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SOME COMMENTS ON INJECTIVITY AND P-INJECTIVITY

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Abstract. A generalization of injective modules (noted GI-modules), distinct from p-injective modules, is introduced. Rings whose p-injective modules are GI are characterized. If M is a left GI-module, $E = \mathrm{End}({}_AM)$, then $E/J(E)$ is von Neumann regular, where $J(E)$ is the Jacobson radical of the ring E. A is semi-simple Artinian if, and only if, every left A-module is GI. If A is a left p. p., left GI-ring such that every non-zero complement left ideal of A contains a non-zero ideal of A, then A is strongly regular. Sufficient conditions are given for a ring to be either von Neumann regular or quasi-Frobenius. Quasi-Frobenius and von Neumann regular rings are characterized. Kasch rings are also considered.

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z, Y will stand respectively for the Jacobson radical, the left singular ideal and the right singular ideal of A. A is called *semi-primitive* or *semi-simple* (resp. (a) left *non-singular*; (b) *right non-singular*) if $J = (0)$ (resp. (a) $Z = (0)$; (b) $Y = (0)$). An ideal of A will always mean a two-sided ideal of A. A is called *left* (resp. *right*) *quasi-duo* if every maximal left (resp. right) ideal of A is an ideal of A. It well-known that J, Z, Y are ideals of A. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent elements.

Following C. Faith, write "A is VNR" if A is a von Neumann regular ring [**8**]. A is called fully (resp. (1) fully left; (2) fully right) idempotent if every ideal (resp.(1) left ideal ; (2) right ideal) of A is idempotent.

It is well-known that A is VNR if and only if every left (right) A-module is flat (Harada ((1956); Auslander (1957)). Also, A is VNR if and only if every left (right) A-module is p-injective ([**2**], [**4**], [**12**], [**22**], [**23**]). Note that the Harada-Auslander's characterization may be weakened as follows: A is VNR if and only if every singular right A-module is flat (cf. [**38**, p. 147]).

Recall that a left A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M ([**8**, p. 122], [**20**, p. 577], [**21**, p. 340], [**26**]). A is called a left p-injective ring if ${}_AA$ is p-injective. P-injectivity is similarly defined on the right side. A generalization of p-injectivity, noted YJ-injectivity, is introduced in [**29**](cf. also [**22**], [**39**]). YJ-injectivity is also called GP-injectivity by other authors (cf. [**4**], [**6**], [**15**]).

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${}_AM$ is called YJ-injective if, for any $0 \neq a \in A$, there exists a positive integer n such that $a^n \neq 0$ and every left A-homomorphism of Aa^n into M extends to one of A into M [**29**]. A is called a left YJ-injective ring if ${}_AA$ is YJ-injective. YJ-injectivity is similarly defined on the right side.

Note that A is left YJ-injective if and only if for every $0 \neq a \in A$, there exists a positive integer n such that a^nA is a non-zero right annihilator [**29**, Lemma 3].

Also, if A is right YJ-injective, then $Y = J$ [**28**, Proposition 1] (this is the origin of the notation). In recent years, p-injectivity and YJ-injectivity have drawn the attention of many authors ([**2**], [**4**], [**6**], [**8**, Theorem 6.4], [**11**], [**15**], [**16**], [**17**], [**20**], [**22**], [**23**], [**40**]).

We have consider the following generalization of injective modules.

Definition 1. A left A-module M is called GI (generalized injective) if, given any left submodule C of M which is isomorphic to a non-zero complement left submodule of M, any monomorphisms g, f of C into M, there exists a left A-homomorphism $h : M \to M$ such that $hf = g$. Write "A is a left GI-ring" if ${}_AA$ is GI.

Note that any simple left A-module is GI. Consequently, GI-modules generalize effectively injective modules.

GI-modules need not be p-injective (otherwise, any arbitrary ring would be fully left and right idempotent!).

The converse is not true either, as shown by the following result.

Theorem 1. *The following conditions are equivalent:*

(1) *A is a left Noetherian ring whose p-injective left modules are injective;*
(2) *Every p-injective left A-module is GI.*

Proof. (1) implies (2) evidently.

