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ON SYMMETRIC GROUP S_3 ACTIONS ON SPIN 4-MANIFOLDS

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ABSTRACT. Let X be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$ and non-positive signature $\sigma(X)$. In this paper we use Seiberg-Witten theory to prove that if X admits an odd type symmetric group S_3 action preserving the spin structure, then $b_2^+(X) \geq |\sigma(X)|/8 + 3$ under some non-degeneracy conditions. We also obtain some information about $\mathrm{Ind}_{\tilde{S}_3} D$, where $\tilde{S}_3$ is the extension of S_3 by Z_2.

1. INTRODUCTION

Let X be a smooth, closed, connected spin 4-manifold. We denote by $b_2(X)$ the second Betti number and denote by $\sigma(X)$ the signature of X. In [**11**], Y. Matsumoto conjectured the following inequality

$$b_2(X) \geq \frac{11}{8}|\sigma(X)|. \tag{1}$$

This conjecture is well known and has been called the $\frac{11}{8}$-conjecture. All complex surfaces and their connected sums satisfy the conjecture (see [**13**]).

From the classification of unimodular even integral quadratic forms and the Rochlin's theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold X is

$$-2kE_8 \oplus mH, \qquad k \geq 0,$$

where E_8 is the 8×8 intersection form matrix and H is the hyperbolic matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Thus, $m = b_2^+(X)$ and $k = -\sigma(X)/16$ and so the inequality (1) is equivalent to $m \geq 3k$. Since $K3$ surface satisfies the equality with $k = 1$ and $m = 3$, the coefficient $\frac{11}{8}$ is optimal, if the $\frac{11}{8}$-conjecture is true.

Donaldson has proved that if $k > 0$ then $m \geq 3$ [**4**]. In early 1995, using the Seiberg-Witten theory introduced by Seiberg and Witten [**15**], Furuta [**7**] proved

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that

$$b_2(X) \geq \frac{5}{4}|\sigma(X)| + 2. \tag{2}$$

This estimate has been dubbed the $\frac{10}{8}$-theorem. In fact, if the intersection form of X is definite, i.e., $m = 0$, then Donaldson proved that $b_2(X)$ and $\sigma(X)$ are zero [**4, 5**]. Thus, Furuta assumed that m is not zero. Inequality (2) follows by a surgery argument from the non-positive signature, $b_1(X) = 0$ case:

Theorem 1.1 (Furuta [**7**])**.** *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ with non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Then,*

$$2k + 1 \leq m$$

if $m \neq 0$.

His key idea is to use a finite dimensional approximation of the monopole equation. Later Furuta and Kametani [**7**] used equivariant e-invariants and improved the above $\frac{10}{8}$-theorem as following.

Theorem 1.2 (Furuta and Kametani [**7**])**.** *Suppose that X is a closed oriented spin 4-manifold. If $\sigma(X) < 0$,*

$$b_2^+(X) \geq \begin{cases} 2(-\sigma(X)/16) + 1, & -\sigma(X)/16 \equiv 0, 1 \mod 4, \\ 2(-\sigma(X)/16) + 2, & -\sigma(X)/16 \equiv 2 \mod 4, \\ 2(-\sigma(X)/16) + 1, & -\sigma(X)/16 \equiv 3 \mod 4. \end{cases}$$

The above inequality was also proved by N. Minami [**12**] by using an equivariant join theorem to reduce the inequality to a theorem of Stolz [**14**].

Throughout this paper we will assume that m is not zero and $b_1(X) = 0$, unless stated otherwise.

A $Z/2^p$-action is called a spin action if the generator of the action $\tau : X \to X$ lifts to an action $\hat{\tau} : P_{\mathrm{Spin}} \to P_{\mathrm{Spin}}$ of the Spin bundle P_{Spin}. Such an action is of even type if $\hat{\tau}$ has order 2^p and is of odd type if $\hat{\tau}$ has order 2^{p+1}.

In [**2**], Bryan (see also [**6**]) used Furuta's technique of "finite dimensional approximation" and the equivariant K-theory to improve the above bound by p under the assumption that X has a spin odd type $Z/2^p$-action satisfying some non-degeneracy conditions analogous to the condition $m \neq 0$. More precisely, he proved

Theorem 1.3 (Bryan [**2**])**.** *Let X be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$. Assume that $\tau : X \to X$ generates a spin smooth $Z/2^p$-action of odd type. Let X_i denote the quotient of X by $Z/2^i \subset Z/2^p$. Then*

$$2k + 1 + p \leq m$$

if $m \neq 2k + b_2^+(X_1)$ and $b_2^+(X_i) \neq b_2^+(X_j) > 0$ for $i \neq j$.

