrm{Con}\mathcal{A}$ and $\Theta_x := \Theta \cap [x,1]^2$ for all $x \in A$, then in the case $b, c \ge a$ the relations

$$b \vee c = (bc)c, \qquad b \wedge c = (((ba)(ca))(ca))a,$$

$$b^a = ba$$

imply that $\Theta_a \in \mathrm{Con}([a,1], \vee, \wedge, {}^a, a, 1)$. Clearly, $b \ge a$ implies also

$$\Theta_b = \Theta \cap [b,1]^2 = \Theta \cap ([a,1]^2 \cap [b,1]^2) = (\Theta \cap [a,1]^2) \cap [b,1]^2 = \Theta_a \cap [b,1]^2$$

proving $(\Theta_x; x \in A) \in \mathrm{CCF}(\mathcal{S})$. Moreover, as $a \vee b = (ab)b$, the following three assertions are equivalent:

$$(a, a \vee b) \in \Theta_a,\ (a \vee b, b) \in \Theta_b, \qquad (a, a \vee b),\ (a \vee b, b) \in \Theta,$$

$$(a,b) \in \Theta.$$

Conversely, assume $(\Theta_x; x \in A) \in \mathrm{CCF}(\mathcal{S})$ and define

$$\Theta := \{(x,y) \in A^2 \mid (x, x \vee y) \in \Theta_x \text{ and } (x \vee y, y) \in \Theta_y\}.$$

Then Θ is reflexive and symmetric.

Further, notice that

$$\text{(1)} \qquad a, b \le c, d \text{ and } (c,d) \in \Theta_a \text{ imply } (c,d) \in \Theta_b.$$

Indeed, we have

$$(c,d) \in \Theta_a \cap [a \vee b, 1]^2 = \Theta_{a \vee b} = \Theta_b \cap [a \vee b, 1]^2 \subseteq \Theta_b.$$

Now, in order to prove that Θ is transitive, take any $(a,b), (b,c) \in \Theta$. Then

$$a \vee b \vee c = (a \vee b) \vee (b \vee c)\ \Theta_b\ (a \vee b) \vee b = a \vee b.$$

Since $b, a \le a \vee b \vee c, a \vee b$, using (1) we obtain

$$a \vee b \vee c\ \Theta_a\ a \vee b.$$

This implies

$$a \vee c = a \vee (a \vee c)\ \Theta_a\ (a \vee b) \vee (a \vee c) = a \vee b \vee c\ \Theta_a\ a \vee b\ \Theta_a\ a,$$

i. e. $(a, a \vee c) \in \Theta_a$. Now

$$a \vee b \vee c = (a \vee b) \vee (b \vee c) \; \Theta_b \; b \vee (b \vee c) = b \vee c.$$

Since $b, c \leq a \vee b \vee c, b \vee c$, using (1) we obtain

$$a \vee b \vee c \; \Theta_c \; b \vee c.$$

This implies

$$a \vee c = (a \vee c) \vee c \; \Theta_c \; (a \vee c) \vee (b \vee c) = a \vee b \vee c \; \Theta_c \; b \vee c \; \Theta_c \; c,$$

i. e. $(a \vee c, c) \in \Theta_c$. Therefore, we obtain $(a, c) \in \Theta$ proving that Θ is transitive.
Next we show that Θ is a right congruence on $\mathcal{A}$. Assume $(a, b) \in \Theta$. Then

$$a \vee b \vee c = (a \vee b) \vee (a \vee c) \; \Theta_a \; a \vee (a \vee c) = a \vee c$$

and hence $a \vee b \vee c \; \Theta_c \; a \vee c$. Moreover,

$$a \vee b \vee c = (a \vee b) \vee (b \vee c) \; \Theta_b \; b \vee (b \vee c) = b \vee c$$

and hence $a \vee b \vee c \; \Theta_c \; b \vee c$. Together we obtain

$$a \vee c \; \Theta_c \; a \vee b \vee c \; \Theta_c \; b \vee c$$

and therefore

$$ac = (a \vee c)^c \; \Theta_c \; (b \vee c)^c = bc.$$

Now

$$ac \vee bc \; \Theta_c \; bc \vee bc = bc \text{ implies } ac \vee bc \; \Theta_{bc} \; bc$$

and

$$ac \vee bc \; \Theta_c \; ac \vee ac = ac \text{ implies } ac \vee bc \; \Theta_{ac} \; ac.$$