Assume (2). Let M be a p-injective left A-module, E the injective hull of ${}_AM$. Write $Q = {}_AM \oplus {}_AE$ and $S = \{(y, o); y \in M\}$. Then ${}_AS$ is a direct summand of ${}_AQ$ and ${}_AS \approx {}_AM$. If $i : M \to E$ is the inclusion map: $j : M \to Q$ and $k : E \to Q$ the canonical injections, since ${}_AQ$ is the direct sum of two p-injective left A-modules, then Q is p-injective and by hypothesis, ${}_AQ$ is GI. There exists a left A-homomorphism $h : Q \to Q$ such that $hki = j$. If $p : Q \to M$ is the canonical projection, then $v = phk : E \to M$ such that $vi = pj =$ identity map on M. Therefore ${}_AM$ is a direct summand of ${}_AE$ which yields $M = E$ is injective. We have shown that every p-injective left A-module is injective. Since any direct sum of p-injective left A-modules is p-injective, then every direct sum of injective left A-modules is injective which implies that A is left Noetherian [**7**, Theorem 20.1]. Thus (2) implies (1). $\square$

As usual, A is called a left IF-ring if every injective left A-module is flat. The next theorem is motivated by [**38**, Proposition 6].

Theorem 2. *The following conditions are equivalent:*

(1) *A is quasi-Frobenius;*
(2) *A is a left IF-ring whose flat modules are GI;*

(3) *The direct sum of any injective and any projective left A-modules is GI.*

Proof. Assume (1). Since A is left perfect, any flat left A-module F is projective. Now F is injective by [**7**, Theorem 24.20], hence GI. Therefore (1) implies (2).

Assume (2). Let Q be a direct sum of an injective and a projective left A-modules. Then Q is the direct sum of two flat left A-modules which is therefore flat. By hypothesis, ${}_AQ$ is GI and therefore (2) implies (3).

Assume (3). Let P be a non-zero projective left A-module, E the injective hull of ${}_AP$. Write $Q = {}_AP \oplus {}_AE$ and $S = \{(y, 0); y \in P\}$. Then ${}_AS \approx {}_AP$ and ${}_AS$ is a direct summand of ${}_AQ$. By hypothesis, ${}_AQ$ is GI. The proof of Theorem 1 then shows that ${}_AP$ must be injective. By [**7**, Theorem 24.20], A is quasi-Frobenius and (3) implies (1). $\square$

Corollary 2.1. *If flat left A-modules coincide with GI left A-modules, then A is quasi-Frobenius.*

Proof. By hypothesis, A is a left IF-ring. The corollary then follows from Theorem 2 (2). $\square$

The proof of Theorem 1 shows that if the direct sum of any two GI left A-modules is GI, then every GI left A-module is injective. The next proposition then follows.

Proposition 3. *A is semi-simple Artinian if and only if every left A-module is GI.*

Given a left A-module M, $\mathrm{End}(M)$ denotes, as usual, the ring of endomorphisms of ${}_AM$. We now turn to an analogous result of a well-known theorem [**7**, Theorem 19.27].

Theorem 4. *Let M be a GI left A-module. If $E = \mathrm{End}(M)$, $\mathrm{J}(E)$ is Jacobson radical of E, then $E/\mathrm{J}(E)$ is VNR and $\mathrm{J}(E) = \{f \in E/\ker f$ is essential in ${}_AM\}$.*

Proof. Write $E = \mathrm{End}(M)$, $\mathrm{J}(E)$ the Jacobson radical of E. Set $V = \{f \in E/\ker f$ is essential in ${}_AM\}$. It is well-known that V is an ideal of E. We first show that $V \subseteq \mathrm{J}(E)$.

For any $f \in V$, $d \in E$, since $\ker f \cap \ker(1 - df) = 0$, then $\ker(1 - df) = 0$. With $u = 1 - df$, u is an isomorphism of M onto uM. Let $v : uM \to M$ be the inverse isomorphism of u. Since ${}_AM$ is GI, with $j : uM \to M$ the inclusion map, there exists an endomorphism h of ${}_AM$ such that $hj = v$.

