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ON THE CLASS OF ALL RECIPROCAL BASES FOR INTEGERS

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Abstract. In this paper the structure of the class of all reciprocal bases of $\mathbb{N}$ is investigated from metric and topological point of view. For this purpose the method of dyadic values of infinite subsets of $\mathbb{N}$ will be applied.

Introduction

A set $A \subseteq \mathbb{N}$ is called the reciprocal basis for integers (of $\mathbb{N}$), shortly R-basis, provided that for each $s \in \mathbb{N}$ there exist $a_1 < a_2 < \cdots < a_k$ from A such that $s = \sum_{j=1}^{k} \frac{1}{a_j}$ (cf. [**1**], [**2**], [**8**]). It is well-known that the set of all positive integers is an R-basis. It is proved in [**1**] that every arithmetic progression is an R-basis. In the same paper a construction of R-bases of zero density based on the fact that for every integer $a \in \mathbb{N}$ the sequence $S_a = \{a, 2a, 3a, \dots\}$ is an R-basis is given. Obviously this concept is closely related to the concept of egyptian fractions.

Note, that if $A \subseteq \mathbb{N}$ is a reciprocal basis of $\mathbb{N}$ then $\sum_{a\in A} a^{-1} = \infty$ and consequently A is infinite.

Denote by $\mathcal{B}_r$ the class of all reciprocal bases of $\mathbb{N}$ and by $\mathcal{U}$ the class of all infinite subsets of $\mathbb{N}$. Then

$$\mathcal{B}_r \subseteq \mathcal{U}. \tag{1}$$

We will use the concept of dyadic values of sets $A \in \mathcal{U}$ for the study of "the magnitude" of $\mathcal{B}_r$ in $\mathcal{U}$.

If $A = \{a_1 < a_2 < \cdots < a_k < \cdots\} \in \mathcal{U}$, then we put $\rho(A) = \sum_{k=1}^{\infty} 2^{-a_k} = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, where $(\varepsilon_k)_1^\infty$ is the characteristic function of the set A (i.e. $\varepsilon_k = 1$ if $k \in A$ and $\varepsilon_k = 0$ if $k \in \mathbb{N} \setminus A$). In this way we get an injective mapping ρ of $\mathcal{U}$ onto the interval $(0,1]$. If $\mathcal{S} \subseteq \mathcal{U}$, then we set $\rho(\mathcal{S}) = \{\rho(A) : A \in \mathcal{S}\}$. The magnitude of the set $\rho(\mathcal{S}) \subseteq (0,1]$ enables us to judge the magnitude of the class $\mathcal{S}$ (cf. [**4**, p. 17]).

The magnitude of $\rho(\mathcal{S})$ can be investigated from the metric point of view (Lebesgue measure, Hausdorff dimension) and also from the topological point of view (Baire's categories).

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We recall the following fact about dyadic expansions of real numbers. Each $x \in (0,1]$ can be uniquely expressed in the form $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$, where $\varepsilon_k(x) = 0$ or 1 and for an infinitely many k's we have $\varepsilon_k(x) = 1$.

If $m \in \mathbb{N}$ is fixed then the whole interval $(0,1]$ can be written in the form $(0,1] = \cup_{j=0}^{2^m-1} (\frac{j}{2^m}, \frac{j+1}{2^m}] = \cup_{j=0}^{2^m-1} i_m^{(j)}$. To every interval $i_m^{(j)}$ $(0 \le j \le 2^m - 1)$ corresponds a sequence $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0$ of numbers $0, 1$ in such a manner that if $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k} \in i_m^{(j)}$, then $\varepsilon_k(x) = \varepsilon_k^0$ $(k = 1, 2, \dots, m)$. We say shortly that the interval $i_m^{(j)}$ and the sequence $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0$ are associated.

1. Topological properties of the set $\rho(\mathcal{B}_r)$

We will show that from the topological point of view the class $\mathcal{B}_r$ is a very large subclass of $\mathcal{U}$ (see (1)).

Let $s \in \mathbb{N}$. Denote by $H(m,s)$ the union of all intervals $i_m^{(j)}$ (m fixed) with the following property: The interval $i_m^{(j)}$ is associated with a sequence $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0$ such that for a suitable set M, $M \subseteq \{k \le m : \varepsilon_k^0 = 1\}$ we have $s = \sum_{k \in M} \frac{1}{k}$.