In the paper [**9**], Kim gave the same bound for smooth, spin, even type $Z/2^p$-action on X satisfying some non-degeneracy conditions analogous to Bryan's.

In the paper [**10**], Liu gave the bound for even type spin S_3 action on 4-manifolds, that is

Theorem 1.4. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Then,*

$$2k + 2 \leq m$$

if $b_2^+(X/ < x_1 >) > 0$, $b_2^+(X/ < x_2 >) > 0$ and $b_2^+(X) \neq b_2^+(X/ < x_1 >)$.

The purpose of this paper is to study the spin symmetric group S_3 actions of odd type on spin 4-manifolds, we prove that $b_2^+(X) \geq |\sigma(X)|/8 + 3$ under some non-degeneracy conditions. We also obtain some results about $\mathrm{Ind}_{\tilde{S_3}} D$, where $\tilde{S_3}$ is the extension of S_3 by Z_2.

We organize the remainder of this paper as follows. In Section 2, we give some preliminaries to prove the main theorem. In Section 3, we use equivariant K-theory and representation theory to study the G-equivariant properties of the moduli space. In the last section we give our main results.

2. Notations and preliminaries

We assume that we have completed every Banach spaces with suitable Sobolev norms. Let $S = S^+ \oplus S^-$ denote the decomposition of spinor bundles into positive and negative spinor bundles. Let $D : \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator, and $\rho : \Lambda_C^* \to \mathrm{End}_C(S)$ be the Clifford multiplication. The Seiberg-Witten equations are for a pair $(a, \phi) \in \Omega^1(X, \sqrt{-1}R) \times \Gamma(S^+)$ and they are

$$D\phi + \rho(a)\phi = 0, \qquad \rho(d^+a) - \phi \otimes \phi^* + \frac{1}{2}|\phi|^2\,\mathrm{id} = 0, \qquad d^*a = 0.$$

Let

$$V = \Gamma(\sqrt{-1}\Lambda^1 \oplus S^+),$$
$$W' = (S^- \oplus \sqrt{-1}\,\mathrm{su}\,(S^+) \oplus \sqrt{-1}\Lambda^0).$$

We can think of the equation as the zero set of a map

$$\mathcal{D} + \mathcal{Q} :\ V \to W,$$

where $\mathcal{D}(a, \phi) = (D\phi, \rho(d^+a), d^*a))$, $\mathcal{Q}(a, \phi) = (\rho(a)\phi, \phi \otimes \phi^* - \frac{1}{2}|\phi|^2\,\mathrm{id}\,, 0)$, and W is defined to be the orthogonal complement to the constant functions in W'.

Now it is time to describe the group of symmetries of the equations. Define $\mathrm{Pin}(2) \subset SU(2)$ to be the normalizer of $S^1 \subset SU(2)$. Regarding $SU(2)$ as the group of unit quaternions and taking S^1 to be elements of the form $\mathrm{e}^{\sqrt{-1}\theta}$, then $\mathrm{Pin}(2)$ consists of the form $\mathrm{e}^{\sqrt{-1}\theta}$ or $\mathrm{e}^{\sqrt{-1}\theta} J$. We define the action of $\mathrm{Pin}(2)$ on V and W as follows: since S^+ and S^- are $SU(2)$ bundles, $\mathrm{Pin}(2)$ naturally acts on $\Gamma(S^\pm)$ by multiplication on the left. Z_2 acts on $\Gamma(\Lambda_C^*)$ by multiplication by ± 1 and this pulls back to an action of $\mathrm{Pin}(2)$ by the natural map $\mathrm{Pin}(2) \to Z_2$. A calculation shows that this pullback also describes the induced action of $\mathrm{Pin}(2)$ on $\sqrt{-1}\,\mathrm{su}(S^+)$. Both $\mathcal{D}$ and $\mathcal{Q}$ are seen to be $\mathrm{Pin}(2)$ equivariant maps.