Then

$$hu(m) = hj(u(m)) = v(u(m)) = m \qquad \text{for all } m \in M$$

which implies that hu is the identity map on M. Therefore $1 - df$ is left invertible in E for every $d \in E$, proving that $f \in \mathrm{J}(E)$.

Now, let $\overline{0} \neq \overline{g} \in E/\mathrm{J}(E)$, $g \in E$. Then $g \notin V$ (because $V \subseteq \mathrm{J}(E)$). By Zorn's Lemma, there exists a non-zero complement submodule K of M such that $\ker g \oplus K$ is an essential submodule of ${}_AM$. If $r : K \to M$ is the restriction of g to K, then r is a monomorphism and consequently $r : K \to r(K)$ is an isomorphism. Let $s : r(K) \to K$ be the inverse isomorphism. Then $sr =$ identity map on K.

Since K is a non-zero complement submodule of M, if $i : K \to M$ is the canonical injection, then $is : r(K) \to M$ and is extends to an endomorphism t of ${}_AM$. For any $k \in K$,

$$t(g(k)) = t(r(k)) = isr(k) = k$$

which implies that $K + \ker g \subseteq \ker(gtg - g)$ and hence $gtg - g \in V \subseteq \mathrm{J}(E)$. Therefore $\overline{g}\overline{t}\overline{g} = \overline{g} \in E/\mathrm{J}(E)$ which proves that $E/\mathrm{J}(E)$ is VNR.

Now suppose there exists $w \in \mathrm{J}(E)$ such that $w \notin V$. Then the above proof shows that there exists $z \in E$ such that $y = w - wzw \in V$. But there exists $q \in E$ such that $(1 - zw)q = 1$. Therefore $y = w(1 - zw)$ yields $yq = w \cdot 1 = w$, whence $w \in V$ (since V is an ideal of E), which is a contradiction! Therefore $\mathrm{J}(E) \subseteq V$ and finally, $J(E) = V = \{f \in E/\ker f \text{ is essential in } {}_AM\}$. □

Proposition 5. *If A is a left GI-ring, then every non-zero-divisor of A is invertible in A. Consequently, A coincides with its classical left (and right) quotient ring.*

Proof. Let c be a non-zero divisor of A. Define $f : Ac \to A$ by $f(ac) = a$ for all $a \in A$. Then f is a well-defined left A-homomorphism which is a monomorphism. Now ${}_AAc \approx {}_AA$ and if $Ac \to A$ is the inclusion map, since ${}_AA$ is GI, there exists a left A-homomorphism $h : A \to A$ such that $hi = f$. If $h(1) = u \in A$, then

$$1 = f(c) = hi(c) = h(c) = ch(1) = cu.$$

Then $c = cuc$ which yields $c(1-uc) = 0$, whence $uc = 1$. Therefore c is invertible in A and consequently, A coincides with its classical left (and right) quotient ring. □

Call A a left TC-ring if every non-zero complement left ideal of A contains a non-zero ideal of A.

Corollary 5.1. *If A is a left TC, left p.p., left GI-ring, then A is strongly regular.*

Proof. Since A is left non-singular, left TC, then A is reduced by [**34**, Lemma 1]. Now A is a reduced left p.p. ring which implies that every element a of A is of the form $a = ce$, where c is a non-zero-divisor and e is a central idempotent in A [**30**, Theorem 2]. By Proposition 5, c is invertible in A. Then

$$a = ce = cec^{-1}c = cec^{-1}ce \qquad \text{(since } e \text{ is central)}$$

which yields $a = ac^{-1}a$. Therefore A is VNR and since A is reduced, then A is strongly regular. □

In [**17**, Example 2.4], the given ring A has the following property: for every $y \in J$, the Jacobson radical of A, $l(y) = r(y)$. This motivates the next result.

Proposition 6. *The following conditions are equivalent:*

(1) *A is strongly regular;*
(2) *A is a left quasi-duo ring whose simple left modules are either YJ-injective or flat and for every $u \in J$, $l(u) = r(u)$.*

Proof. (1) implies (2) evidently.

