Acta Math. Univ. Comenianae
Vol. LXXVI, 2(2007), pp. 273–278

ON THREE INEQUALITIES SIMILAR TO HARDY-HILBERT'S INTEGRAL INEQUALITY

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Abstract. The following three inequalities, which are similar to Hardy-Hilbert integral inequality are proved.

$$\int_0^\infty\int_0^\infty \frac{x^{\beta/q}y^{\alpha/p}F(x)G(y)}{\max\{x^\lambda, y^\lambda\}}\mathrm{d}x\mathrm{d}y \le \frac{\lambda}{(\alpha+1)^{1/p}(\beta+1)^{1/q}}\frac{p^{1-1/p}q^{1-1/q}}{(p-1)(q-1)} \cdot \left(\int_0^\infty f^p(t)\mathrm{d}t\right)^{1/p}\left(\int_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

where

$$F(x)=\int_0^x f(t)\mathrm{d}t, \qquad G(y)=\int_0^y g(t)\mathrm{d}t.$$

$$\int_0^\infty\int_0^\infty \frac{x^{1/p}y^{1/q}F(x)G(y)}{(x+y)^4}\mathrm{d}x\mathrm{d}y < B^{1/p}(p,p)B^{1/q}(q,q)\frac{pq}{(p-1)(q-1)} \cdot \left(\int_0^\infty f^p(t)\mathrm{d}t\right)^{1/p}\left(\int_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{x^{1/p}y^{1/q}z^{1/r}\sqrt{xyzF(x)G(y)H(z)}}{(x+y+z)^8}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$< K_pK_qK_r\left(\int_0^\infty f^{p/2}(t)\mathrm{d}t\right)^{1/p}\left(\int_0^\infty g^{q/2}(t)\mathrm{d}t\right)^{1/q}\left(\int_0^\infty h^{r/2}(t)\mathrm{d}t\right)^{1/r}.$$

where

$$H(z)=\int_0^z h(t)\mathrm{d}t, \qquad K_p=\sqrt{\frac{p/2}{p/2-1}}B^{1/p}\left(\frac{p}{2},\frac{p}{2}\right)B^{1/p}(p,p).$$

Received June 13, 2006.
2000 *Mathematics Subject Classification.* Primary 26D15.

1. Introduction

Let $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^2(t)\mathrm{d}t < \infty \qquad \text{and} \qquad 0 < \int_0^\infty g^2(t)\mathrm{d}t < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\,\mathrm{d}y < \pi \left(\int_0^\infty f^2(t)\mathrm{d}t \int_0^\infty g^2(t)\mathrm{d}t \right)^{1/2}, \tag{1}$$

where the constant factor π is the best possible (cf. Hardy et al. [**3**]). Inequality (1) is well known as Hilbert's integral inequality. This inequality had been extended by Hardy [**1**] as follows

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$ satisfy

$$0 < \int_0^\infty f^p(t)\mathrm{d}t < \infty \qquad \text{and} \qquad 0 < \int_0^\infty g^q(t)\mathrm{d}t < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\,\mathrm{d}y < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(t)\mathrm{d}t \right)^{1/p} \left(\int_0^\infty g^q(t)\mathrm{d}t \right)^{1/q}, \tag{2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (2) is called Hardy-Hilbert's integral inequality and is important in analysis and application (cf. Mitrinovic et al. [**4**]).

B. Yang gave the following extension of (2) as follows:

Theorem ([**5**])**.** *If $\lambda > 2 - \min\{p, q\}$, $f, g \geq 0$, satisfy*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)\mathrm{d}t < \infty \qquad \textit{and} \qquad 0 < \int_0^\infty t^{1-\lambda} g^q(t)\mathrm{d}t < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\,\mathrm{d}y < k_\lambda(p) \left(\int_0^\infty t^{1-\lambda} f^p(t)\mathrm{d}t \right)^{1/p} \left(\int_0^\infty t^{1-\lambda} g^q(t)\mathrm{d}t \right)^{1/q}, \tag{3}$$

where the constant factor $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible, B is the beta function.

The object of this paper is that to give some new inequalities similar to that of Hardy-Hilbert's inequality.

We need the following result for our aim

Theorem A ([**2**]). *Let f be a nonnegative integrable function. Define*

$$F(x) = \int_a^x f(t)\mathrm{d}t.$$

Then

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\mathrm{d}x, \qquad p > 1.$$

2. New results

We state and prove the following:

Theorem 1. *Let $f, g \geq 0$, $F(x) = \int_0^x f(t)\mathrm{d}t$, $G(y) = \int_0^y g(t)\mathrm{d}t$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $p = \lambda - \alpha - 1 > 1$, $q = \lambda - \beta - 1 > 1$, $\alpha, \beta > -1$ then*

$$\int_0^\infty \int_0^\infty \frac{x^{\beta/q} y^{\alpha/p} F(x)G(y)}{\max\{x^\lambda, y^\lambda\}} \mathrm{d}x\,\mathrm{d}y \leq \frac{\lambda}{(\alpha+1)^{1/p}(\beta+1)^{1/q}} \frac{p^{1-1/p} q^{1-1/q}}{(p-1)(q-1)} \cdot \left(\int_0^\infty f^p(t)\mathrm{d}t\right)^{1/p} \left(\int_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