It is a well-known fact that (every) integer s can be represented as a sum of reciprocal values of some distinct integers (cf. [**1**]). So for s there exists an m such that $H(m,s) \ne \emptyset$. Put $H(s) = \cup_{m=1}^{\infty} H(m,s)$. Hence $H(s) \ne \emptyset$.

The following auxiliary result will be used in what follows.

Lemma 1.1. *We have*

$$\rho(\mathcal{B}_r) = \bigcap_{s=1}^{\infty} H(s) = \bigcap_{s=1}^{\infty} \bigcup_{m=1}^{\infty} H(m,s). \tag{2}$$

Proof. 1) Let $x \in \rho(\mathcal{B}_r)$. We show that x belongs to the right-hand side of (2).

Since $x \in \rho(\mathcal{B}_r)$, we have $x = \rho(A)$, where $A = \{a_1 < a_2 < \dots < a_k < \dots\} \subseteq \mathbb{N}$, A being an R-basis. Hence there are numbers $a_{j_1} < a_{j_2} < \dots < a_{j_t}$ from A such that $s = \frac{1}{a_{j_1}} + \dots + \frac{1}{a_{j_t}}$. Put $m = a_{j_t} \in \mathbb{N}$. The sequences $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0$ of 0's and 1's satisfying the conditions $\varepsilon_{a_{j_i}}^0 = 1$ $(i = 1, 2, \dots, t)$ are associated with some intervals $i_m^{(l)}$ (m fixed) and these intervals are subsets of the set $H(m,s)$. Hence x belongs to $H(s)$. This is true for an arbitrary $s \in \mathbb{N}$, therefore x belongs to the right-hand side of (2).

2) Let $x = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j}$ belong to the right-hand side of (2). Put $A = \{j : \varepsilon_j = 1\}$. Then $x = \rho(A)$. We will show that x belongs to $\rho(\mathcal{B}_r)$. For this it suffices to show that $A \in \mathcal{B}_r$.

Let $v \in \mathbb{N}$. We show that v can be expressed as a sum of reciprocal values of a finite number of distinct elements of A.

Since $x \in H(v)$, there is an $m \in \mathbb{N}$ such that $x \in H(m,v)$. By the definition of the set $H(m,v)$ there exists an interval $i_m^{(l)}$ $(l \in \{0, 1, \dots, 2^m - 1\})$ such that $x \in i_m^{(l)}$ and $i_m^{(l)}$ is associated with a sequence $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0$ of 0's and 1's such that for a set $M \subseteq \{k \le m : \varepsilon_k^0 = 1\}$ we have

$$v = \sum_{k \in M} \frac{1}{k}. \tag{3}$$

For the dyadic expansion $x = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j}$ we have $\varepsilon_j = \varepsilon_j^0$ $(j = 1, 2, \ldots, m)$ and so the set M consists of some k's, $k \le m$ such that $\varepsilon_k = 1$. Hence these k's belong to the set A and the number v can be expressed by (3) as a sum of reciprocal values of some distinct elements of A. Since v is an arbitrary positive integer, we see that $A \in \mathcal{B}_r$. □

Let $\mathcal{S} \subseteq \mathcal{U}$. Denote by $c\mathcal{S}$ the class $\mathcal{U} \setminus \mathcal{S}$ (complement of $\mathcal{S}$ in $\mathcal{U}$). Hence $c\mathcal{B}_r = \mathcal{U} \setminus \mathcal{B}_r$. The class $c\mathcal{B}_r$ is the class of all infinite sets $A \subseteq \mathbb{N}$ that are not R-bases. Hence for each $A \in c\mathcal{B}_r$ there exists at least one $s \in \mathbb{N}$ such that s cannot be expressed as a finite sum of reciprocal values of distinct elements of A.

In what follows the interval $(0, 1]$ will be considered as a metric space with the Euclidean metric.