Assume (2). Suppose there exists $0 \neq v \in J$ such that $v^2 = 0$. If $I = AvA + l(v)$, suppose that $I \neq A$. Let M be a maximal left ideal of A containing I. If ${}_AA/M$ is YJ-injective, since $v^2 = 0$, every left A-homomorphism of Av into A/M extends to one of A into A/M.

Define

$$g : Av \to A/M \quad \text{by} \quad g(av) = a + M \qquad \text{for all } a \in A.$$

Then

$$1 + M = g(v) = vy + M \qquad \text{for some } y \in A.$$

Since $vy \in J \subseteq M$, then $1 \in M$, which contradicts $M \neq A$.

If ${}_AA/M$ is flat, then $v \in I \subseteq M$ implies that $v = vd$ for some $d \in M$ [**3**, p. 458]. Now $(1-d) \in r(v) = l(v) \subseteq M$ which yields $1 \in M$, again a contradiction! Therefore $I = A$. Then $1 = s + t$, $s \in AvA$, $t \in l(v)$ and $v = sv$. Since $s \in J$, $1 - s$ is left invertible in A which yields $v = 0$, contradicting our original hypothesis. We have shown that J must be a reduced ideal of A.

Now suppose that $J \neq 0$. If $0 \neq w \in J$, since J is reduced, for any positive integer m,

$$l(w^m) = l(w) = r(w) = r(w^m).$$

Set $W = AwA + l(w)$. If $W \neq A$, let N be a maximal left ideal of A containing W. If ${}_AA/N$ is YJ-injective, there exists a positive integer n such that every left A-homomorphism of Aw^n into A/N extends to one of A into A/N. We may define a left A-homomorphism

$$h : Aw^n \to A/N \quad \text{by} \quad h(aw^n) = a + N \qquad \text{for all } a \in A.$$

Then

$$1 + N = h(w^n) = w^n z + N \qquad \text{for some } z \in A.$$

Now $w^n z \in J \subseteq N$ implies that $1 \in N$, contradicting $N \neq A$. If ${}_AA/N$ is flat, $w = wc$ for some $c \in N$.

Now $1 - c \in r(w) = l(w) \subseteq N$ implies that $1 \in N$, again a contradiction! Therefore $W = A$ and $1 = p + q$, $p \in AwA$, $q \in l(w)$, whence $w = pw$.

Now $1 - p$ is left invertible in A which yields $w = 0$, contradicting our first hypothesis. We have proved that $J = 0$. Since A is left quasi-duo, then A must be a reduced ring (cf. the proof of "(2) implies (3)" in [**27**, Theorem 2.1]). Now A is a left quasi-duo reduced ring whose simple left modules are either YJ-injective or flat which yields A strongly regular by a result of Chen and Ding [**5**, Corollary 7]. Thus (2) implies (1). □

In the above proposition, the expression "$l(u) = r(u)$" is not superfluous as shown by the following example.

Example. If A denotes the 2×2 upper triangular matrix ring over a field, then A is a left and right quasi-duo, Artinian, hereditary ring whose simple one-sided modules are either injective or projective but not semi-prime (indeed, the Jacobson radical J of A is non-zero and $J^2 = 0$).

Singular modules play an important role in the theory of modules and rings. It is well-known that A is a left non-singular ring if and only if A has a VNR maximal left quotient ring Q. In that case, ${}_AQ$ is the injective hull of ${}_AA$ and Q is a left self-injective ring. If A is left non-singular, then for any injective left A-module M, the singular submodule $Z(M)$ is injective [**25**, Theorem 4]. If A is left self-injective regular, then for any essentially finitely generated left A-module M, $Z(M)$ is a direct summand of ${}_AM$ [**39**, Corollary 10]. The right singular ideal will be crucial in the next result. Recall that M is a maximal right annihilator ideal of A if $M = r(S)$ for some non-zero subset S of A such that for any right annihilator R which strictly contains M, $R = A$. In that case, $M = r(s)$ for any $0 \neq s \in S$.