Let X be a smooth closed spin 4-manifold and suppose that X admits a spin structure preserving action by a compact Lie group (or finite group) G. We may assume a Riemannian metric on X so that G acts by isometries. If the action is of even type, both $\mathcal{D}$ and $\mathcal{Q}$ are $\tilde{G} = \mathrm{Pin}(2) \times G$ equivariant maps.

Now we define V_λ to be the subspace of V spanned by the eigenspaces $\mathcal{D}^*\mathcal{D}$ with eigenvalues less than or equal to $\lambda \in R$. Similarly, we define W_λ using $\mathcal{D}\mathcal{D}^*$. The virtual G-representation $[V_\lambda \otimes C] - [W_\lambda \otimes C] \in R(\tilde{G})$ is the $\tilde{G}$-index of $\mathcal{D}$ and can be determined by the $\tilde{G}$-index and is independent of $\lambda \in R$, where $R(\tilde{G})$ is the complex representation of $\tilde{G}$. In particular, since $V_0 = \ker D$ and $W_0 = \operatorname{Coker} D \oplus \operatorname{Coker} d^+$, we have

$$[V_\lambda \otimes C] - [W_\lambda \otimes C] = [V_0 \otimes C] - [W_0 \otimes C] \in R(\tilde{G}).$$

Note that $\operatorname{Coker} d^+ = H^2_+(X, R)$.

The G-action on X can always be lifted to $\hat{G}$-actions on spinor bundles, where $\hat{G}$ is the following extension

$$1 \to Z_2 \to \hat{G} \to G \to 1.$$

Recall that the G-action is of even type if $\hat{G}$ contains a subgroup isomorphic to G, otherwise it is of odd type.

For S_3 action of odd type, it is easy to know that the extension of S_3 by Z_2 is isomorphic to the group

$$\tilde{S_3} = \langle a,\, b | a^6 = 1,\, b^2 = a^3,\, ba = a^{-1}b\rangle.$$

The group $\tilde{S_3}$ has 12 elements and can be partitioned into 6 conjugacy classes: the identity element 1, $\{b, a^2b\}$, $\{a^2, a^4\}$, $\{a, a^5, a^4b\}$, $\{a^3\}$, and $\{ab, a^3b, a^5b\}$.

The character table for $\tilde{S_3}$ is as following

	1	a^3	a^2	b	a	ab
η_0	1	1	1	1	1	1
η_1	1	-1	1	-1	i	$-\mathrm{i}$
η_2	1	1	1	1	-1	-1
η_3	1	-1	1	-1	$-\mathrm{i}$	i
η_4	2	2	-1	-1	0	0
η_5	2	-2	-1	1	0	0

3. The index of $\mathcal{D}$ and the character formula for the K-theory degree

The virtual representation $[V_{\lambda,C}] - [W_{\lambda,C}] \in R(\tilde{G})$ is the same as $\mathrm{Ind}(\mathcal{D}) = [\ker \mathcal{D}] - [\operatorname{Coker} \mathcal{D}]$. Furuta determines $\mathrm{Ind}(\mathcal{D})$ as a $\mathrm{Pin}(2)$ representation; denoting the restriction map $r:\ R(\tilde{G}) \to R(\mathrm{Pin}(2))$, Furuta shows

$$r(\mathrm{Ind}(\mathcal{D})) = 2kh - m\tilde{1}$$

where $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Thus $\mathrm{Ind}(\mathcal{D}) = sh - t\tilde{1}$ where s and t are polynomials such that $s(1) = 2k$ and $t(1) = m$. For a spin odd S_3 action, $\tilde{G} = \mathrm{Pin}(2) \times \tilde{S}_3$, we can write

$$s(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = a_0 + b_0\eta_1 + c_0\eta_2 + d_0\eta_3 + e_0\eta_4 + f_0\eta_5,$$

and

$$t(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = a_1 + b_1\eta_1 + c_1\eta_2 + d_1\eta_3 + e_1\eta_4 + f_1\eta_5,$$

such that $a_0 + b_0 + c_0 + d_0 + 2e_0 + 2f_0 = 2k$ and $a_1 + b_1 + c_1 + d_1 + 2e_1 + 2f_1 = m = b_2^+(X)$.