Theorem 1.1. *The set $\rho(\mathcal{B}_r)$ is an $F_{\sigma\delta}$-set in $(0, 1]$.*

Proof. We use Lemma 1.1. Recall that the set $H(m, s)$ is a union of a finite number of intervals $i_m^{(l)}$ (m fixed). Therefore $H(m, s)$ is an F_σ-set in $(0, 1]$. Then the right-hand side of (2) is an $F_{\sigma\delta}$-set in $(0, 1]$. The same holds for $\rho(\mathcal{B}_r)$. □

Remark. By the definition of $c\mathcal{B}_r$ and injectivity of the mapping $\rho : \mathcal{U} \to (0, 1]$ we get

$$\rho(c\mathcal{B}_r) = (0, 1] \setminus \rho(\mathcal{B}_r). \tag{4}$$

From this and from Theorem 1.1 follows that the set $\rho(c\mathcal{B}_r)$ is a $G_{\delta\sigma}$-set in $(0, 1]$.

We have shown that the both sets $\rho(\mathcal{B}_r)$, $\rho(c\mathcal{B}_r)$ belong to the second Borel class. We will determine their Baire's categories.

Theorem 1.2. *The set $\rho(\mathcal{B}_r)$ is a residual set in $(0, 1]$.*

Proof. It suffices to prove that the set $\rho(c\mathcal{B}_r)$ is a dense set of the first Baire category in $(0, 1]$. The density of the set $\rho(c\mathcal{B}_r)$ follows from the fact that the set $\rho(\mathcal{K})$, $\mathcal{K}$ being the class of all $A \subseteq \mathbb{N}$ with $\sum_{a \in A} a^{-1} < \infty$, is dense in $(0, 1]$ (cf. [**6**, Theorem 3]). We have $\mathcal{K} \subseteq c\mathcal{B}_r$ so that $\rho(\mathcal{K}) \subseteq \rho(c\mathcal{B}_r)$ and the density of $\rho(c\mathcal{B}_r)$ follows.

We prove that the set $\rho(c\mathcal{B}_r)$ is a set of the first category in $(0, 1]$.

By (4) and (2) we get

$$\rho(c\mathcal{B}_r) = (0, 1] \setminus \bigcap_{s=1}^{\infty} \bigcup_{m=1}^{\infty} H(m, s) = \bigcup_{s=1}^{\infty} \bigcap_{m=1}^{\infty} cH(m, s). \tag{5}$$

(Where $cH(m, s) = (0, 1] \setminus H(m, s)$.)

In virtue of (5) it suffices to prove that each of the sets $\cap_{m=1}^{\infty} cH(m, s)$ $(s = 1, 2, \ldots)$ is nowhere dense in $(0, 1]$.

Fix $s \in \mathbb{N}$. On account of the well-known criterion of nowheredensity of a set in metric space (cf. [**3**, p. 37]) it suffices to show that the following statement holds:

Every non-empty interval $I \subset (0,1]$ contains an interval $J \subseteq I$ such that $J \cap cH(m^{'}, s) = \emptyset$ for an $m^{'} \in \mathbb{N}$.

Let $I \subset (0,1]$ be an interval. Choose the numbers m, d, $m \in \mathbb{N}$, $0 \le d \le 2^m - 1$ in such a way that $i_m^{(d)} \subset I$. We show that there is a subinterval $i_{m+v}^{(t)}$ of $i_m^{(d)}$ such that

$$i_{m+v}^{(t)} \cap cH(m+v, s) = \emptyset. \tag{6}$$

holds.

If $i_m^{(d)} \subseteq H(m,s)$ then we put $v = 0$ and $t = d$.

Let $i_m^{(d)} \subseteq H(m,s)$ does not hold. Since the set $\{m+1, m+2, \dots, m+k, \dots\}$ is an R-basis (cf. [**2**]) there exist n_k $(k = 1, 2, \dots, j)$, $m+1 \le n_1 < n_2 < \dots < n_j$, such that

$$s = \sum_{k=1}^{j} \frac{1}{n_k}. \tag{7}$$

Put $n_j = m + v$ (i.e. $v = n_j - m$) and $\varepsilon_k^0 = 1$ for $k = m+1, m+2, \dots, m+v$. Let the interval $i_m^{(d)}$ be associated with the sequence $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0$ of 0's and 1's. Construct the interval $i_{m+v}^{(t)}$ which is associated with the sequence $\varepsilon_1^0, \varepsilon_2^0, \dots, \varepsilon_m^0, \varepsilon_{m+1}^0, \dots, \varepsilon_{m+v}^0$ of numbers 0, 1. Then $i_{m+v}^{(t)} \subseteq i_m^{(d)}$ and from (7) we get (6). □