Proposition 7. *Let A be right YJ-injective such that each finitely generated right ideal is either a projective right annihilator or a maximal right annihilator. Then A is either VNR or quasi-Frobenius.*

Proof. First suppose that $Y \neq 0$. For any $0 \neq y \in Y$, since $r(y)$ is an essential right ideal of A, then yA cannot be a projective right annihilator. Therefore yA is a maximal right annihilator.

If $u \notin yA$, then $yA + uA = A$, whence $Y = yA$. We have just shown that Y is a minimal right ideal of A. If $a \in A$, $a \notin Y$, then $aA + yA = A$. This shows that Y must be a maximal right ideal of A. Since Y cannot contain a non-zero idempotent, then Y is an essential right ideal of A. For any non-zero proper right ideal I of A, $I \cap Y \neq 0$ which implies that $I \cap Y = Y$ by the minimality of Y. Therefore $Y \subseteq I$ which yields $Y = I$ by the maximality of Y. We have proved that Y is the unique non-zero proper right ideal of A. A is therefore right Artinian local with $J = Y$.

Let V denote a minimal left ideal of A. If $V = Av$, $v \in A$, either $V^2 = 0$ or V is a direct summand of ${}_AA$. If $v^2 = 0$, since A is right YJ-injective, Av is a left annihilator by [**29**, Lemma 3]. If V is a direct summand of ${}_AA$, then V is again a left annihilator. We have shown that every minimal left ideal of A must be a left annihilator. Since, by hypothesis, every finitely generated right ideal of A is a right annihilator, then A is quasi-Frobenius by [**18**, Proposition 1].

Now suppose that $Y = 0$. If $0 \neq b \in A$ such that bA is a maximal right annihilator, since $Y = 0$, bA cannot be an essential right ideal of A. Therefore $bA \cap cA = 0$ for some $0 \neq c \in A$. Now $bA \oplus cA = A$ (bA being a maximal right annihilator). Then every principal right ideal of A must be projective.

Now for any $0 \neq d \in A$, there exists a positive integer m such that Ad^m is a non-zero left annihilator [**29**, Lemma 3]. Since d^mA is a projective right A-module, then $r(d^m)$ is a direct summand of A_A. Therefore $Ad^m = l(r(Ad^m)) = l(r(d^m))$ is a direct summand of ${}_AA$. We have just proved that every left A-module must be YJ-injective. By [**40**, Theorem 9], A is VNR. □

Proposition 8. *The following conditions are equivalent for a ring A with centre C:*

(1) *A is VNR;*

(2) *A is a semi-prime ring whose essential left ideals are idempotent and for every maximal ideal M of C, A/AM is a VNR ring.*

Proof. (1) implies (2) evidently.

Assume (2). If $d \in C$ such that $d^2 = 0$, then $(Ad)^2 = Ad^2 = 0$ implies that $d = 0$. C is therefore a reduced ring. For any $c \in C$, let K be a complement left ideal of A such that $L = (Ac + l(c)) \oplus K$ is an essential left ideal of A. Then $Kc = cK \subseteq Ac \cap K = 0$ implies that $K \subseteq l(c)$, whence $K \subseteq K \cap (Ac + l(c)) = 0$. Therefore $L = Ac + l(c)$ and by hypothesis, $L = L^2$.

Now $c = \sum_{i=1}^{n} (a_i c + u_i)(b_i c + v_i)$, $a_i, bi \in A$, $ui, vi \in l(c)$, and

$$c - \sum_{i=1}^{n} a_i c b_i c = \sum_{i=1}^{n} (a_i c v_i + u_i b_i c + u_i v_i) = \sum_{i=1}^{n} u_i v_i,$$

since $a_i c v_i = a_i v_i c = 0$, $u_i b_i c = u_i c b_i = 0$. If $w \in Ac \cap l(c)$, $w = dc$, $d \in A$, $dc^2 = wc = 0$ and therefore $cAdc = 0$ which implies that $(Adc)^2 = 0$. Since A is semi-prime, $Adc = 0$ which yields $w = dc = 0$.