For any element $g \in \tilde{S}_3$, denote by $\langle g\rangle$ the subgroup of $\tilde{S}_3$ generated by g. Then we have

$$\begin{aligned}
\dim(H^+(X)^{\tilde{S}_3}) &= a_1 = b_2^+(X/\tilde{S}_3),\\
\dim(H^+(X)^{\langle a^3\rangle}) &= a_1 + c_1 + 2e_1 = b_2^+(X/\langle a^3\rangle),\\
\dim(H^+(X)^{\langle a^2\rangle}) &= a_1 + b_1 + c_1 + d_1 = b_2^+(X/\langle a^2\rangle),\\
\dim(H^+(X)^{\langle b\rangle}) &= a_1 + c_1 = b_2^+(X/\langle b\rangle),\\
\dim(H^+(X)^{\langle a\rangle}) &= a_1 + e_1 + f_1 = b_2^+(X/\langle a\rangle),\\
\dim(H^+(X)^{\langle ab\rangle}) &= a_1 + e_1 + f_1 = b_2^+(X/\langle ab\rangle),
\end{aligned}$$

The Thom isomorphism theory in equivariant K-theory for a general compact Lie group is a deep theory proved using elliptic operator [**1**]. The subsequent character formula of this section uses only elementary properties of the Bott class.

Let V and W be complex Γ representations for some compact Lie group Γ. Let BV and BW denote balls in V and W and let $f : BV \to BW$ be a Γ-map preserving the boundaries SV and SW. $K_\Gamma(V)$ is by definition $K_\Gamma(BV, SV)$, and by the equivariant Thom isomorphism theorem, $K_\Gamma(V)$ is a free $R(\Gamma)$ module with generator the Bott class $\lambda(V)$. Applying the K-theory functor to f we get a map

$$f^* : K_\Gamma(W) \to K_\Gamma(V)$$

which defines a unique element $\alpha_f \in R(\Gamma)$ by the equation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$. The element α_f is called the K-theory degree of f.

Let V_g and W_g denote the subspaces if V and W fixed by an element $g \in \Gamma$ and let $V_g^\perp$ and $W_g^\perp$ be the orthogonal complements. Let $f^g : V_g \to W_g$ be the restriction of f and let $d(f^g)$ denote the ordinary topological degree of f^g (by definition, $d(f^g) = 0$ if $\dim V_g \neq \dim W_g$). For any $\beta \in R(\Gamma)$, let $\lambda_{-1}\beta$ denote the alternating sum $\Sigma(-1)^i\lambda^i\beta$ of exterior powers.

T. tom Dieck proves the following character formula for the degree α_f:

Theorem ([**3**])**.** *Let $f : BV \to BW$ be a Γ-map preserving boundaries and let $\alpha_f \in R(\Gamma)$ be the K-theory degree. Then*

$$\mathrm{tr}_g(\alpha_f) = d(f^g)\,\mathrm{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp))$$

where tr_g is the trace of the action of an element $g \in \Gamma$.

This formula is very useful in the case where $\dim V_g \neq \dim W_g$ so that $d(f^g) = 0$.

Recall that $\lambda_{-1}(\Sigma_i a_i r_i) = \prod_i (\lambda_{-1} r_i)^{a_i}$ and that for a one dimensional representation r, we have $\lambda_{-1} r = (1 - r)$. A two dimensional representation such as h has $\lambda_{-1} h = (1 - h + \Lambda^2 h)$. In this case, since h comes from an $SU(2)$ representation, $\Lambda^2 h = \det h = 1$ so $\lambda_{-1} h = (2 - h)$.

In the following by using the character formula to examine the K-theory degree α_{f_λ} of the map $f_\lambda : BV_{\lambda,C} \to BW_{\lambda,C}$ coming from the Seiberg-Witten equations. We will abbreviate α_{f_λ} as α and $V_{\lambda,C}$ and $W_{\lambda,C}$ as just V and W. Let $\phi \in S^1 \subset \mathrm{Pin}(2) \subset G$ be the element generating a dense subgroup of S^1, and recall that there is the element $J \in \mathrm{Pin}(2)$ coming from the quaternion. Note that the action of J on h has two invariant subspaces on which J acts by multiplication with $\sqrt{-1}$ and $-\sqrt{-1}$.