2. Metric properties of the set $\rho(\mathcal{B}_r)$

We have seen that the set $\rho(\mathcal{B}_r)$ belongs to the second Borel class (Theorem 1.1). So it is Lebesgue measurable and it would be desirable to determine $\lambda(\rho(\mathcal{B}_r))$ – the Lebesgue measure of $\rho(\mathcal{B}_r)$. Unfortunately we are not able to do this and therefore it remains as an open problem. It is interesting that we can determine the Hausdorff dimension (cf. [**5**]) of the set $\rho(\mathcal{B}_r)$. But unfortunately from this result we cannot derive the magnitude of $\lambda(\rho(\mathcal{B}_r))$.

Theorem 2.1. *We have* $\dim \rho(\mathcal{B}_r)) = 1$.

Proof. It is well-known that there exists a set $A \in \mathcal{B}_r$ with $d(A) = 0$ (cf. [**2**]), where $d(A)$ denotes the asymptotic density of A, i.e. $d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$, $A(n) = |A \cap \{1, 2, \dots, n\}|$.

Obviously every set $D \supseteq A$, $D \subseteq \mathbb{N}$ belongs again to $\mathcal{B}_r$. Denote by $\mathcal{S}(A)$ the set $\{D \subseteq \mathbb{N} : A \subseteq D\}$. Then we have $\mathcal{S}(A) \subseteq \mathcal{B}_r$ and so

$$\dim \rho(\mathcal{S}(A)) \le \dim \rho(\mathcal{B}_r). \tag{8}$$

In virtue of (8) it suffices to show that

$$\dim \rho(\mathcal{S}(A)) = 1. \tag{9}$$

We will prove it using the following result which is an easy consequence of [**7**, Theorem 2.7]:

(S) Let M be a set of positive integers and (ε_j^0), $j \in M$ be a fixed sequence of 0's and 1's. Denote by $Z = Z(M, (\varepsilon_j^0), j \in M)$ the set of all $x = \sum_{j=1}^{\infty} \varepsilon_j(x) 2^{-j} \in (0,1]$ for which $\varepsilon_j(x) = \varepsilon_j^0$ if $j \in M$ and $\varepsilon_j(x) = 0$ or 1 if $j \in \mathbb{N} \setminus M$. Then

$$\begin{aligned} \dim(Z) &= \liminf_{n\to\infty} \frac{\log \prod\limits_{j \le n,\, j \in \mathbb{N}\setminus M} 2}{n \log 2} \\ &= \liminf_{n\to\infty} \frac{\mathbb{N} \setminus M\ (n)}{n} = 1 - \overline{d}(M), \end{aligned} \tag{10}$$

where $\overline{d}(M) = \limsup\limits_{n\to\infty} \frac{M(n)}{n}$.

Put in (S) (see (10)): $M = A$, $\varepsilon_j^0 = 1$ for $j \in A$. Then $Z(M, (\varepsilon_j^0), j \in M) = \rho(\mathcal{S}(A))$ and from (10) we obtain $\rho(\mathcal{S}(A)) = 1 - \overline{d}(A) = 1 - d(A) = 1$. Hence (9) holds. $\square$

Remark. There are infinite sets $A \subseteq \mathbb{N}$ with $\sum_{a\in A} a^{-1} = \infty$ and zero asymptotic density that do not belong to $\mathcal{B}_r$. If $A = \{a_1 < a_2 < \cdots < a_k < \cdots\} \subseteq \mathbb{N}$, $\gcd(a_i, a_j) = 1$ for $i \ne j$, then the set A does not belong to $\mathcal{B}_r$, because it is easy to show that the number 1 cannot be expressed in the form $1 = \frac{1}{a_{i_1}} + \frac{1}{a_{i_2}} + \cdots + \frac{1}{a_{i_m}}$, $i_1 < i_2 < \cdots < i_m$. Taking the set of all prime numbers for A we get a set of zero density with $\sum_{a\in A} a^{-1} = \infty$ which does not belong to $\mathcal{B}_r$.

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