Now $c - \sum_{i=1}^{n} a_i c b_i c = \sum_{i=1}^{n} u_i v_i \in Ac \cap l(c) = 0$ which yields $c = \sum_{i=1}^{n} a_i c b_i c = czc$, where $z = \sum_{i=1}^{n} a_i b_i \in A$. Set $y = c^2 z^3$. Then

$$cyc = (czc)zczc = (czc)zc = c \qquad \text{and} \qquad c^2 z = zc^2 = czc = c.$$

For every $b \in A$,

$$zc^2 b = cb = bc = bc^2 z = c^2 bz$$

and hence $z^3 c^2 b = c^2 b z^3$ which shows that

$$yb = c^2 z^3 b = z^3 c^2 b = c^2 b z^3 = bc^2 z^3 = by,$$

whence $y \in C$. Therefore C is VNR. Then (2) implies (1) by [**1**, Theorem 3]. □

Proposition 9. *The following conditions are equivalent for a commutative ring A:*

(1) *A is VNR;*
(2) *For each non-zero principal ideal P of A, there exists a positive integer n such that P^n is generated by a non-zero idempotent;*
(3) *For each non-zero principal ideal P of A, there exists a positive integer n such that P^n is a non-zero flat complement ideal of A.*

Proof. It is clear that (1) implies (2) while (2) implies (3).

Assume (3). First suppose that A is not reduced. Then there exists $0 \neq b \in A$ such that $b^2 = 0$. By hypothesis, Ab is a non-zero flat complement ideal of A. Now $Ab \approx A/l(b)$ and since $b \in l(b)$, then $b = bd$ for some $d \in l(b)$ [**3**, p. 458]. Therefore $bd = db = 0$ implies that $b = 0$, a contradiction! We have shown that A must be reduced.

By [**33**, Proposition 1], every complement ideal of A is an annihilator. By hypothesis, for any $0 \neq a \in A$, there exists a positive integer n such that Aa^n

is a non-zero complement ideal of A and hence Aa^n is an annihilator. By [**29**, Lemma 3], A is YJ-injective. Then (3) implies (1) by [**29**, Lemma 5]. $\square$

Question. Is A VNR if every finitely generated right ideal of A is a flat complement right ideal of A?

Recall that A is a right coherent ring if every finitely generated right ideal of A is finitely presented. For example, VNR rings are coherent.

Proposition 10. *If A is a commutative ring, then every factor ring of A is an IF-ring if and only if every factor ring of A is a coherent p-injective ring.*

Proof. Suppose that every factor ring B of A is a coherent p-injective ring. Then every factor ring B is a self FP-injective ring by [**35**, Proposition 3]. By [**13**, Corollary 2.5], every finitely generated ideal of B is an annihilator. Since B is coherent, then B is an IF-ring by [**9**, Theorem 2.1]. The converse is well-known. $\square$

Proposition 11. *The following conditions are equivalent:*

(1) *Every factor ring of A is QF;*
(2) *A has the following properties: (a) A satisfies the maximum condition on left annihilators; (b) Every finitely generated left ideal of A is principal; (c) A left A-module M is p-injective if and only if M is flat.*

Proof. Assume (1). Then A is a principal left ideal ring which is QF [**7**, Proposition 25.4.6B]. Now every p-injective left A-module M is injective, which implies that M is flat [**7**, Theorem 24.12]. If ${}_AN$ is flat, since A is left perfect, then ${}_AN$ is projective [**21**, p. 392] which implies that ${}_AN$ is injective [**7**, Theorem 24.20]. Therefore ${}_AN$ is p-injective and (1) implies (2).

Assume (2). Then A is a left p-injective ring. Since A is a left IF-ring by (c), then A is right p-injective. $\square$

Since A is left p-injective with maximum condition on left annihilators, then A is right Artinian [**22**, p. 34]. Then (2) implies (1) by [**18**, Proposition 2] and [**7**, Proposition 25.4.6B].

(Condition (a) is not superfluous since any VNR ring satisfies Conditions 2 (b), (c).)

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