4. The main results

Consider $\alpha = \alpha_{f_\lambda} \in R(\mathrm{Pin}(2) \times \tilde{S_3})$, it has the following form

$$\alpha = \alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i.$$

where $\alpha_i = l_i + m_i\eta_1 + n_i\eta_2 + p_i\eta_3 + q_i\eta_4 + r_i\eta_5$, $i \geq 0$ and $\tilde{\alpha_0} = \tilde{l_0} + \tilde{m_0}\eta_1 + \tilde{n_0}\eta_2 + \tilde{p_0}\eta_3 + \tilde{q_0}\eta_4 + \tilde{r_0}\eta_5$.

Since ϕ acts non-trivially on h and trivially on $\tilde{1}$, then we have

$$\begin{aligned}\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_\phi &- \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{1})_\phi \\ &= -(a_1 + b_1 + c_1 + d_1 + 2e_1 + 2f_1) = -b_2^+(X).\end{aligned}$$

So if $b_2^+(X) \neq 0$, $\mathrm{tr}_\phi\, \alpha = 0$.

ϕa acts non-trivially on $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$ but trivially on $a_1\tilde{1}$. Besides, the action of a on $e_1\eta_4$ and $f_1\eta_5$ both have a one-dimensional invariant subspace, then we have

$$\begin{aligned}\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))h)_{\phi a} &- \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))\tilde{1})_{\phi a} \\ &= -(a_1 + e_1 + f_1) = -b_2^+(X/\langle a\rangle).\end{aligned}$$

So if $a_1 + e_1 + f_1 = b_2^+(X/\langle a\rangle) \neq 0$, $\mathrm{tr}_{\phi a}\, \alpha = 0$.

Since ϕa^2 acts non-trivially on $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$, and trivially on $a_1\tilde{1}$, $b_1\eta_1\tilde{1}$ and $d_1\eta_3\tilde{1}$, then we have

$$\begin{aligned}\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))_{\phi a^2} &- \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))_{\phi a^2} \\ &= -(a_1 + b_1 + c_1 + d_1) = -b_2^+(X/\langle a^2\rangle).\end{aligned}$$

So if $a_1 + b_1 + c_1 + d_1 = b_2^+(X/\langle a^2\rangle) \neq 0$, $\mathrm{tr}_{\phi a^2}\, \alpha = 0$.

ϕa^3 acts non-trivially on $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$ but trivially on $a_1\tilde{1}$ and $c_1\eta_2\tilde{1}$. Besides, the action of a^3 on $e_1\eta_4$ has a two-dimensional invariant subspaces, so we have

$$\begin{aligned}\dim(V(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))_{\phi a^3} &- \dim(W(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5))_{\phi a^3}\\ &= -(a_1+c_1+2e_1) = -b_2^+(X/\langle a^3\rangle).\end{aligned}$$

So if $a_1+c_1+2e_1 = b_2^+(X/\langle a^3\rangle) \neq 0$, $\operatorname{tr}_{\phi a^3}\alpha = 0$.

Since ϕb acts non-trivially on $V(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)h$ and trivially on $a_1\tilde{1}$ and $c_1\eta_2\tilde{1}$, then we have

$$\begin{aligned}\dim(V(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)h)_{\phi b} &- \dim(W(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)\tilde{1})_{\phi b}\\ &= -(a_1+c_1) = -b_2^+(X/\langle b\rangle).\end{aligned}$$

So if $a_1+c_1 = b_2^+(X/\langle b\rangle) \neq 0$, $\operatorname{tr}_{\phi b}\alpha = 0$.

ϕab acts non-trivially on $V(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)h$ but trivially on $a_1\tilde{1}$. Besides, the action of ab on $e_1\eta_4$ and $f_1\eta_5$ both have a one-dimensional invariant subspace, then we have

$$\begin{aligned}\dim(V(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)h)_{\phi ab} &- \dim(W(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)\tilde{1})_{\phi ab}\\ &= -(a_1+e_1+f_1) = -b_2^+(X/\langle ab\rangle).\end{aligned}$$

So if $a_1+e_1+f_1 = b_2^+(X/\langle ab\rangle) \neq 0$, $\operatorname{tr}_{\phi ab}\alpha = 0$.