Together these relations show $(ac, bc) \in \Theta$ proving that Θ is a right congruence on $\mathcal{A}$. From this it follows that $(a, b) \in \Theta$ implies $(a \vee c, b \vee c) \in \Theta$ since $(ac, bc) \in \Theta$ and hence $(a \vee c, b \vee c) = ((ac)c, (bc)c) \in \Theta$.
Next we show that Θ is a left congruence on $\mathcal{A}$. Assume $(a, b) \in \Theta$. Then

$$a \vee b \vee c = (a \vee b) \vee (a \vee c) \; \Theta_a \; a \vee (a \vee c) = a \vee c$$

and hence

$$(a \vee b \vee c)^a \; \Theta_a \; (a \vee c)^a$$

which implies

$$(a \vee b \vee c)^a \; \Theta_{(a \vee b \vee c)^a} \; (a \vee c)^a.$$

Thus

$$(a \vee b \vee c)^a \vee (a \vee c)^a = (a \vee c)^a \; \Theta_{(a \vee b \vee c)^a} \; (a \vee b \vee c)^a$$

and

$$(a \vee b \vee c)^a \vee (a \vee c)^a = (a \vee c)^a \; \Theta_{(a \vee c)^a} \; (a \vee c)^a$$

showing $((a\vee b\vee c)^a, (a\vee c)^a) \in \Theta$. Analogously, we obtain $((a\vee b\vee c)^b, (b\vee c)^b) \in \Theta$. Using (CC) we get

$$\begin{aligned} ca &= (c \vee a)^a = (a \vee c)^a \ \Theta \ (a \vee b \vee c)^a = (a \vee b \vee c)^a \vee a \ \Theta \ (a \vee b \vee c)^a \vee b \\ &= ((a \vee b \vee c)^a \vee a) \vee b = (a \vee b \vee c)^a \vee (a \vee b) = (a \vee b \vee c)^{a \vee b} \\ &= (a \vee b \vee c)^b \vee (a \vee b) = (a \vee b \vee c)^b \vee (b \vee a) = ((a \vee b \vee c)^b \vee b) \vee a \\ &= (a \vee b \vee c)^b \vee a \ \Theta \ (a \vee b \vee c)^b \vee b = (a \vee b \vee c)^b \ \Theta \ (b \vee c)^b = (c \vee b)^b \\ &= cb. \end{aligned}$$

As Θ is transitive, the above relations imply $(ca, cb) \in \Theta$, proving that Θ is a left congruence on $\mathcal{A}$.
Hence $\Theta \in \mathrm{Con}\mathcal{A}$. Moreover, the following are equivalent:

$$\begin{array}{ll} (b,c) \in \Theta \cap [a,1]^2, & b, c \geq a, \quad (b, b \vee c) \in \Theta_b, \ (b \vee c, c) \in \Theta_c, \\ (b, b \vee c), (b \vee c, c) \in \Theta_a, & (b,c) \in \Theta_a. \end{array}$$

This fact shows that the mappings induced by the formulas stated in the theorem are mutually inverse bijections between $\mathrm{Con}\mathcal{A}$ and $\mathrm{CCF}(\mathcal{S})$. It is obvious, that in fact they are isomorphisms between $(\mathrm{Con}\mathcal{A}, \subseteq)$ and $(\mathrm{CCF}(\mathcal{S}), \leq)$. □

Corollary 1. *Every congruence on $\mathcal{A}$ is uniquely determined by its restrictions to the intervals $[x,1]$, $x \in A$.*

The following lemma explains how the congruences on an orthoimplication algebra are determined by their kernels.

Lemma 1. $\Theta = \{(x,y) \in A^2 \mid xy, yx \in [1]\Theta\}$ *for all* $\Theta \in \mathrm{Con}\mathcal{A}$.