Proof. We have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{x^{\beta/q} y^{\alpha/p} F(x)G(y)}{\max\{x^\lambda, y^\lambda\}} \mathrm{d}x\,\mathrm{d}y \\
&\qquad \int_0^\infty \int_0^\infty \frac{y^{\alpha/q} F(x)}{\left(\max\{x^\lambda, y^\lambda\}\right)^{1/p}} \cdot \int_0^\infty \int_0^\infty \frac{x^{\beta/q} G(y)}{\left(\max\{x^\lambda, y^\lambda\}\right)^{1/q}} \mathrm{d}x\,\mathrm{d}y \\
&\qquad \leq \left(\int_0^\infty \int_0^\infty \frac{y^\alpha F^p(x)}{\max\{x^\lambda, y^\lambda\}} \mathrm{d}x\,\mathrm{d}y\right)^{1/p} \left(\int_0^\infty \int_0^\infty \frac{x^\beta G^q(y)}{\max\{x^\lambda, y^\lambda\}} \mathrm{d}x\,\mathrm{d}y\right)^{1/q} \\
&\qquad = M^{1/p} N^{1/q}.
\end{aligned}$$

Observe that

$$\begin{aligned}
M &= \int_0^\infty F^p(x)\mathrm{d}x \int_0^\infty \frac{y^\alpha}{\max\{x^\lambda, y^\lambda\}} \mathrm{d}x\,\mathrm{d}y = \int_0^\infty F^p(x)\mathrm{d}x \left(\int_0^x \frac{y^\alpha}{x^\lambda}\mathrm{d}y + \int_x^\infty \frac{y^\alpha}{y^\lambda}\mathrm{d}y\right) \\
&= \frac{\lambda}{(\alpha+1)(\lambda-\alpha-1)} \int_0^\infty \frac{F^p(x)}{x^{\lambda-\alpha-1}} \mathrm{d}x = \frac{\lambda}{(\alpha+1)p} \int_0^\infty \left(\frac{F(x)}{x}\right)^p \mathrm{d}x
\end{aligned}$$

$$< \frac{\lambda}{(\alpha+1)}\frac{p^{p-1}}{(p-1)^p}\int\limits_0^\infty f^p(x)\mathrm{d}x,$$

in view of Theorem A.

Similarly, It can be shown that

$$N < \frac{\lambda}{(\beta+1)}\frac{q^{q-1}}{(q-1)^q}\int\limits_0^\infty g^q(x)\mathrm{d}x.$$

Combining these inequalities to obtain

$$\int\limits_0^\infty\int\limits_0^\infty \frac{x^{\beta/q}y^{\alpha/p}F(x)G(y)}{\max\{x^\lambda, y^\lambda\}}\mathrm{d}x\,\mathrm{d}y \le \frac{\lambda}{(\alpha+1)^{1/p}(\beta+1)^{1/q}}\frac{p^{1-1/p}q^{1-1/q}}{(p-1)(q-1)} \cdot \left(\int\limits_0^\infty f^p(t)\mathrm{d}t\right)^{1/p}\left(\int\limits_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

□

Theorem 2. *Let* $f, g \ge 0$, $F(x) = \int_0^x f(t)\mathrm{d}t$, $G(y) = \int_0^y g(t)\mathrm{d}t$, $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, *then, we have*

$$\int\limits_0^\infty\int\limits_0^\infty \frac{x^{1/p}y^{1/q}F(x)G(y)}{(x+y)^4}\mathrm{d}x\mathrm{d}y < B^{1/p}(p,p)B^{1/q}(q,q)\frac{pq}{(p-1)(q-1)} \cdot \left(\int\limits_0^\infty f^p(t)\mathrm{d}t\right)^{1/p}\left(\int\limits_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

Proof.

$$\begin{aligned}
&\int\limits_0^\infty\int\limits_0^\infty \frac{x^{1/p}y^{1/q}F(x)G(y)}{(x+y)^4}\mathrm{d}x\mathrm{d}y \\
&\quad= \int\limits_0^\infty\int\limits_0^\infty \frac{y^{1/q}F(x)}{(x+y)^2}\cdot\frac{x^{1/p}G(y)}{(x+y)^2}\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \left(\int\limits_0^\infty\int\limits_0^\infty \frac{y^{p/q}F^p(x)}{(x+y)^{2p}}\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\left(\int\limits_0^\infty\int\limits_0^\infty \frac{x^{q/p}G^q(y)}{(x+y)^{2q}}\mathrm{d}x\,\mathrm{d}y\right)^{1/q} \\
&\quad= P^{1/p}Q^{1/q}.
\end{aligned}$$

Now, we consider

$$P=\int_0^\infty\left(\frac{F(x)}{x}\right)^p\mathrm{d}x\int_0^\infty\frac{\left(\frac{y}{x}\right)^{p-1}\frac{1}{x}}{\left(1+\frac{y}{x}\right)^{2p}}\mathrm{d}y=\int_0^\infty\left(\frac{F(x)}{x}\right)^p\mathrm{d}x\int_0^\infty\frac{u^{p-1}}{(1+u)^{2p}}\mathrm{d}u$$
$$<B(p,p)\left(\frac{p}{p-1}\right)^p\int_0^\infty f^p(x)\mathrm{d}x,$$

by Theorem A.