From the above analysis, we know if $b_2^+(X/\langle a\rangle) \neq 0$ and $b_2^+(X/\langle b\rangle >) \neq 0$, we have $\operatorname{tr}_\phi\alpha = \operatorname{tr}_{\phi a}\alpha = \operatorname{tr}_{\phi a^2}\alpha = \operatorname{tr}_{\phi a^3}\alpha = \operatorname{tr}_{\phi b}\alpha = \operatorname{tr}_{\phi ab}\alpha = 0$ which implies that

$$\begin{aligned}0 = \operatorname{tr}_\phi\alpha &= \operatorname{tr}_\phi(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\alpha_i h_i) = \operatorname{tr}_\phi\alpha_0 + \operatorname{tr}_\phi\tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\operatorname{tr}\alpha_i(\phi^i+\phi^{-i})\\ &= (l_0+m_0+n_0+p_0+q_0+r_0) + (\tilde{l_0}+\tilde{m_0}+\tilde{n_0}+\tilde{p_0}+\tilde{q_0}+\tilde{r_0})\\ &\quad + \sum_{i=1}^{\infty}\operatorname{tr}\alpha_i(\phi^i+\phi^{-i}),\\ 0 = \operatorname{tr}_{\phi a}\alpha &= \operatorname{tr}_{\phi a}(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\alpha_i h_i) = \operatorname{tr}_a\alpha_0 + \operatorname{tr}_a\tilde{\alpha_0} + \sum_{i=1}^{\infty}\operatorname{tr}_a\alpha_i(\phi^i+\phi^{-i})\\ &= (l_0+im_0-n_0-ip_0) + (\tilde{l_0}+i\tilde{m_0}-\tilde{n_0}-i\tilde{p_0})\\ &\quad + \sum_{i=1}^{\infty}\operatorname{tr}_a\alpha_i(\phi^i+\phi^{-i}),\\ 0 = \operatorname{tr}_{\phi a^2}\alpha &= \operatorname{tr}_{\phi a^2}(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty}\alpha_i h_i) = \operatorname{tr}_{a^2}\alpha_0 + \operatorname{tr}_{a^2}\tilde{\alpha_0} + \sum_{i=1}^{\infty}\operatorname{tr}_{a^2}\alpha_i(\phi^i+\phi^{-i})\\ &= (l_0+m_0+n_0+p_0-q_0-r_0) + (\tilde{l_0}+\tilde{m_0}+\tilde{n_0}+\tilde{p_0}-\tilde{q_0}-\tilde{r_0})\\ &\quad + \sum_{i=1}^{\infty}\operatorname{tr}_{a^2}\alpha_i(\phi^i+\phi^{-i}),\end{aligned}$$

$$
\begin{aligned}
0 = &\operatorname{tr}_{\phi a^3} \alpha = \operatorname{tr}_{\phi a^3}(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i) = \operatorname{tr}_{a^3} \alpha_0 + \operatorname{tr}_{a^3} \tilde{\alpha_0} + \sum_{i=1}^{\infty} \operatorname{tr}_{a^3} \alpha_i(\phi^i + \phi^{-i}) \\
=&(l_0 - m_0 + n_0 - p_0 + 2q_0 - 2r_0) + (\tilde{l_0} - \tilde{m_0} + \tilde{n_0} - \tilde{p_0} + 2\tilde{q_0} - 2\tilde{r_0}) \\
&+ \sum_{i=1}^{\infty} \operatorname{tr}_{a^3} \alpha_i(\phi^i + \phi^{-i}), \\
0 = &\operatorname{tr}_{\phi b} \alpha = \operatorname{tr}_{\phi b}(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i) = \operatorname{tr}_{b} \alpha_0 + \operatorname{tr}_{b} \tilde{\alpha_0} + \sum_{i=1}^{\infty} \operatorname{tr}_{b} \alpha_i(\phi^i + \phi^{-i}) \\
=&(l_0 - m_0 + n_0 - p_0 - q_0 + r_0) + (\tilde{l_0} - \tilde{m_0} + \tilde{n_0} - \tilde{p_0} - \tilde{q_0} + \tilde{r_0}) \\
&+ \sum_{i=1}^{\infty} \operatorname{tr}_{b} \alpha_i(\phi^i + \phi^{-i}), \\
0 = &\operatorname{tr}_{\phi ab} \alpha = \operatorname{tr}_{\phi ab}(\alpha_0 + \tilde{\alpha_0}\tilde{1} + \sum_{i=1}^{\infty} \alpha_i h_i) = \operatorname{tr}_{ab} \alpha_0 + \operatorname{tr}_{ab} \tilde{\alpha_0} + \sum_{i=1}^{\infty} \operatorname{tr}_{ab} \alpha_i(\phi^i + \phi^{-i}) \\
=&(l_0 - im_0 - n_0 + ip_0) + (\tilde{l_0} - i\tilde{m_0} - \tilde{n_0} + i\tilde{p_0}) \\
&+ \sum_{i=1}^{\infty} \operatorname{tr}_{ab} \alpha_i(\phi^i + \phi^{-i}),
\end{aligned}
$$