Proof. If $(a,b) \in \Theta$ then $ab, ba \in [aa]\Theta = [1]\Theta$ and if, conversely, $a, b \in A$ and $ab, ba \in [1]\Theta$ then $a = 1a \ \Theta \ (ba)a = (ab)b \ \Theta \ 1b = b$ and hence $(a,b) \in \Theta$. □

Now we introduce the notion of a congruence kernel of an orthoimplication algebra.

Definition 6. A subset F of A is called a *congruence kernel* of $\mathcal{A}$ if there exists a congruence Θ on $\mathcal{A}$ with $[1]\Theta = F$. Let $\mathrm{CK}(\mathcal{A})$ denote the set of all congruence kernels of $\mathcal{A}$.

The natural one-to-one correspondence between congruences on orthoimplication algebras and their kernels is established by the following

Theorem 4. *The formulas*

$$F = [1]\Theta$$

and

$$\Theta = \{(x,y) \in A^2 \mid xy, yx \in F\}$$

induce mutually inverse isomorphisms between $(\mathrm{Con}\mathcal{A}, \subseteq)$ *and* $(\mathrm{CK}(\mathcal{A}), \subseteq)$.

Proof. It is an immediate consequence of Lemma 1. □

Remark 5. $(\mathrm{CK}(\mathcal{A}), \subseteq)$ is a complete lattice.

As far as follows let $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ be an arbitrary, but fixed orthomodular lattice.

Congruence kernels of orthomodular lattices are called p-filters. This is the content of

Definition 7 (cf. [**8**])**.** A subset F of L is called a *p-filter* of $\mathcal{L}$ if there exists a congruence Θ on $\mathcal{L}$ with $[1]\Theta = F$. Let $\mathrm{F}(\mathcal{L})$ denote the set of all p-filters of $\mathcal{L}$.

Remark 6. $(\mathrm{F}(\mathcal{L}), \subseteq)$ is a complete lattice.

The following one-to-one correspondence between congruences and p-filters of orthomodular lattices (generalizing the corresponding one for Boolean algebras) is well-known:

Theorem 5 (cf. [**8**])**.** *The formulas*

$$F = [1]\Theta$$

and

$$\Theta = \{(x, y) \in L^2 \mid (x \wedge y) \vee (x' \wedge y') \in F\}$$

induce mutually inverse isomorphisms between $(\mathrm{Con}\mathcal{L}, \subseteq)$ *and* $(\mathrm{F}(\mathcal{L}), \subseteq)$.

For describing congruence kernels of orthoimplication algebras by p-filters of the corresponding orthomodular lattices we need the following concept:

Definition 8. A *compatible filter family* on $\mathcal{S}$ is a family $(F_x; x \in A)$ of p-filters F_x of $([x, 1], \vee, \wedge, ^x, x, 1)$ such that $F_y = F_x \cap [y, 1]$ for all $x, y \in A$ with $x \leq y$. Let $\mathrm{CFF}(\mathcal{S})$ denote the set of all compatible filter families on $\mathcal{S}$. On $\mathrm{CFF}(\mathcal{S})$ we define a binary relation $\leq$ by

$$(F_x; x \in A) \leq (G_x; x \in A) \text{ if } F_x \subseteq G_x \text{ for all } x \in A.$$

Remark 7. $(\mathrm{CCF}(\mathcal{S}), \leq)$ is a complete lattice.

In the proof of the next theorem we need the following easy property of congruence kernels of orthoimplication algebras:

Lemma 2. *If* $F \in \mathrm{CK}(\mathcal{A})$*,* $a \in F$*,* $b \in A$ *and* $a \leq b$ *then* $b \in F$*.*

Proof. If $\Theta \in \mathrm{Con}\mathcal{A}$ with $[1]\Theta = F$ then $b = a \vee b \in [1 \vee b]\Theta = [1]\Theta = F$. □

We are now able to formulate and prove the natural one-to-one correspondence between congruence kernels of orthoimplication algebras and compatible filter families on the corresponding semi-orthomodular lattice.

Theorem 6. *The formulas*

$$F_x = F \cap [x, 1]$$

and

$$F = \bigcup_{x \in A} F_x$$

induce mutually inverse isomorphisms between $(\mathrm{CK}(\mathcal{A}), \subseteq)$ *and* $(\mathrm{CFF}(\mathcal{S}), \leq)$.