Similarly, $\quad Q<B(q,q)\left(\frac{q}{q-1}\right)^q\int\limits_0^\infty g^q(y)\mathrm{d}y.$

Therefore, we have

$$\int_0^\infty\int_0^\infty\frac{x^{1/p}y^{1/q}F(x)G(y)}{(x+y)^4}\mathrm{d}x\mathrm{d}y<B^{1/p}(p,p)B^{1/q}(q,q)\frac{pq}{(p-1)(q-1)}$$
$$\cdot\left(\int_0^\infty f^p(t)\mathrm{d}t\right)^{1/p}\left(\int_0^\infty g^q(t)\mathrm{d}t\right)^{1/q}.$$

This completes the proof of the theorem. □

Theorem 3. *Let* $f,g,h\geq 0,\ \ p,q,r>2\ \ \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1,$

$$F(x)=\int_0^x f(t)\mathrm{d}t,\qquad G(y)=\int_1^y g(t)\mathrm{d}t,\qquad H(z)=\int_0^z h(t)\mathrm{d}t.$$

Then

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{x^{1/p}y^{1/q}z^{1/r}\sqrt{xyzF(x)G(y)H(z)}}{(x+y+z)^8}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$<\ K_pK_qK_r\left(\int_0^\infty f^{p/2}(x)\mathrm{d}x\right)^{1/p}\left(\int_0^\infty g^{q/2}(y)\mathrm{d}y\right)^{1/q}\left(\int_0^\infty h^{r/2}(z)\mathrm{d}z\right)^{1/r}.$$

where

$$K_p=\sqrt{\frac{p/2}{p/2-1}B^{1/p}\left(\frac{p}{2},\frac{p}{2}\right)B^{1/p}(p,p)}.$$

Proof.

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{x^{1/p}y^{1/q}z^{1/r}\sqrt{xyzF(x)G(y)H(z)}}{(x+y+z)^8}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty\left(\frac{y^{(1/2-1/p)}z^{(1/q+1/r)}\sqrt{F(x)}}{(x+y+z)^2}\cdot\frac{z^{(1/2-1/q)}x^{(1/p+1/r)}\sqrt{G(y)}}{(x+y+z)^2}\right.$$
$$\left.\cdot\frac{x^{(1/2-1/r)}y^{(1/p+1/q)}\sqrt{H(z)}}{(x+y+z)^2}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$\leq \left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{y^{(p/2-1)}z^{p-1}F^{p/2}(x)}{(x+y+z)^{2p}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{1/p}$$
$$\cdot\left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{z^{(q/2-1)}x^{q-1}G^{q/2}(y)}{(x+y+z)^{2q}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{1/q}$$
$$\cdot\left(\int_0^\infty\int_0^\infty\int_0^\infty \frac{x^{(r/2-1)}y^{r-1}H^{r/2}(z)}{(x+y+z)^{2r}}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)^{1/r}$$
$$= A^{1/p}B^{1/q}C^{1/r}.$$

Observe that

$$A = \int_0^\infty \frac{F^{p/2}(x)}{x^{p/2}}\mathrm{d}x \int_0^\infty \frac{\left(\frac{y}{x}\right)^{p/2-1}\frac{1}{x}}{\left(1+\frac{y}{x}\right)^p}\mathrm{d}y \int_0^\infty \frac{\left(\frac{z}{x+y}\right)^{p-1}\frac{1}{x+y}}{\left(1+\frac{z}{x+y}\right)^{2p}}\mathrm{d}z$$
$$= \int_0^\infty \left(\frac{F(x)}{x}\right)^{p/2}\mathrm{d}x \int_0^\infty \frac{u^{p/2-1}}{(1+u)^p}\mathrm{d}u \int_0^\infty \frac{v^{p-1}}{(1+v)^{2p}}\mathrm{d}v$$
$$= B\left(\frac{p}{2},\frac{p}{2}\right)B(p,p)\int_0^\infty \left(\frac{F(x)}{x}\right)^{p/2}\mathrm{d}x < K_p^p\int_0^\infty f^{p/2}(x)\mathrm{d}x,$$

in view of Theorem A.

Similarly,

$$B < K_q^q\int_0^\infty g^{p/2}(y)\mathrm{d}y, \qquad C < K_r^r\int_0^\infty h^{r/2}(z)\mathrm{d}z.$$

The proof is complete. □

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