and so on. From these equations, we have $\alpha_0 = -\tilde{\alpha_0}$ and $\alpha_i = 0, i > 0$, that is $\alpha = \alpha_0(1 - \tilde{1})$.

Next we calculate $\operatorname{tr}_J \alpha$. Since J acts non-trivially on both h and $\tilde{1}$, $\dim V_J = \dim W_J = 0$, so $d(f^J) = 1$. Using $\operatorname{tr}_J h = 0$ and $\operatorname{tr}_J \tilde{1} = -1$, by the character formula we have

$$\operatorname{tr}_J(\alpha) = \operatorname{tr}_J(\lambda_{-1}(m\tilde{1} - 2kh)) = \operatorname{tr}_J((1 - \tilde{1})^m (2 - h)^{-2k}) = 2^{m-2k}.$$

Now we calculate $\operatorname{tr}_{Ja} \alpha$. Ja acts non-trivially on both $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$, but trivially on $c_1\eta_2\tilde{1}$. Besides, the action of a on $e_1\eta_4\tilde{1}$ and $f_1\eta_5\tilde{1}$ both have a one-dimensional invariant subspace. So we have

$$\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{Ja} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{1})_{Ja} = -(c_1 + e_1 + f_1).$$

Then, if $c_1 + e_1 + f_1 \neq 0$, $\operatorname{tr}_{Ja} \alpha = 0$

Since Ja^2 acts non-trivially on both $V(\eta_1,\eta_2, \eta_3, \eta_4,\eta_5)h$ and $W(\eta_1, \eta_2,\eta_3,\eta_4, \eta_5)\tilde{1}$, then $d(f^{Ja^2}) = 1$. By tom Dieck formula, we have

$$
\begin{aligned}
\operatorname{tr}_{Ja^2} \alpha &= \operatorname{tr}_{Ja^2}[\lambda_{-1}(a_1 + b_1\eta_1 + c_1\eta_2 + d_1\eta_3 + e_1\eta_4 + f_1\eta_5)\tilde{1} \\
&\quad - \lambda_{-1}(a_0 + b_0\eta_1 + c_0\eta_2 + d_0\eta_3 + e_0\eta_4 + f_0\eta_5)h] \\
&= 2^{(a_1+b_1+c_1+d_1)-(a_0+b_0+c_0+d_0)}.
\end{aligned}
$$

Ja^3 acts non-trivially on $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$, but trivially on $b_1\eta_1\tilde{1}$ and $d_1\eta_3\tilde{1}$. Besides, the action of Ja^3 on $f_1\eta_5\tilde{1}$ has two invariant subspaces. So

$$\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{Ja^3} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{1})_{Ja^3} = -(b_1 + d_1 + 2f_1).$$

Then, if $b_1 + d_1 + 2f_1 \neq 0$, $\mathrm{tr}_{Ja^3}\,\alpha = 0$.

Since Jb acts non-trivially on $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$ but trivially on $b_1\eta_1\tilde{1}$ and $d_1\eta_3\tilde{1}$, then

$$\begin{aligned}&\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{Jb} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{1})_{Jb}\\&= -(b_1 + d_1) = b_2^+(X/\langle a^2\rangle - b_2^+(X/\langle b\rangle).\end{aligned}$$

Then, if $b_1 + d_1 \neq 0$, $\mathrm{tr}_{Jb}\,\alpha = 0$

Jab acts non-trivially on $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$ but trivially on $c_1\eta_2\tilde{1}$. Besides, the action of ab on $e_1\eta_4\tilde{1}$ and $f_1\eta_5\tilde{1}$ both have a one-dimensional invariant subspace. Then we have

$$\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{Jab} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{1})_{Jab} = -(c_1 + e_1 + f_1).$$