Proof. Let $a, b, c, d \in A$. If $F \in \mathrm{CK}(\mathcal{A})$ and $F_x := F \cap [x, 1]$ for all $x \in A$ then there exists a congruence Θ on $\mathcal{A}$ with $[1]\Theta = F$, and $\Theta \cap [a, 1]^2 \in \mathrm{Con}([a, 1], \vee, \wedge, {}^a, a, 1)$, according to Theorem 3. Now, since

$$F_a = F \cap [a, 1] = [1]\Theta \cap [a, 1] = [1](\Theta \cap [a, 1]^2) \in \mathrm{F}(([a, 1], \vee, \wedge, {}^a, a, 1)),$$

in the case $a \leq b$ we obtain

$$F_b = F \cap [b, 1] = F \cap ([a, 1] \cap [b, 1]) = (F \cap [a, 1]) \cap [b, 1] = F_a \cap [b, 1]$$

proving $(F_x; x \in A) \in \mathrm{CFF}(\mathcal{S})$. Moreover,

$$\bigcup_{x \in A} F_x = \bigcup_{x \in A} (F \cap [x, 1]) = F \cap \bigcup_{x \in A} [x, 1] = F \cap A = F.$$

Conversely, assume $(F_x; x \in A) \in \mathrm{CFF}(\mathcal{S})$ and set $F := \bigcup_{x \in A} F_x$. Then for every $x \in A$ there exists a congruence Θ_x on $([x, 1], \vee, \wedge, {}^x, x, 1)$ with $[1]\Theta_x = F_x$. Assume $a \leq b$. Then $F_b = F_a \cap [b, 1] \subseteq F_a$ and hence $\Theta_b \subseteq \Theta_a$ according to Theorem 5 and therefore $\Theta_b \subseteq \Theta_a \cap [b, 1]^2$. Conversely, $(c, d) \in \Theta_a \cap [b, 1]^2$ implies $c, d \geq b$ and $(c \wedge d) \vee (c^a \wedge d^a) \in [1]\Theta_a = F_a$. Using (CC) we obtain

$$(c \wedge d) \vee (c^b \wedge d^b) = (c \wedge d) \vee ((c^a \vee b) \wedge (d^a \vee b)) \in F_a \cap [b, 1] = F_b = [1]\Theta_b$$

according to Lemma 2 whence $(c, d) \in \Theta_b$. Therefore $\Theta_b = \Theta_a \cap [b, 1]^2$ and $(\Theta_x; x \in A) \in \mathrm{CCF}(\mathcal{S})$. Put

$$\Theta := \{(x, y) \in A^2 \mid (x, x \vee y) \in \Theta_x \text{ and } (x \vee y, y) \in \Theta_y\}.$$

According to Theorem 3, $\Theta \in \mathrm{Con}\mathcal{A}$ and $\Theta \cap [x, 1]^2 = \Theta_x$ for all $x \in A$. Now

$$F = \bigcup_{x \in A} F_x = \bigcup_{x \in A} ([1]\Theta_x) = \bigcup_{x \in A} ([1](\Theta \cap [x, 1]^2)) = [1]\Theta \in \mathrm{CK}(\mathcal{A}).$$

Moreover,

$$\begin{aligned} F \cap [a, 1] &= (\bigcup_{x \in A} F_x) \cap [a, 1] = \bigcup_{x \in A} (F_x \cap [a, 1]) = \bigcup_{x \in A} ((F_x \cap [x, 1]) \cap [a, 1]) \\ &= \bigcup_{x \in A} (F_x \cap ([x, 1] \cap [a, 1])) = \bigcup_{x \in A} (F_x \cap [a \vee x, 1]) = \bigcup_{x \in A} F_{a \vee x} \\ &= \bigcup_{x \in A} (F_a \cap [a \vee x, 1]) = F_a \cap \bigcup_{x \in A} [a \vee x, 1] = F_a \cap [a, 1] \\ &= F_a. \end{aligned}$$

The rest of the proof is clear. □

From Theorem 6 we deduce the following nice characterization of congruence kernels of orthoimplication algebras:

Corollary 2. *A subset F of A is a congruence kernel of $\mathcal{A}$ if and only if $F \cap [x, 1] \in \mathrm{F}(([x, 1], \vee, \wedge, {}^x, x, 1))$ for all $x \in A$. If the latter holds then* $(F \cap$

$[x,1]; x \in A) \in \mathrm{CCF}(\mathcal{S})$ *and according to Theorem 6 there exists a* $G \in \mathrm{CK}(\mathcal{A})$ *with* $G \cap [x,1] = F \cap [x,1]$ *for all* $x \in A$ *and hence*

$$F = F \cap A = F \cap \bigcup_{x\in A}[x,1] = \bigcup_{x\in A}(F \cap [x,1]) = \bigcup_{x\in A}(G \cap [x,1])$$
$$= G \cap \bigcup_{x\in A}[x,1] = G \cap A = G \in \mathrm{CK}(\mathcal{A}).$$

Finally, we want to show that the lattice of congruence kernels of an orthoimplication algebra is relatively pseudocomplemented. First we recall the notion of a relative pseudicomplement in a meet-semilattice:

Definition 9. Let $(S, \wedge)$ be a meet-semilattice and $a, b \in S$. An element c of S is called the *relative pseudocomplement* of a with respect to b if c is the greatest element x of S satisfying $a \wedge x \le b$.

In the lattice of p-filters of an orthomodular lattice there exists a nice description of relative pseudocomplements:

Theorem 7. (*cf.* [**3**] *and* [**8**]) $(\mathrm{F}(\mathcal{L}), \subseteq)$ *is relatively pseudocomplemented. If* $F, G \in \mathrm{F}(\mathcal{L})$ *then* $\langle F, G\rangle := \{x \in L \mid x \vee y \in G$ *for all* $y \in F\}$ *is the relative pseudocomplement of* F *with respect to* G *in* $(\mathrm{F}(\mathcal{L}), \subseteq)$.

Using this description and the connection between congruence kernels of orthoimplication algebras and p-filters of the corresponding orthomodular lattices we obtain a nice and simple description of the relative pseudocomplement in the lattice of all congruence kernels of an orthoimplication algebra:

Theorem 8. $(\mathrm{CK}(\mathcal{A}), \subseteq)$ *is relatively pseudocomplemented. If* $F, G \in \mathrm{CK}(\mathcal{A})$ *then* $\langle F, G\rangle := \{x \in A \mid (xy)y \in G$ *for all* $y \in F\}$ *is the relative pseudocomplement of* F *with respect to* G *in* $(\mathrm{CK}(\mathcal{A}), \subseteq)$.

Proof. Let $a, b \in A$. If $b \in \langle F, G\rangle \cap [a,1]$ then for every $y \in F \cap [a,1]$ we get $b \vee y = (by)y \in G \cap [a,1]$, and hence $b \in \langle F \cap [a,1], G \cap [a,1]\rangle$. If, conversely, $b \in \langle F \cap [a,1], G \cap [a,1]\rangle$ then $b \in [a,1]$ and $b \vee y \in F \cap [a,1]$ for all $y \in F$ according to Lemma 2 and hence

$$(by)y = b \vee y = b \vee (b \vee y) \in G \cap [a,1] \subseteq G$$

showing $b \in \langle F, G\rangle \cap [a,1]$. Hence

$$\langle F, G\rangle \cap [a,1] = \langle F \cap [a,1], G \cap [a,1]\rangle \in \mathrm{F}(([a,1], \vee, \wedge, {}^a, a, 1)).$$

According to Corollary 2, $\langle F, G\rangle \in \mathrm{CK}(\mathcal{A})$. If $a \in F \cap \langle F, G\rangle$ then $a \in F$ and $(ay)y \in G$ for all $y \in F$ and hence $a = (aa)a \in G$ proving $F \cap \langle F, G\rangle \subseteq G$. Conversely, if $a \in H \in \mathrm{CK}(\mathcal{A})$ and $F \cap H \subseteq G$ then $(ay)y = a \vee y \in F \cap H \subseteq G$ for all $y \in F$ according to Lemma 2 and hence $a \in \langle F, G\rangle$ showing $H \subseteq \langle F, G\rangle$. This completes the proof of the theorem. $\square$

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