Then by assuming $b_2^+(X/\langle a^2\rangle - b_2^+(X/\langle b\rangle) \neq 0$ and $b_2^+(X/\langle a^3\rangle - b_2^+(X/\langle b\rangle) \neq 0$, we have $\mathrm{tr}_{Ja}\,\alpha = 0$, $\mathrm{tr}_{Ja^3}\,\alpha = 0$, $\mathrm{tr}_{Jb}\,\alpha = 0$, $\mathrm{tr}_{Jab}\,\alpha = 0$

By direct calculation, we have

$$\mathrm{tr}_J\,\alpha_0 = l_0 + m_0 + n_0 + p_0 + 2q_0 + 2r_0 = 2^{m-2k-1}, \tag{3}$$

$$\mathrm{tr}_{a^2}\,\alpha_0 = l_0 + m_0 + n_0 + p_0 - q_0 - r_0 = 2^{(a_1+b_1+c_1+d_1)-(a_0++b_0+c_0+d_0)-1}, \tag{4}$$

$$\mathrm{tr}_a\,\alpha_0 = l_0 + im_0 - n_0 - ip_0 = 0, \tag{5}$$

$$\mathrm{tr}_{a^3}\,\alpha_0 = l_0 - m_0 + n_0 - p_0 + 2q_0 - 2r_0 = 0, \tag{6}$$

$$\mathrm{tr}_b\,\alpha_0 = l_0 - m_0 + n_0 - p_0 - q_0 + r_0 = 0, \tag{7}$$

$$\mathrm{tr}_{ab}\,\alpha_0 = l_0 - im_0 - n_0 + ip_0 = 0, \tag{8}$$

Here we use $\mathrm{tr}_{Jg}\,\alpha = \mathrm{tr}_g(2 \cdot \alpha_0) = 2 \cdot \mathrm{tr}_g\,\alpha_0$ where g is any element of $\tilde{S_3}$.

From (3), (5), (6) and (8), we get $l_0 + q_0 = 2^{m-2k-3}$. So we have the following main result.

Theorem 4.1. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\sigma(X)/16$ and $m = b_2^+(X)$. If X admits a spin odd type S_3 action, then $2k + 3 \leq m$, if $b_2^+(X/\langle a\rangle) \neq 0$, $b_2^+(X/\langle b\rangle) \neq 0$, $b_2^+(X/\langle a^2\rangle) - b_2^+(X/\langle b\rangle) \neq 0$ and $b_2^+(X/\langle a^3\rangle) - b_2^+(X/\langle b\rangle) \neq 0$.*

Besides, from the above six equations, we get

$$q_0 = r_0 = [2^{m-2k-2} - 2^{(a_1+b_1+c_1+d_1)-(a_0+b_0+c_0+d_0)-2}]/3$$

$$l_0 = m_0 = n_0 = p_0 = [2^{m-2k-3} - 2^{(a_1+b_1+c_1+d_1)-(a_0+b_0+c_0+d_0)-2}]/3$$

Since $q_0 \in Z$, then $2^{m-2k-2} - 2^{(a_1+b_1+c_1+d_1)-(a_0+b_0+c_0+d_0)-2} \in 3Z \subset Z$. From Theorem 4.1, we know $2^{m-2k-2} \in Z$. So $2^{(a_1+b_1+c_1+d_1)-(a_0+b_0+c_0+d_0)-2} \in Z$, i.e., $(a_1 + b_1 + c_1 + d_1) \geq (a_0 + b_0 + c_0 + d_0) + 2$. Hence, we have

Theorem 4.2. *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If X admits a spin odd type S_3 action, then*

$$b_2^+(X/\langle a^2\rangle) \geq \dim((Ind_{\tilde{S_3}}D)^{\langle a^2\rangle}) + 2,$$

if $b_2^+(X/\langle a\rangle) \neq 0$, $b_2^+(X/\langle b\rangle) \neq 0$, $b_2^+(X/\langle a^2\rangle) - b_2^+(X/\langle b\rangle) \neq 0$ *and* $b_2^+(X/\langle a^3\rangle) - b_2^+(X/\langle b\rangle) \neq 0$. *Moreover, under this condition, the K-theory degree* $\alpha = \alpha_0(1-\tilde{1})$ *for some* $\alpha_0 = l_0(1+\eta_1+\eta_2+\eta_3) + q_0(\eta_4+\eta_5)$.

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